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Minimal Graphs with Crossing Number at Least k

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It is easily seen that, for each k, there is a graph G whose crossing number is at least k and an edge e of G such that $G-e$ is planar. We show that there is a positive constant c such that, if G is a graph with crossing number k, then there exists an edge e of G such that $G-e$ has crossing number at least ck. © 1993 Academic Press, Inc.

1. INTRODUCTION

Recent work has dealt with the set $\mathcal{M}_k$ of graphs which are minimal with respect to having crossing number at least k. In [5, 2] are exhibited infinitely many different 3-connected graphs in $\mathcal{M}_k$, for each $k \geq 2$. In [4], it is shown there are exactly 8 cubic graphs in $\mathcal{M}_2$, all of which have crossing number equal to 2. This is extended in [7], where it is shown that every graph in $\mathcal{M}_2$ has crossing number equal to 2 with the single exception of $C_3 \times C_3$, which has crossing number equal to 3.

The crossing number of the complete bipartite graph $K_{3,n}$ is known to be $\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$ and an optimal drawing has one edge crossing $\lceil n/2 \rceil - 1$ others. (See [1].) Since $K_{3,n}$ is edge-transitive, it is, therefore, in $\mathcal{M}_k$ for $k = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor - \lceil n/2 \rceil + 2$ and its crossing number is actually $\lceil n/2 \rceil - 2 \approx \sqrt{k}$ greater than this.

All these examples suggest the following question: Is there a function $f(k)$ such that, if G is in $\mathcal{M}_k$, then the crossing number of G is at most $f(k)$? It is our purpose to show that f exists and is at most $2.5k + 16$.

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We also show that, if $\mathcal{M}_k$ has infinitely many r-regular simple graphs, then $r = 4$ or 5. We exhibit an infinite class of 4-regular, 4-connected simple graphs in $\mathcal{M}_3$.

2. Cycles in Nearly Planar Graphs

In this section we show that every "nearly planar" graph has a cycle containing at most one vertex of large degree—this is central to the proof that the members of $\mathcal{M}_k$ have bounded crossing number.

THEOREM 1. *Let G be a simple connected planar graph with minimum degree at least 3. Then there is a face boundary C in G of length at most 5 such that at most one vertex of C has degree 12 or more.*

Proof. We proceed as suggested by Lebesgue [3]—the dual of our argument is used in [6]. For each face F of an embedding of G in the plane, let the *weight* of F be the sum

$$w(F) = \sum_{v \sim F} \frac{1}{d(v)},$$

where $d(v)$ is the degree of the vertex v and $v \sim F$ means v is incident with F. (A vertex v occurs in the sum for $w(F)$ as often as the boundary of F goes through v.) Obviously, for any face F, $w(F) \leqslant l(F)/3$, where $l(F)$ is the length of the boundary of F.

It is readily seen that $\sum_F w(F)$ is the number p of vertices of G. (The $\sum_F$ means to sum over all faces of G.) Also, if q is the number of edges of G, then $2q = \sum_F l(F)$. Let r denote the number of faces of G.

By Euler's formula, $2 = p - q + r = \sum_F \{w(F) - l(F)/2 + 1\}$, so there is a face F such that $w(F) - l(F)/2 + 1 > 0$. If $l(F) \geqslant 6$, we have that $w(F) - l(F)/2 + 1 \leqslant 1 - l(F)/6 \leqslant 0$. Hence, there is a face F such that $l(F) \leqslant 5$ and $w(F) - l(F)/2 + 1 > 0$. Let C denote the boundary of F.

If two or more vertices of C have degree 12 or more, then

$$w(F) \leqslant \frac{l(F) - 2}{3} + \frac{2}{12}$$

so that

$$0 < w(F) - \frac{l(F)}{2} + 1 \leqslant \frac{1}{2} - \frac{l(F)}{6} \leqslant \frac{1}{2} - \frac{3}{6} = 0,$$

a contradiction. Therefore, at most one vertex of C has degree 12 or more. ∎

THEOREM 2. *Let G be a simple graph with minimum degree at least 3. Assume G has a set E of t edges such that $G-E$ is planar. Then G has a cycle C with a vertex v such that*

$$\sum_{u \in V(C)\setminus\{v\}} (d(u)-2) \leqslant 36+t.$$

Proof. We proceed by induction on t. If $t=0$, then G is planar. Applying Theorem 1 to a connected component H of G, there is a face F of H having at most one vertex of degree 12 or more incident with F. Because the minimum degree is at least 2, the boundary of F contains the required cycle C.

