The dimension of attractor of the 2D $g$-Navier–Stokes equations

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Abstract

The $g$-Navier–Stokes equations in spatial dimension 2 were introduced by Roh as
\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f,
\]
with the continuity equation
\[
\nabla \cdot (gu) = 0,
\]
where $g$ is a suitable smooth real valued function. Roh proved the existence of global solutions and the global attractor, for the spatial periodic and Dirichlet boundary conditions. Roh also proved that the global attractor $\mathcal{A}_g$ of the $g$-Navier–Stokes equations converges (in the sense of upper continuity) to the global attractor $\mathcal{A}_1$ of the Navier–Stokes equations as $g \to 1$ in the proper sense.

In this paper, we will estimate the dimension of the global attractor $\mathcal{A}_g$, for the spatial periodic and Dirichlet boundary conditions. Then, we will see that the upper bounds for the dimension of
the global attractors $\mathcal{A}_g$ converge to the corresponding upper bounds for the global attractor $\mathcal{A}_1$ as $g \to 1$ in the proper sense.

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1. Introduction

The study of the Navier–Stokes equations on thin domains originates in a series of papers by Hale and Raugel [10–12] concerning the reaction–diffusion and damped wave equations on thin domains. Then, Raugel andSell [22,23] proved global existence of strong solutions for large initial data and forcing terms in thinthree-dimensional domains for the purely periodic boundary conditions and the periodic-Dirichlet boundary conditions, that is, periodic conditions in the thin vertical direction and homogeneous Dirichlet conditions on the lateral boundary condition $\Gamma_l = \partial \Omega \times (0, \varepsilon)$, where $\Omega \subset \mathbb{R}^2$.

As in [10], an essential tool in their proof is the vertical mean operator $M$ which allows the decomposition of every function $U$ on $\Omega_\varepsilon = \Omega \times (0, \varepsilon)$ into the sum of a function $M U = v(x_1, x_2)$ which does not depend on the vertical variable and a function $(I - M) U = w(x_1, x_2, x_3)$ with vanishing vertical mean and thus to use more precise Sobolev and Poincaré inequalities. Then, they showed that the reduced 3D Navier–Stokes evolutionary equations by $v$ incorporates the 2D Navier–Stokes equations on $\Omega$. Later, by using same tool as Raugel and Sell with improved Agmon inequalities, Temam and Ziane [29,30] generalized the results of [22,23] to other boundary conditions and, in the case of the free boundary conditions, to thin spherical domains. One can refer [1,14,20,21] for more thin domain problems.

In [22,23], Raugel and Sell used the thin three-dimensional domain as $\Omega_\varepsilon = \Omega \times (0, \varepsilon)$, where $\Omega \subset \mathbb{R}^2$. In [25], Roh applied their methods on $\Omega_g = \Omega_2 \times (0, g)$, where $\Omega_2$ is a bounded region in the plane and $g = g(x_1, x_2)$ is a smooth function defined on $\Omega_2$ with $0 \leq g(x_1, x_2) \leq M$, for $(x_1, x_2) \in \Omega_2$. Now, we consider the 3D Navier–Stokes equations,

$$\frac{\partial U}{\partial t} - \nu \Delta U + (U \cdot \nabla)U + \nabla \Phi = F \quad \text{in } \Omega_g,$$

$$\nabla \cdot U = 0 \quad \text{in } \Omega_g,$$

with the boundary condition

$$U \cdot n = 0 \quad \text{on } \partial_{\text{top}} \Omega_g \cup \partial_{\text{bottom}} \Omega_g,$$

where

$$\partial_{\text{top}} \Omega_g = \{(x_1, x_2, x_3) \in \Omega_g : x_3 = g(x_1, x_2)\},$$

$$\partial_{\text{bottom}} \Omega_g = \{(x_1, x_2, x_3) \in \Omega_g : x_3 = 0\}.$$

The lateral boundary condition corresponding to $\partial \Omega_2$ does not affect to the derivation of the 2D $g$-Navier–Stokes equations. But, in this paper we consider the periodic and Dirichlet boundary conditions to study the 2D $g$-Navier–Stokes equations.
Now we define \( v(x_1, x_2) \) as
\[
v_i(x_1, x_2) = M_{U_i}(x_1, x_2, x_3) = \frac{1}{g(x_1, x_2)} \int_0^{g(x_1, x_2)} U_i(x_1, x_2, x_3) \, dx_3,
\]
where \( i = 1, 2, 3 \) and for \( u = (u_1, u_2) = (v_1, v_2) \), we get the following proposition.

