The disjoint curve property and genus 2 manifolds

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Abstract

Genus 2 manifolds are a convenient and accessible place to introduce an interesting condition on Heegaard splittings, called the disjoint curve property. This paper will describe the disjoint curve property and its ramifications for understanding genus 2 manifolds, and use it to find a necessary condition for a genus 2 manifold to be hyperbolic. © 1999 Elsevier Science B.V. All rights reserved.

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Definition. Let \((H_1, H_2, F)\) be a Heegaard splitting of a 3-manifold \(M\), where \(H_1\) and \(H_2\) are handlebodies and \(F = \partial H_1 = \partial H_2\). The genus of the Heegaard splitting is the genus of the surface \(F\). The Heegaard splitting is reducible if there exists an essential simple closed curve \(c\) on \(F\) such that \(c\) bounds (imbedded) disks \(D_1\) in \(H_1\) and \(D_2\) in \(H_2\). The splitting is stabilized if there exist essential simple closed curves \(c_1\) and \(c_2\) on \(F\) such that \(c_i\) bounds an (imbedded) disk \(D_i\) in \(H_i\) and \(c_1\) and \(c_2\) intersect in a single point. A stabilized splitting of genus at least 2 is reducible. The splitting is weakly reducible if there exist essential simple closed curves \(c_1\) and \(c_2\) on \(F\) such that \(c_i\) bounds an (imbedded) disk \(D_i\) in \(H_i\) and \(c_1\) and \(c_2\) are disjoint. A splitting that is not weakly reducible is strongly irreducible. As this paper will deal largely with genus 2 Heegaard splittings, it is worth noting the following well-known proposition:

Proposition 1. A weakly reducible genus 2 Heegaard splitting is reducible.

Proof. Let \((H_1, H_2, F)\) be a genus 2 Heegaard splitting of a 3-manifold \(M\). Let \(c_1\) and \(c_2\) be essential simple closed curves on \(F\) such that \(c_i\) bounds an (imbedded) disk \(D_i\) in \(H_i\)

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and \( c_1 \) and \( c_2 \) are disjoint. Assume \( c_1 \) and \( c_2 \) are non-separating (the argument is similar, but easier, if one is separating). Let \( F' = F - (c_1 \cup c_2) \). \( F' \) is a 4-punctured sphere. Let \( c \) be a curve on \( F' \) separating the copies of \( c_1 \) from the copies of \( c_2 \). \( c \) can be obtained by banding together two copies of \( c_i \). Hence \( c \) bounds a disk in \( H_i \). So \((H_1, H_2, F)\) is reducible.

The notion of weak reducibility was introduced by Casson and Gordon [2]. It has proved a very useful tool. They showed that if \( M \) has a weakly reducible Heegaard splitting, either \( M \) is Haken or the splitting is reducible. Subsequent to their results, most attention has concentrated on the strongly irreducible case.

We introduce a ‘weaker’ form of reducibility:

**Definition.** Let \((H_1, H_2; F)\) be a Heegaard splitting of a 3-manifold \( M \). The Heegaard splitting has the *disjoint curve property* if there exist essential simple closed curves \( c, a \) and \( b \) on \( F \) such that \( c \) is disjoint from \( a \) and \( b \), \( a \) bounds a disk in \( H_1 \), and \( b \) bounds a disk in \( H_2 \).

**Notes.**

1. If the splitting is assumed to be strongly irreducible, \( a \) and \( b \) must necessarily intersect.
2. If a strongly irreducible Heegaard splitting fails to have the disjoint curve property, then every pair of curves such as \( a \) and \( b \) must cut the splitting surface into disks, i.e., they must fill the surface. In particular, any such pair of curves \( a \) and \( b \) must intersect in many points.

**Theorem 2.** Let \((H_1, H_2, F)\) be a strongly irreducible Heegaard splitting of a closed orientable 3-manifold \( M \). If \( M \) contains an essential torus \( T \), then \((H_1, H_2, F)\) has the disjoint curve property.