Suppose, now, that $t>0$. Let $e=uw\in E$, let $G'=G-e$ and let $E'=E\setminus\{e\}$. We now consider several cases.

Case 1

G' has no vertex of degree 2. Apply the inductive assumption to G' to get a cycle C such that

$$\sum_{u \in V(C)\setminus\{v\}} (d(u)-2) \leqslant 36+t-1.$$

If e is not a chord of C, then C is the required cycle for G. If, on the other hand, e is a chord of C, then we may choose one of the two cycles in $C+e$ containing e as our cycle for G. It is easily checked that either of these will work.

Case 2

G' has at least one vertex of degree 2. Let G'' be the graph obtained from G' by suppressing any vertices of degree 2. Note that G'' has no loop.

Subcase 2.1. G'' has no parallel edges. Then, by the induction hypothesis, G'' has a cycle C such that

$$\sum_{u \in V(C)\setminus\{v\}} (d(u)-2) \leqslant 36+t-1.$$

Again, if e is a chord of the cycle C' if G corresponding to C, then we pick one of the two cycles in $C'+e$ containing e.

Subcase 2.2. G'' has parallel edges. Let x, y be the vertices of a cycle of length 2 in G''. We can assume that G has the path (x, u, y). If G also has the path (x, w, y), then we let C_1, C_2 be the cycles (x, u, w, x), (y, u, w, y), respectively. If G does not contain the path (x, w, y), then let C_3 be the cycle (x, u, y, x). If either $d(x)\leqslant 4$ or $d(y)\leqslant 4$, then one of the cycles C_1, C_2, and C_3 will be the desired cycle in G.

Otherwise, we can replace any set of parallel edges with a single edge and get a simple graph G^* with minimum degree at least 3. Apply the inductive assumption to G^* to get the cycle C. The edge e is not a chord of the corresponding cycle C' in G, so C' is the required cycle. ∎

3. Crossing-Critical Graphs

In this section, we prove the existence of the function $f(k)$. The crossing number of a graph G is denoted $cr(G)$.

THEOREM 3. *Let G be a graph minimal with respect to having crossing number at least k. Then $cr(G) \leqslant 2.5k + 16$.*

Proof. First, suppress any vertices of degree 2. This will affect neither the crossing number of G nor the minimality. Therefore, we may assume G has minimum degree at least 3. If G is not simple, then it must have parallel edges. If e and e' are parallel, delete e and draw $G-e$ with at most $k-1$ crossings. By drawing e next to the drawing of e', we get a drawing of G with at most $2(k-1)$ crossings. Thus, we can assume G is simple with minimum degree at least 3.

Observe that G has a set S of at most k edges such that $G-S$ is planar. For if e is any edge of G, then the minimality of G implies $cr(G-e) \leqslant k-1$. Let S consist of e together with one edge from each of the crossing pairs in a drawing of $G-e$ having at most $k-1$ crossings.

Now put $t=k$ and let C be a cycle and v a vertex as in Theorem 2. Let $e=vw$ be an edge of C incident with v. Again, by the minimality of G, there is a drawing of $G-e$ in the plane with at most $k-1$ crossings. Let P be the path in $C-e$ joining v and w. Some edges of P may cross other edges of P. We may consider the drawing of P as a planar graph H with vertices of degrees 2 and 4. Let P' be a shortest path in H joining v and w. There are two ways of drawing e close to P', one for each side of P'.

In total, these two drawings of e cross each edge not in P incident with an internal vertex of P at most once. (There are at most $k+36$ such crossings.) For an edge of P crossing another edge of P at a vertex of P', these drawings produce a total of 2 crossings. This conclusion also holds for any edge not in P crossing an edge of P'. (These provide a total of at most $2(k-1)$ crossings.)