**Proposition 1.1** [25]. Assume that \( \nabla \cdot U = 0 \) in \( \Omega_g \) and that (1.1) is valid. Then one has
\[
\nabla^2 \cdot (g u) = \frac{\partial (g u_1)}{\partial x_1} + \frac{\partial (g u_2)}{\partial x_2} = \nabla g \cdot u + g (\nabla^2 u) = 0 \quad \text{in} \quad \Omega_2,
\]
where \( \nabla_2 = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) \) and \( \nabla g = \left( \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \right) \).

**Remark 1.2.** In the above proposition, for (1.1), we can assume \( U = 0 \) on \( \partial_{\text{bottom}} \Omega_g \) and \( U \cdot n = 0 \) on \( \partial_{\text{top}} \Omega_g \).

Now, good properties of the 2D \( g \)-Navier–Stokes equations can lead to initiate the study of the Navier–Stokes equations on thin three-dimensional domain \( \Omega_g \), because the reduced 3D Navier–Stokes evolutionary equations by
\[
v(x_1, x_2) = (v_1, v_2, v_3) = \frac{1}{g(x_1, x_2)} \int_0^{g(x_1, x_2)} U(x_1, x_2, x_3) \, dx_3,
\]
can be solved by the 2D \( g \)-Navier–Stokes equations of \( u = (v_1, v_2) \). In fact, \( v_3 \) depends on \( (v_1, v_2) \). One can refer [25] for the detail.

This is the basis for our study of the 2D \( g \)-Navier–Stokes equations,
\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.2}
\]
\[
\frac{1}{g} \nabla \cdot (g u) = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.3}
\]
\[
u u(\cdot, 0) = u_0(\cdot) \quad \text{in} \quad \Omega, \tag{1.4}
\]
where \( g = g(x_1, x_2) \) is a suitable real-valued smooth function on the domain \( \Omega \subset \mathbb{R}^2 \). Here \( \nu \) and \( f \) are given, and the velocity \( u \) and the pressure \( p \) are the unknowns. If \( g = 1 \), the Eqs. (1.2)–(1.3) become the usual two-dimensional Navier–Stokes equations.

For the general theory of the Navier–Stokes equations and infinite-dimensional dynamical systems, one can refer [5,24,27,28]. The global attractor of the Navier–Stokes equations was first obtained for bounded domains in the works of Ladyzhenskaya [16] and Foias and Temam [9]. Then, the latter work showed the finite dimensionality of the attractor in the sense of the Hausdorff dimension (see [4,6,28]).

**Definition 1.3.** Let \( X \) be a compact set in a Hausdorff space \( H \). The Hausdorff dimension of \( X \) in \( H \) is the number
\[
\dim_H X = \inf \{ d : \mu_H(X, d) = 0 \},
\]
where \( \mu_H(X, d) = \lim_{\varepsilon \to 0^+} \mu_H(X, d, \varepsilon) \), \( \mu_H(X, d, \varepsilon) = \inf_{U \subset X} V_d(U) \). Here the infimum is taken over all coverings \( U \) of the set \( X \) by balls \( B(x_i, r_i) \) with centers at \( x_i \) and radii \( r_i \leq \varepsilon \), and

\[
V_d(U) = \sum r_i^d.
\]

**Definition 1.4.** The fractal dimension of \( X \) in \( H \) is the number

\[
\dim_F X = \limsup_{\varepsilon \to 0} \frac{\log N(X, \varepsilon)}{-\log \varepsilon},
\]

where \( N(X, \varepsilon) \) is the minimum number of \( \varepsilon \) balls required to cover \( X \).

It is well known by definitions that \( \dim_H(X) \leq \dim_F(X) \).

Suppose that a continuous map \( S \) acts in \( H \) and let a compact set \( X \subset H \) be strictly invariant: \( SX = X \). It was shown in [28] that if the differential \( DS \) uniformly contracts \( d \)-dimensional volumes on \( X \) then

\[
\dim_H X \leq d.
\]

As for the fractal dimension, it was also shown in [5] and [7] that

\[
\dim_F X \leq cd, \quad c = \text{const} > 1.
\]

With an additional condition for the maps \( S \), the estimate \( \dim_F X \leq d \) was proved by Chepyzhov and Ilyin in [2]. Then recently in [3], they proved \( \dim_F X \leq d \) in general case and applied to estimates the fractal dimension of attractors of the Navier–Stokes equations:

\[
\dim_F A \leq \frac{1}{\sqrt{2\pi}} \left( \frac{\lambda_1 |\Omega|^{1/2} \|f\|_1}{\lambda_1 v^2} \right) \leq \frac{1}{2\pi^{3/2}} \frac{\|f\| |\Omega|}{v^2}.
\]

Their improved estimate from [2] and [15] is due to the improved Lieb–Thirring inequality of [13,17].