**Proof.** Let \( T \) be an essential torus in \( M \). By Kobayashi [5], \( T \) can be isotoped to intersect \( F \) in a collection of (at least 2) simple closed curves which are essential on \( T \). These curves split \( T \) into annuli. Further, none of the annuli are boundary parallel in \( H_1 \) or \( H_2 \). Let \( A_1 \) and \( A_2 \) be two adjacent annuli on \( T \), sharing the curve \( c \) on their boundaries. Since \( T \) is essential, \( A_1 \) and \( A_2 \) are incompressible. Since they are imbedded in the handlebodies \( H_1 \) and \( H_2 \) they are boundary compressible. Let \( D_1 \) and \( D_2 \) be the disks obtained by boundary compressions on \( A_1 \) and \( A_2 \), respectively. \( D_1 \) and \( D_2 \) are essential since \( A_1 \) \( A_2 \) are not boundary parallel. \( D_1 \) and \( D_2 \) are both disjoint from \( c \), hence \( c \) is a disjoint curve for the Heegaard splitting. \( \square \)

**Corollary 3.** Let \((H_1, H_2, F)\) be a Heegaard splitting of a closed orientable 3-manifold \( M \). If \((H_1, H_2, F)\) does not have the disjoint curve property, then \( M \) is atoroidal.

So there is a simple condition on a Heegaard splitting of \( M \) which guarantees that \( M \) is atoroidal.
We now focus on the genus 2 case, to understand genus 2 splittings with the disjoint curve property. We will assume all curves on the splitting surface intersect transversely.

**Definition.** Let $T$ be a separating torus in a 3-manifold $M$ (= a Lens space, $S^3$, or $S^1 \times S^2$), bounding solid tori, $T_1$ and $T_2$, on both sides. Let $K$ be a knot in $M$. $K$ is a 1-bridge braid in $M$ if $K$ can be isotoped to intersect $T$ transversely in two points, such that $K$ intersects each $T_i$ in a boundary parallel arc.

**Theorem 4.** Let $(H_1, H_2, F)$ be a genus 2 Heegaard splitting of a 3-manifold $M$. Suppose $(H_1, H_2, F)$ has the disjoint curve property. Then either:

1. $(H_1, H_2, F)$ is a stabilization;
2. $M$ is reducible (hence a connect sum of Lens spaces);
3. $M$ contains an essential torus—this case was analyzed by Kobayashi in [6];
4. $M$ is obtained by surgery on a 1-bridge braid on an unknotted torus in a Lens space, $S^3$ or $S^1 \times S^2$.

**Proof.** Let $c$, $a$ and $b$ be essential simple closed curves on $F$ such that $c$ is disjoint from $a$ and $b$, $a$ bounds a disk $D_1$ in $H_1$, and $b$ bounds a disk $D_2$ in $H_2$.

**Definition.** A simple closed curve $c$ on $\partial H_i$ is a core of $H_i$ if there exists a simple closed curve $c_0$ on $\partial H_i$ such that $c$ intersects $c_0$ in a single point and $c_0$ bounds a disk $E$ in $H_i$.

**Case 1.** $c$ is not a core in $H_1$ and $c$ is not a core in $H_2$. Cut $H_i$ along $D_i$. If $D_i$ is non-separating, let $T_i$ be the resulting solid torus. If $D_i$ is separating, let $T_i$ be the solid torus containing $c$ in its boundary.

Let $M^*$ be the manifold obtained by gluing $T_1$ and $T_2$ together along an annular neighborhood of $c$. $M^*$ is contained in $M$. Since $c$ is not a core in $H_1$ or $H_2$, $M^*$ is a Seifert-fibered space, with base space a disk, with two singular fibers. The boundary of $M^*$ is a torus $T$. Let $M^{**} = M - M^*$.

If $T$ is incompressible in $M^{**}$, condition (3) holds.
If $M^{**}$ is a solid torus, condition (0), (1) or (2) hold.