Therefore, one of the two choices for e crosses at most $3k/2+17$ edges of $G-e$. This choice yields a drawing of G with at most $2.5k+16$ crossings. (Note that in this drawing, some edge of G may cross e more than once. Standard arguments show that such multiple crossings can be eliminated without increasing the number of crossings.) ∎

COROLLARY 1. *If G is any graph with crossing number k, then there is an edge e of G such that $cr(G-e) \geq (2k-37)/5$.*

We have proved $f(k) \leq 2.5k + 16$. Let $\mathscr{C}$ be an infinite class of edge-transitive graphs whose crossing number is bounded below by a quadratic function of the number of edges. (It is not difficult to see that the nonplanar complete bipartite graphs and regular complete m-partite graphs all have this property.) There exist two positive constants c_1 and c_2 such that if $G \in \mathscr{C}$ and e is an edge of G, then

$$cr(G) - c_1 \sqrt{cr(G)} < cr(G-e) < cr(G) - c_2 \sqrt{cr(G)}.$$

In particular, there is a positive constant c such that $f(k) > k + c\sqrt{k}$ for infintely many k. It is our guess that this lower bound is the correct answer, i.e., $f(k) \leq k + c' \sqrt{k}$ holds for some constant c'.

We now apply Theorem 3 to crossing critical graphs with fixed minimum degree.

COROLLARY 2. *Let G be a simple graph that is minimal with respect to having crossing number at least k. If all vertices of G have degree at least 7, then G has at most $5k + 20$ vertices.*

Proof. By Theorem 3, G has a drawing with $m \leq 2.5k + 16$ crossings. We may think of this drawing as a planar graph G' with $p = |V(G)|$ vertices of degree at least 7 and m vertices of degree 4. As

$$|E(G')| \; |E(G)| + 2m \geq \tfrac{7}{2} p + 2m$$

and

$$|E(G')| \leq 3(m+p) - 6,$$

the result follows. ∎

We now indicate how the 7 in Corollary 2 can be improved to 6; the ideas in the proof of the improvement are similar but the details are tedious.

COROLLARY 3. *For each natural number k, there exists a natural number n_k such that any simple graph G having minimum degree 6 in $\mathscr{M}_k$ has at most n_k vertices.*

Sketch of Proof. Let G and G' be as in the proof of Corollary 2. We assume that G has many vertices and prove that G has an edge e such $cr(G-e) \geq k$. As G' has average degree less than 6, almost all vertices of G' have degree 6 and almost all faces are bounded by triangles. Therefore, there is a vertex v such that all vertices at distance at most $100k^2$ from v in G' have degree 6 and are incident only with triangles. Then v is surrounded by $100k^2$ disjoint nested chordless cycles $C_1, C_2, ..., C_{100k^2}$ such that all vertices of C_i have distance i from v and such that all vertices of distance i from v are "inside" C_i—by inside we mean the same side as v. Observe that if we delete the vertices of C_i we get a graph with exactly one more component than G has, while if we delete the vertices of C_i and of C_j, $1 \leqslant i < j-1 \leqslant 100k^2 - 1$, then we obtain a graph with two more components than G.

Choose $q \in \{2k+1, 2k+2, ..., 100k^2 - 2k\}$ such that G' has no vertex of degree 4 between C_{q-2k} and C_{q+2k}. Let e be any edge on C_q. We claim that $cr(G-e) \geqslant k$. For if not, then $G-e$ has a drawing G'' with at most $k-1$ crossings, involving at most $2k-2$ edges. So, for some $1 \leqslant i \leqslant 2k$ and some $1 \leqslant j \leqslant 2k$, both C_{q-i} and C_{q+j} are drawn in G'' with no crossings. Let r (respectively, t) denote the number of crossings, in G'', of the subgraph of G consisting of C_{q-i} and all edges and vertices inside C_{q-i} in the drawing G' (respectively C_{q+j} and all edges and vertices outside C_{q+j}). Evidently, these subgraphs are disjoint, so $r+t \leqslant k-1$. It follows that we can draw G with r crossings inside C_{q-i}, t crossings outside C_{q+j} and no crossings between C_{q-i} and C_{q+j}. Therefore, $cr(G) \leqslant r+t < k$, a contradiction. ▮

A reformulation of Corollary 3 is that if G is a simple graph of minimum degree 6 such that G has many vertices compared to its crossing number, then G has an edge e such that $cr(G-e) = cr(G)$.