In this paper, we will estimate the fractal dimension of the global attractor of the 2D \( g \)-Navier–Stokes equations for the spatial periodic and Dirichlet boundary conditions. For the proofs we will follow the standard method of Temam [28] and estimate the fractal dimension due to the results of [3].

For the theory of the 2D \( g \)-Navier–Stokes equations, Roh [25] proved the existence of the global solutions, and the existence of the global attractor when \( |\nabla g|_\infty \) is small enough. Then, in [26], Roh proved that the global attractor \( A_g \) of the \( g \)-Navier–Stokes equations converges (in the sense of upper continuity) to the global attractor \( A_1 \) of the Navier–Stokes equations as \( g \to 1 \) in the proper sense.

This paper is organized as follows. In Section 2, we first introduce some notations and preliminary results for the \( g \)-Navier–Stokes equations. In Section 3, we estimate the dimension of the global attractors. In Section 4, we see the behavior of the dimension of the global attractors as \( g \to 1 \) in the proper sense.
2. Preliminary

In this section we will introduce useful notations and preliminary results for 2D $g$-Navier–Stokes equations, without the proofs. One can refer to [25] and [26] for the details.

2.1. Periodic boundary conditions

Let $\Omega = (0, 1) \times (0, 1)$ and we assume that the function $g(x) = g(x_1, x_2)$ satisfies the following properties:

1. $g(x) \in C^\infty_{\text{per}}(\Omega)$ and
2. there exist constants $m_0 = m_0(g)$ and $M_0 = M_0(g)$ such that, for all $x \in \Omega$,
   \[ 0 < m_0 \leq g(x) \leq M_0. \]

Note that the constant function $g \equiv 1$ satisfies these conditions.

We denote by $L^2(\Omega, g)$ the space with the scalar product and the norm given by
\[
(u, v)_g = \int_{\Omega} (u \cdot v)g \, dx \quad \text{and} \quad |u|^2_g = (u, u)_g,
\]
as well as $H^1(\Omega, g)$ with the norm
\[
\|u\|_{H^1(\Omega, g)} = \left[ (u, u)_g + \sum_{i=1}^{n} (D_i u, D_i u)_g \right]^{1/2},
\]
where $D_i = \frac{\partial}{\partial x_i}$.

Then for the functional setting of the problems (1.2)–(1.4), we use the following functional spaces
\[
H_g = \text{Cl}_{L^2_{\text{per}}(\Omega, g)} \left\{ u \in C^\infty_{\text{per}}(\Omega): \nabla \cdot gu = 0, \int_{\Omega} u \, dx = 0 \right\},
\]
\[
V_g = \left\{ u \in H^1_{\text{per}}(\Omega, g): \nabla \cdot gu = 0, \int_{\Omega} u \, dx = 0 \right\},
\]
where $H_g$ is endowed with the scalar product and the norm in $L^2(\Omega, g)$, and $V_g$ is the spaces with the scalar product and the norm given by
\[
((u, v))_g = \int_{\Omega} (\nabla u \cdot \nabla v)g \, dx \quad \text{and} \quad \|u\|_g^2 = ((u, u))_g.
\]

Also, we define the orthogonal projection $P_g$ as
\[
P_g : L^2_{\text{per}}(\Omega, g) \rightarrow H_g
\]
and we have that \( Q \subseteq H_g^{1} \), where
\[
Q = \text{Cl}_{L^2_{\text{per}}(\Omega,g)} \{ \nabla \phi : \phi \in C^1(\overline{\Omega}, R) \}.
\]

Then, we define the \( g \)-Laplacian operator
\[
-\Delta_g u \equiv -\frac{1}{g}(\nabla \cdot g \nabla)u = -\Delta u - \frac{1}{g}(\nabla g \cdot \nabla)u
\]
to have the linear operator
\[
A_g u = P_g \left[ -\frac{1}{g} \left( \nabla \cdot (g \nabla u) \right) \right],
\]
which satisfies the following proposition.

**Proposition 2.1** (Roh [25]). For the linear operator \( A_g \), the following hold:

1. \( A_g \) is a positive, self-adjoint operator with compact inverse, where the domain of \( A_g \),
   \[ \mathcal{D}(A_g) = V_g \cap H^2(\Omega). \]
2. There exist countable eigenvalues of \( A_g \) satisfying
   \[ 0 < \lambda_g \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots, \]
   where \( \lambda_g = 4\pi^2 m_0 / M_0 \) and \( \lambda_1 \) is the smallest eigenvalue of \( A_g \). In addition, there
   exists the corresponding collection of eigenfunctions \( \{ e_1, e_2, e_3, \ldots \} \) which forms an
   orthonormal basis for \( H_g \).