Suppose $T$ is compressible in $M^{**}$, but $M^{**}$ is not a solid torus. Let $S$ be the 2-sphere obtained by compressing $T$ in $M^{**}$. If $S$ is essential in $M$, then (1) holds. If $S$ is not essential in $M$, then it bounds a 3-ball containing $M^*$, and hence $c'$, a core of the solid torus $T_1$. $c'$ is also a core of $H_1$. It follows from an argument due to Frohman [3] that the Heegaard splitting is weakly reducible, and so reducible.

**Case 2.** $c$ is a core in $H_1$. Let $K$ be a core of $H_2$, disjoint from $b$. Let $G$ be an essential disk in $H_2$, disjoint from $b$, intersecting $K$ once, such that

\[
\text{algebraic intersection } #(\partial G, c) = \text{geometric intersection } #(\partial G, c).
\]

If $b$ is separating, choose $K$ in the component of $H_2$ which contains $c$. Push $K$ slightly into the interior of $H_2$. Let $A$ be an annulus with one boundary component equal to $c$ on
\( \partial H_2 \) and one boundary component \( r \) on the boundary of a neighborhood of \( K \). Notice that \( r \) surgery on \( K \) yields a manifold \( N \) with a natural genus 2 Heegaard splitting, \((H_1, H'_2, F)\) where \( H'_2 \) is obtained from \( H_2 \) by \( r \)-surgery on \( K \). This new Heegaard splitting is stabilized; \( c \) now bounds a disk \( E \) in \( H'_2 \), and \( c \) intersects the boundary \( c' \) of a disk \( E' \) in \( H_1 \) in a single point. Thus \( N \) is either a Lens space, \( S^3 \) or \( S^1 \times S^2 \).

Compressing \( H'_2 \) along the disk \( E \) yields an unknotted solid torus \( W \) in \( N \). \( \partial W \) is punctured twice by \( K' \), the dual of \( K \), in \( N \). The arc of \( K' \) exterior to \( W \) is boundary parallel to \( \partial W \) via the remnants of the disk \( E' \). The arc of \( K' \) interior to \( W \) is boundary parallel to \( \partial W \) via the remnants of the disk \( G \). So \( K' \) is a 1-bridge braid on the unknotted solid torus \( W \), and condition (4) holds. \( \square \)

**Note.** Hempel has obtained results similar to Theorems 2–4 independently [4].

**Theorem 5.** Let \((H_1, H_2, F)\) be a genus 2 Heegaard splitting of a 3-manifold \( M \). Suppose there exist essential disks \( D_1 \) and \( D_2 \) in \( H_1 \) and \( H_2 \), respectively such that \( \partial D_1 \) intersects \( \partial D_2 \) in at most three points. Then

1. \((H_1, H_2, F)\) has the disjoint curve property, and
2. \( M \) is not hyperbolic.

**Proof.**

1. If \( \partial D_1 \) intersects \( \partial D_2 \) in one point, the Heegaard splitting is stabilized.

2. Suppose \( \partial D_1 \) intersects \( \partial D_2 \) in exactly two points. Then \( D_1 \) and \( D_2 \) cannot both be separating. There exists an essential simple closed curve \( c \) on \( F \) disjoint from \( \partial D_1 \) and \( \partial D_2 \). If \( c \) is not a core on either \( H_1 \) or \( H_2 \), we are done by the argument in Case 1 of Theorem 4.

**Case 1.** \( D_1 \) is separating and \( c \) is a core on \( H_1 \). Then there exists a compressing disk \( E \) for \( H_1 \) which intersects \( \partial D_2 \) in at most one point, so the splitting is either reducible or stabilized.