4. Regular Crossing Critical Graphs

In this secton, we focus on regular simple graphs in $\mathcal{M}_k$.

Theorem 4. (1) *If there are infinitely many r-regular simple graphs that are minimal with respect to having crossing number at least k, then $r=4$ or* 5.

(2) *There are infinitely many* 4*-regular* 4*-connected simple graphs that are minimal with respect to having crossing number* 3.

Proof. (1) Clearly $r \geqslant 3$. If $r \geqslant 6$, then Corollary 3 ensures the finiteness. If $r=3$, then the finiteness is ensured by the Robertson–Seymour theory, since, in any infinite sequence of 3-regular graphs, one contains a subdivision of another.

(2) Let H_m be the graph obtained from a cycle C_{2m} of length $2m$ by adding, for each two diametically opposite edges e and e' on C_{2m}, a new

vertex joined to the four ends of e and e'. Note that H_3 is isomorphic to $C_3 \times C_3$. We now prove that if $m \geqslant 3$, then $cr(H_m) \geqslant 3$.

The proof is by induction on m, with the case $m = 3$ being the base. Although the result in this case is well known, we present an elegant proof due to Dan McQuillan. We require the observation that if e and e' are disjoint edges of $C_3 \times C_3$, then there is a 3-cycle T and a 4-cycle S such that T and S are vertex-disjoint and one of e and e' is in T while the other is in S.

Let D be a drawing of $C_3 \times C_3$ in the plane with a minimum number of crossings. If e and e' are a pair of edges that cross in D, then they are disjoint. By the above remark, there are disjoint 3- and 4-cycles T and S, respectively, one containing e and the other containing e'. Evidently T and S cross each other at least twice.

The graph $C_3 \times C_3 - E(S)$ is a subdivision of K_5, so D must have a third crossing.

For the induction step, let $m \geqslant 4$ and let $B_1, B_2, ..., B_m$ be the "bow ties" consisting of the two triangles containing diametrically opposite edges of C_{2m}. Let D_m be an optimal drawing of H_m and let n_i be the sum, taken over the edges e of B_i, of the number of crossings of e in D_m.

As $H_m - E(B_i)$ has no isthmus, either $n_i = 0$ or $n_i \geqslant 2$, for $i = 1, 2, ..., m$. Since $cr(H_m) \geqslant (1/2) \sum_{i=1}^{m} n_i$, if $\sum_{i=1}^{m} n_i > 4$, then $cr(H_m) \geqslant 3$, as required.

If $\sum_{i=1}^{m} n_i \leqslant 4$, then, for some i, $n_i = 0$. Delete the centre vertex of B_i and contract the two remaining edges of B_i to get a drawing of H_{m-1} with no more crossings than in D_m. Hence, $cr(H_m) \geqslant cr(H_{m-1}) \geqslant 3$.

To see that $cr(H_m) = 3$, draw C_{2m} with one crossing, at a diametrically opposite pair of edges. It is now straightforward to add in the "bow ties" with only 2 more crossings.

For the minimality, there are only two types of edges in H_m: those on C_{2m} and the others. The drawing above has one of each type crossed. ∎

We do not know if, for some $k \geqslant 2$, there are infinitely many 5-regular simple graphs in $\mathcal{M}_k$. Perhaps an appropriate modification of the graphs H_m results in such a family.

The finiteness result of Theorem 4(1) does not extend to graphs with multiple edges. For if we replace every edge in H_m by p parallel edges, then the resulting $4p$-regular graph belongs to $\mathcal{M}_{3p^2}$.

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