**Remark 2.2.** Due to [8], we can estimate \( \lambda_n \) by using the \( n \)th eigenvalue of the Stokes
operator. We recall the following lemma from [8].

**Lemma 2.3.** Consider the eigenvalue problem of a self-adjoint second order partial differential equation,
\[
L[u] + \lambda \rho u = (pu_x)_x + (pu_y)_y + \lambda \rho u = 0.
\]
If the coefficient \( \rho \) varies at every point in the same sense, then, for every boundary
condition, the \( n \)th eigenvalue changes in the opposite sense. If the coefficient \( p \) changes
everywhere in the same sense, every eigenvalue changes in this same sense.

For our problem, we consider three different equations:

1. case I: \( p = m_0 \) and \( \rho = M_0 \),
2. case II: \( p = M_0 \) and \( \rho = m_0 \),
3. case III: \( p = g \) and \( \rho = g \).

Then, by above lemma, for \( n \)th eigenvalue \( \lambda_n \) of the 2D \( g \)-Stokes operator, we have
\[
\frac{m_0}{M_0} \lambda'_n \leq \lambda_n \leq \frac{M_0}{m_0} \lambda'_n,
\]
where \( \lambda'_n \) is \( n \)th eigenvalue of the Stokes operator.
We also consider the trilinear form
\[ b_g(u, v, w) = \sum_{i,j=1}^{n} \int_{\Omega} u_i(D_i v_j) w_j g \, dx = \left( P_g(u \cdot \nabla) v, w \right)_g, \]
where \( u, v, w \) lie in appropriate subspaces of \( L^2(\Omega, g) \). Then, the form \( b_g \) satisfies
\[ b_g(u, v, w) = -b_g(u, w, v) \quad \text{for } u, v, w \in H_g \]
and hence \( b_g(u, v, v) = 0 \). For \( (u, v) \in V_g \times V_g \), we denote by \( B_g \) the bilinear form defined by
\[ \langle B_g(u, v), w \rangle_{V'_g V_g} = b_g(u, v, w), \quad w \in V_g, \]
and we let \( B_g u = B_g(u, u) \).

We denote a linear operator \( R \) on \( V_g \) by
\[ Ru = P_g \left[ \frac{1}{g} (\nabla g \cdot \nabla) u \right] \quad \text{for } u \in V_g, \]
and have \( R \) as a continuous linear operator from \( V_g \) into \( H_g \) such that
\[ \left| (Ru, u) \right| \leq \frac{|\nabla g|_{\infty} m_0}{m_0 |\nabla g|_{\infty}^{1/2}} \|u\|_g^2 \quad \text{for } u \in V_g. \]

We now rewrite Eqs. (1.2)–(1.4) as abstract evolution equations,
\[
\begin{align*}
\frac{du}{dt} + \nu A_g u + B_g u + \nu Ru &= P_g f, \\
 u(0) &= u_0.
\end{align*}
\] (2.3) (2.4)

Then we obtain the result of existence and uniqueness of solutions of Eqs. (2.3) and (2.4).

**Proposition 2.4** (Roh [25]). Let \( f \in L^\infty(0, \infty; L^2(\Omega)) \) be given. Then for every \( u_0 \in H_g \) there is precisely one weak solution (of class \( LH \)) \( u = u(t) \) on \([0, \infty)\) of Eqs. (2.3) and (2.4), satisfying \( u(0) = u_0 \). Moreover, we have \( u(t) \in C[0, \infty; H_g] \). Also, let \( u = u(t) \) be any weak solution of Eqs. (2.3) and (2.4) on \([0, \infty)\) with initial condition \( u(0) = u_0 \in H_g \). Then for each \( t_0 > 0 \), \( v(t) = u(t + t_0) \) is a strong solution of Eqs. (2.3) and (2.4) on \([0, \infty)\) with initial condition \( v(0) = u(t_0) \) and \( D_t u \in L^2_{\text{loc}}(0, \infty; H_g) \).

For the existence of the global attractors, we let \( \sigma_w(t, u_0) = S_w(t)u_0 \) denote the semiflows on \( H_g \) generated by a weak solution of Eqs. (2.3) and (2.4) with the data \( u_0 \in H_g \). Likewise, let \( \sigma_s(t, u_0) = S_s(t)u_0 \) denote the semiflows on \( V_g \) generated by a strong solution of Eqs. (2.3) and (2.4) with the data \( u_0 \in V_g \).