**Case 2.** \( D_1 \) is non-separating and \( c \) is a core on \( H_1 \). Recall that \( \partial D_2 \) is disjoint from the core \( c \). Consider the graph \( G \) on the annulus \( A = (F - (\partial D_1 \cup c)) \) formed by contracting the two copies of \( D_1, E_1 \) and \( E_2 \), to points, and taking the arcs of \( \partial D_2 \) as edges. The two vertices of \( G \) are both order two. Examining \( G \), one can see that either there exists a compressing disk \( E \) for \( H_1 \) which intersects \( \partial D_2 \) in at most one point, or \( G \) consists of two essential loops in \( A \). If \( G \) is two essential loops, let \( c' \) be an essential curve in \( A \) separating the two loops of \( G \). Then there exists a properly imbedded annulus \( X \) in \( H_1 \), with boundary \( c \cup c' \), disjoint from \( \partial D_1 \) and \( \partial D_2 \), which is incompressible, non-boundary parallel, and non-separating in \( H_1 \). Note that \( c \) and \( c' \) are not parallel on \( F \).

**Subcase (a).** \( D_2 \) is non-separating in \( H_2 \). Then \( c \cup c' \) bound an essential non-separating annulus \( Y \) in \( H_2 \). \( X \cup Y \) is a non-separating torus in \( M \).

**Subcase (b).** \( D_2 \) is separating in \( H_2 \). Since \( c \) and \( c' \) are not parallel on \( F \), \( \partial D_2 \) must separate \( c \) and \( c' \). If either \( c \) or \( c' \) is a core of \( H_2 \), the arguments from case 1 apply. Suppose neither is a core. Let \( T_1 \) and \( T_2 \) be the solid tori obtained by cutting \( H_2 \) open along \( D_2 \). Let
Let $M^*$ be the manifold $[T_1 \cup T_2 \cup (\text{nhd}(X))]$. Apply the arguments from Theorem 4 case 1 to $M^*$.

(III) Suppose $\partial D_1$ intersects $\partial D_2$ in exactly three points. Then both $D_1$ and $D_2$ are non-separating.

**Claim.** There exists an essential simple closed curve $c$ on $F$ disjoint from $\partial D_1$ and $\partial D_2$.

**Proof.** Let $R = F - \partial D_1$. $\partial D_2$ intersects $R$ in three arcs, $x$, $y$, and $z$. Suppose there are no simple closed curves disjoint from $\partial D_1$ and $\partial D_2$. Then $x$, $y$, and $z$ must cut $R$ into a disk. At least one of the arcs, say $x$, must connect the two boundary components of $R$. Cut $R$ open along $x$ to obtain $R'$, a once-punctured torus. The remaining two arcs of $\partial D_2$, $y$ and $z$, cut $R'$ into a disk. Thus the endpoints of these arcs are linked on $\partial R'$. Since the endpoints of $y$ and $z$ are linked, $y$ and $z$ must also connect the two boundary components of $R$. When $F$ is reconstructed from $R$ by re-attaching the two boundary components, exactly one of the three arcs will form a single simple closed curve while the other two form a second simple closed curve. This contradicts the connectivity of $\partial D_2$. Hence there is a simple closed curve $c$ disjoint from $\partial D_1$ and $\partial D_2$.

**Case 1.** $c$ is not a core in $H_1$ or $H_2$. The argument is the same as for Theorem 4, Case 1.

**Case 2.** $c$ is a core in $H_1$. Cut $H_1$ open along $D_1$, to obtain a solid torus $Q$. $c$ is a core of $Q$, and the image of $D_1$ on $\partial Q$ is two disks $E_1$ and $E_2$. $\partial D_2$ is a collection of at most three arcs imbedded in $\partial Q$ with endpoints on $E_1$ and $E_2$, and $\partial D_2$ is disjoint from $c$. [\[\]

**Claim** (with thanks to Y.Q. Wu). There exists an essential disk $J$ for $Q$ such that $\partial J$ intersects $\partial D_2$ in at most two points.