**Definition 2.5.** Let \( \mathcal{U}_0 \) be an attractor for a given \( \kappa \)-contracting semiflow \( S_0(t) \). Let \( S_0(t) \) be imbedded into a continuous family of semiflows \( S_\lambda(t) \), where \( \lambda \in \Lambda \) and \( S_{\lambda_0}(t) = S_0(t) \). We will say that the family \( S_\lambda(t) \) is robust at \( \mathcal{U}_0 \) provided that, for every \( \epsilon > 0 \), there is a
neighborhood $O = O(\varepsilon)$ of $\lambda_0$ in $\Lambda$ such that for each $\lambda \in O$, the semiflow $S_\lambda(t)$ has an attractor $U_\lambda$ and

$$U_\lambda \subset N_\varepsilon(U_0) \quad \text{for all } \lambda \in O.$$ 

**Proposition 2.6** (Roh [25]). We assume that $|\nabla g|_\infty^2 < m_0^3 \pi^2 / M_0$ and the forcing term $f \in L^2(\Omega)$ is a time-independent function. Then, for $u_0 \in H^g$, 

$$\sigma_w(t, u_0) = S_w(t)u_0$$

is a semiflow on $H^g$ which is point dissipative and compact for $t > 0$. Furthermore, there exists a global attractor $A_w$ for $S_w(t)$ and the semiflow $S_w(t)$ is robust at $A_w$ for every $f \in L^2(\Omega)$.

If $u_0 \in V^g$ then 

$$\sigma_s(t, u_0) = S_s(t)u_0$$

is a semiflow on $V^g$ which is point dissipative and compact for $t > 0$. Moreover, there exists a global attractor $A_s$ for $S_s(t)$ which is compact in the space $V^g$. In addition, we note that $A_s = A_w$ for fixed $f \in L^2(\Omega)$.

2.2. Dirichlet boundary conditions

In this section we will see Dirichlet boundary conditions on bounded domain $\Omega \subset R^2$. First for the function $g$ we assume that $g(x) \in C^\infty(\Omega)$ with for all $x \in \Omega$, 

$$0 < m_0 \leq g(x) \leq M_0.$$ 

Then, for the functional setting we consider the following functional spaces:

$$H^g = Cl_{L^2(\Omega, g)}\{u \in C_0^\infty(\Omega) : \nabla \cdot gu = 0\},$$

$$V^g = \{u \in H_0^1(\Omega, g) : \nabla \cdot gu = 0\}.$$ 

Also, we define the orthogonal projection $P_g$ as 

$$P_g : L^2(\Omega, g) \to H^g$$

and have $Q = H^g_\perp$, where 

$$Q = Cl_{L^2(\Omega, g)}\{\nabla \phi : \phi \in C^1(\bar{\Omega}, R)\}.$$ 

Here, one note that since the norm in $L^2(\Omega)$ is equivalent to the norm in $L^2(\Omega, g)$, we can say 

$$Q = Cl_{L^2(\Omega)}\{\nabla \phi : \phi \in C^1(\bar{\Omega}, R)\}.$$ 

So, the space $Q$ for the pressure does not depend on the function $g(x)$.

Next, we have same results for Proposition 2.1 with 

$$\lambda_g \geq \frac{cm_0}{M_0},$$ 

where $c$ is the Poincaré constant on $\Omega$.

Also, we can obtain same results for Propositions 2.4 and 2.6 when $|\nabla g|_\infty^2 < cm_0^3 / 4 M_0$, where $c$ is the Poincaré constant on $\Omega$. 
3. The dimension of the attractor

In this section, we estimate the dimension of the global attractor of 2D $g$-Navier–Stokes equations. For the estimation, we will follow the presentation given by Temam [28] and use the recent results of Chepyzhov and Ilyin [2,3]. We first briefly recall the formal scheme from [28] for estimating the sums of the first $m$ global Lyapunov exponents, that is, the number $q(m)$.

If the semigroup $S(t)$ generated by the equation
\[ \partial_t u = F(u), \quad u(0) = u_0 \]
has an invariant set $X$, then
\[
q(m) \leq \limsup_{t \to \infty} \sup_{u_0 \in X} \sup_{\xi_j \in H_\varepsilon} \left( \frac{1}{t} \int_0^t \mathrm{Tr} \ F'_u(S(t)u_0) \circ Q_m(\tau) \ d\tau \right).
\]
Here $Q_m(\tau)$ is the orthogonal projection in $H$ onto $\text{Span}(U_1(\tau), \ldots, U_m(\tau))$, where $U_i$ are the solutions of the first variation equation
\[ \partial_t U_i = F'_u(S(t)u_0) \cdot U_i(t), \quad U_i(0) = \xi_i. \]

Now, to estimate the fractal dimension of an invariant set $X$, we use the following proposition proved by Chepyzhov and Ilyin.

**Proposition 3.1** [2]. If $q(m) \leq f(m)$, where $f(d)$ is a concave function of the continuous variable $d$ and $f(d_*) = 0$, then
\[ \dim_F X \leq d_* . \]

Recently, in [3], they proved that Proposition 3.1 still holds when $f$ is concave at least in the interval $d_* - 1 < d < d_* + 1$. Also, for the general function $f$, they obtained $\dim_F X \leq d_* + 1$.

For the boundary conditions, we consider the spatial periodic and Dirichlet boundary conditions.

3.1. Periodic boundary conditions

For the periodic boundary conditions, we follow the presentation of [28] to estimate the upper bound of the first $m$ global Lyapunov exponents $q_m(t)$. Then, we use Proposition 3.1. Unlike the Dirichlet boundary conditions, we have little problem to apply Chepyzhov and Ilyin method for the periodic boundary condition as the reader can find in next section.

For the periodic boundary condition we will see in next chapter that the upper bound of $q_m(t)$ converge to the upper bound (in [28]) of $\tilde{q}_m(t)$ as $g \to 1$ in $W^{1,\infty}(\Omega)$, where $\tilde{q}_m(t)$ is the first $m$ global Lyapunov exponents for the Navier–Stokes equations. As a result we can see the convergence of the upper bound of the dimension as $g \to 1$ in $W^{1,\infty}$.

We write Eq. (2.3) in the abstract form
\[ u' = F(u) , \]
(3.1)
where \( F(u) = -\nu A_g u - B_g u - \nu R u + P_g f \) and we see that the first variation equation
\[
U' = F'(u) U
\]
is equivalent to
\[
\frac{dU}{dt} + \nu A_g U + B_g(u, U) + B_g(U, u) + \nu R(U) = 0. \tag{3.3}
\]
Equation (3.3) is supplemented as usual by the initial condition
\[
U(0) = \xi, \quad \xi \in H_g. \tag{3.4}
\]
We obtain the following properties rigorously (see Temam [28] and Roh [25]):

- If \( u \) is the solution of (3.1), then the initial- and boundary-value problem (3.3) and (3.4) possesses a unique solution \( U \in L^2(0, T; V_g) \cap C([0, T]; H_g) \), \( \forall T > 0 \).
- For every \( t > 0 \), the function \( u_0 \to S(t)u_0 \) is Fréchet differentiable in \( H_g \) at \( u_0 \) with differential \( L(t, u_0) : \xi \in H_g \to U(t) \in H_g \), where \( U \) is the solution of (3.3) and (3.4).

Before proceeding with the estimate of the Lyapunov exponents, we compute here a bound for the energy dissipation flux \( \varepsilon \) defined by
\[
\varepsilon = \nu \lambda_1 \limsup_{t \to \infty} \frac{1}{t} \int_0^t \| u(s) \|^2_g ds, \tag{3.5}
\]
where \( u \) is the velocity and \( \lambda_1 \) represents the first eigenvalue of \( A_g \).

By taking an inner product (3.1) with \( u \), we obtain
\[
\frac{1}{2} \frac{d}{dt} |u|^2_g + \nu \| u \|^2_g \leq |f|^2_g |u|_g - \left( \frac{\nu}{g} (\nabla g \cdot \nabla) u, u \right)_g
\leq \frac{|f|^2_g}{2\nu \lambda_1} + \frac{\nu}{2} \| u \|^2_g + \frac{\nu}{m_0} |\nabla g|_{\infty} \| u \|_g |u|_g
\]
which implies
\[
\frac{d}{dt} |u|^2_g + \nu \| u \|^2_g \leq \frac{|f|^2_g}{\nu \lambda_1} + \frac{2\nu}{m_0 \lambda_g^{1/2}} |\nabla g|_{\infty} \| u \|^2_g
\]
for \( \lambda_g = 4\pi^2 m_0 / M_0 \) in Proposition 2.1, and then
\[
\nu \left( 1 - \frac{2|\nabla g|_{\infty}}{m_0 \lambda_g^{1/2}} \right) \frac{1}{t} \int_0^t \| u(s) \|^2_g ds \leq \frac{|f|^2_g}{\nu \lambda_1} + \frac{|u_0|^2_g}{t}.
\]
So, we obtain
\[
\frac{1}{t} \int_0^t \| u(s) \|^2_g ds \leq \left( \frac{|f|^2_g}{\nu \lambda_1} + \frac{|u_0|^2_g}{vt} \right) \left( 1 - \frac{2|\nabla g|_{\infty}}{m_0 \lambda_g^{1/2}} \right)^{-1}, \tag{3.6}
\]
when $1 - 2|\nabla g|_\infty/ m_0 \lambda_g^{1/2} > 0$.