**Proof.** Recall that $\partial D_2$ is disjoint from the core $c$, and consider the graph $G$ on $\partial Q$ formed by contracting $E_1$ and $E_2$ to points and taking the arcs of $\partial D_2$ as edges. Since the two vertices of $G$ are both order three, there are only three possibilities for $G$, and in all cases there exists a meridian disk $J$ of $Q$ intersecting $\partial D_2$ in at most two points. [\[\]

We can use this idea to examine tunnel number one knots in the 3-sphere:

Let $K$ be a hyperbolic tunnel number one knot in $S^3$ with tunnel $t$. Let $H_1$ be the compression body obtained by taking a $(2\delta)$-neighborhood of $K \cup t$ and removing a $(\delta)$-neighborhood of $K$ from it. $\partial H_1 = F$ separates $S^3$ into two pieces. Let $H_2$ be the piece which does not contain $K$. $H_2$ is a handlebody by construction.

**Corollary 6.** Let $m \subset \partial H_1$ be a meridian of $t$. Then every compressing disk $D_2$ for $H_2$ crosses $m$ at least 4 times.

**Proof.** Consider the generalized Heegaard splitting $(H_1, H_2, F)$ of the complement of $K$. $m$ is the boundary of a compressing disk $D_1$ in $H_1$. Let $K(r)$ be the 3-manifold obtained by $r$-Dehn surgery on $K$. By Thurston’s hyperbolic surgery theorem, $K(r)$ is hyperbolic.
for all but a finite number of slopes $r$. Choose $s$ such that $K(s)$ is hyperbolic. Applying Theorem 5 to $K(s)$, it follows that every compressing disk $D_2$ for $H_2$ crosses $m = \partial D_1$ at least 4 times. □

**Note.** Adams obtained Corollary 6 in the case that the unknotting tunnel is a vertical geodesic [1, Corollary 5.4].

**Theorem 7.** Let $(H_1, H_2, F)$ be a genus 2 Heegaard splitting of a 3-manifold $M$. Suppose $(H_1, H_2, F)$ has the disjoint curve property with curves $a$, $b$ and $c$. If $a$ intersects $b$ in at most four points, then $M$ is not hyperbolic.

**Proof.** If $a$ intersects $b$ in at most three points, we are done by Theorem 5. So assume $a$ intersects $b$ in exactly four points.

If $c$ is not a core in $H_1$ and it is not a core in $H_2$ we are done.

Assume $c$ is a core in $H_1$.

Subcase (a). Suppose $D_1$ is separating. Cut $H_1$ open along $D_1$. Let $Q$ be the solid torus component with core $c$. The image of $D_1$ on $\partial Q$ is a disk $E$. $\partial D_2$ is a collection of two arcs imbedded in $\partial Q$ with endpoints on $E$, and $\partial D_2$ is disjoint from $c$. Consider the graph $G$ on $\partial Q$ formed by contracting $E$ to a point and taking the arcs of $\partial D_2$ as edges. For all possibilities for $G$, there exists a compressing disk $J$ for $H_1$ intersecting $\partial D_2$ in at most two points. This case is covered by Theorem 5.

Subcase (b). Suppose $D_1$ is non-separating. Cut $H_1$ open along $D_1$ to obtain a solid torus $Q$. $c$ is a core of $Q$, and the image of $D_1$ on $\partial Q$ is two disks $E_1$ and $E_2$. $\partial D_2$ is a collection of four arcs imbedded in $\partial Q$ with endpoints on $E_1$ and $E_2$, and $\partial D_2$ is disjoint from $c$. Consider the graph $G$ on $\partial Q$ formed by contracting $E_1$ and $E_2$ to points and taking the arcs of $\partial D_2$ as edges.

If $G$ contains at most one loop at each vertex, then there exists a compressing disk $J$ for $H_1$ intersecting $\partial D_2$ in at most three points.

Suppose $G$ consists of two loops at each vertex. If any of the loops are not essential in $\partial Q - c$, then there exists a compressing disk $J$ for $H_1$ intersecting $\partial D_2$ in at most three points. Suppose $G$ consists of two loops at each vertex, all essential in $\partial Q - c$. Then the argument proceeds exactly as in Theorem 5 (II), Case 2. □

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**References**