Therefore, we have

$$
\varepsilon \leq \frac{1}{\nu} \left( 1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_g^{1/2}} \right)^{-1} |f|^2_g = \nu^3 \lambda_1^2 G^2 \left( 1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_g^{1/2}} \right)^{-1}.
$$

More generally, we can consider, instead of a specific trajectory $S(t)u_0$, all the trajectories for $u_0$ in a bounded functional-invariant set $X \subset H$. In this case $\varepsilon$ is defined as

$$
\varepsilon = \nu \lambda_1 \limsup_{t \to \infty} \sup_{u_0 \in X} \frac{1}{t} \int_0^t \|u(s)\|_g^2 ds
$$

and (3.7) remains valid.

We consider, for $m \in \mathbb{N}$,

$$
\left| U_1(t) \wedge \cdots \wedge U_m(t) \right|_{\wedge^m H_g} = \left| \xi_1 \wedge \cdots \wedge \xi_m \right|_{\wedge^m H_g} \exp \int_0^t \text{Tr} F'(S(\tau)u_0) \circ Q_m(\tau) d\tau,
$$

where $S(\tau)u_0 = u(\tau)$ is the solution of (3.1); $U_1, \ldots, U_m$ are $m$ solutions of (3.3)–(3.4) corresponding to the initial data $\xi_1, \ldots, \xi_m$; $Q_m(\tau) = Q_m(\tau, u_0; \xi_1, \ldots, \xi_m)$ is the orthogonal projector in $H_g$ onto the space the spanned by $U_1(\tau), \ldots, U_m(\tau)$.

At a given time $\tau$, let $\varphi_j(\tau), j = 1, \ldots, m$, be an orthonormal basis of $Q_m(\tau)H_g = \text{Span}[U_1(\tau), \ldots, U_m(\tau)]: \varphi_j(\tau) \in V_g$ for $j = 1, \ldots, m$ since $U_1(\tau), \ldots, U_m(\tau) \in V_g$ (a.e. $\tau \in \mathbb{R}_+$), and we have

$$
\text{Tr} F'(S(\tau)u_0) \circ Q_m(\tau) = \sum_{j=1}^m \left( \text{Tr} F'(u(\tau)) \circ Q_m(\tau) \varphi_j(\tau), \varphi_j(\tau) \right)_g
$$

$$
= \sum_{j=1}^m \left( F'(u(\tau)) \varphi_j(\tau), \varphi_j(\tau) \right)_g,
$$

($\cdot, \cdot)_g$ denoting the scalar product in $H_g$. Omitting temporarily the dependence on $\tau$, we write

$$
\left( F'(u) \varphi_j, \varphi_j \right)_g = -\nu (A_g \varphi_j, \varphi_j)_g - \left( B_g(\varphi_j, u), \varphi_j \right)_g - \left( \frac{\nu}{g} (\nabla g \cdot \nabla) \varphi_j, \varphi_j \right)_g
$$

$$
= -\nu \|\varphi_j\|_g^2 - b_g(\varphi_j, u, \varphi_j) - \left( \frac{\nu}{g} (\nabla g \cdot \nabla) \varphi_j, \varphi_j \right)_g,
$$

and

$$
\sum_{j=1}^m \left( F'(u) \varphi_j, \varphi_j \right)_g = -\nu \sum_{j=1}^m \|\varphi_j\|_g^2 - \sum_{j=1}^m b_g(\varphi_j, u, \varphi_j) - \sum_{j=1}^m \left( \frac{\nu}{g} (\nabla g \cdot \nabla) \varphi_j, \varphi_j \right)_g.
$$

Using the explicit expression of $b_g$, we obtain
\[
\sum_{j=1}^{m} b_g(\varphi_j, u, \varphi_j) = \int_{\Omega} \sum_{j=1}^{m} \sum_{i,k=1}^{2} \varphi_{ji}(x) D_i u_k(x) \varphi_{jk}(x) g(x) \, dx \\
\leq \int_{\Omega} |\nabla u(x)| |\rho(x)| \, dx,
\]
(3.10)

where

\[
|\nabla u(x)| = \left\{ \sum_{i,k=1}^{2} |D_i u_k(x)|^2 \right\}^{1/2}, \quad \rho(x) = \sum_{j=1}^{m} \left| \sqrt{g} \varphi_j(x) \right|^2.
\]

Therefore, we get

\[
\left| \sum_{j=1}^{m} b_g(\varphi_j, u, \varphi_j) \right| \leq \int_{\Omega} |\nabla u(x)| |\rho(x)| \, dx \\
\leq (\text{with the Schwarz inequality}) \\
\leq \|u\| |\rho|.
\]
(3.11)

Also, we obtain

\[
\left| \sum_{j=1}^{m} \left( \frac{v_g}{g} (\nabla \cdot \nabla) \varphi_j, \varphi_j \right) \right|_g \leq \sum_{j=1}^{m} \frac{v|\nabla g|_{\infty}}{m_0 \lambda_1} \|\varphi_j\|_g |\varphi_j|_g \\
\leq \sum_{j=1}^{m} \frac{v|\nabla g|_{\infty}}{m_0 \lambda_1^{1/2}} \|\varphi_j\|_g^2,
\]
(3.12)

We recall that the dependence on \(\tau\) has been omitted and in fact \(u = u(x, \tau), \rho = \rho(x, \tau), \text{ etc.}\) From (3.8)–(3.12), we have established the following inequality:

\[
\text{Tr} F'(u(\tau)) \circ Q_m(\tau) \leq -v \left( 1 - \frac{|\nabla g|_{\infty}}{m_0 \lambda_1^{1/2}} \right) \sum_{j=1}^{m} \|\varphi_j\|_g^2 + \|u\| |\rho|.
\]
(3.13)

Now, we have the constant \(c'_2\) of (3.29) of Chapter VI in [28], depending only on the shape of \(\Omega\) such that

\[
|\rho|^2 \leq c'_2 \sum_{j=1}^{m} |\nabla (\sqrt{g} \varphi_j)|^2 \leq c'_2 \sum_{j=1}^{m} \left( \frac{|\nabla g|^2_{\infty}}{4m_0} + |\nabla g|_{\infty} + M_0 \right) |\nabla \varphi_j|^2 \\
\leq c'_2 d_4 \sum_{j=1}^{m} \|\varphi_j\|^2 \leq c'_2 d_4 \sum_{j=1}^{m} \|\varphi_j\|_g^2,
\]
(3.14)

where \(d_4 = |\nabla g|_{\infty}^2 / 4m_0 + |\nabla g|_{\infty} + M_0\).

Hence
$$\operatorname{Tr} F'(u(\tau)) \circ Q_m(\tau) \leq -\nu \left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) \sum_{j=1}^m \|\varphi_j\|_g^2 + \|u\| \left(\frac{c'^2d_4}{m_0} \sum_{j=1}^m \|\varphi_j\|_g^2\right)^{1/2}$$

$$\leq (\text{with the Schwarz inequality})$$

$$\leq -\frac{\nu}{2} \left(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) \sum_{j=1}^m \|\varphi_j\|_g^2 + \frac{c'^2d_4}{2m_0^2} \|u\|_g^2.$$  (3.15)

Now, we see the lower bound of the eigenvalues of the $g$-Stokes operator $A_gu = P_g(-\frac{1}{g}(\nabla \cdot g \nabla)u)$.

**Remark 3.2.** We note that if for spatial periodic boundary conditions with $\int_\Omega u \, dx = 0$, $u$ and $\lambda$ satisfies the following eigenvalue problem of self-adjoint second order partial differential equation,

$$L[u] + \lambda gu = (gu_x)_x + (gu_y)_y + \lambda gu = 0,$$  (3.16)

then $u$ and $\lambda$ are eigenvector and eigenvalue of $A_gu = P_g(-\frac{1}{g}(\nabla \cdot g \nabla)u)$. Also, we have the following lemma from Chapter VI in Courant and Hilbert [8],

**Lemma 3.3.** For the differential equation $L[u] + \lambda gu = 0$, under any of the boundary conditions considered, the number $A(\lambda)$ of eigenvalues less than a given number $\lambda$ for a square-domain $\Omega$ is asymptotically equal to

$$\frac{\lambda}{4\pi} \int_\Omega dx \, dy$$

in other words, the relation

$$\lim_{\lambda \to \infty} \frac{A(\lambda)}{\lambda} = \frac{1}{4\pi} \int_\Omega dx \, dy.$$  (3.17)

So, for the eigenvalues $\lambda_j$ of $A_g$, we obtain

$$\lambda_j \sim c\lambda_1 j \quad \text{as} \quad j \to \infty.$$  (3.18)

Then Temam [28, Lemma VI.2.1] shows the existence of a dimensionless constant $c_1$ depending only on the shape of $\Omega$ and $g$ such that

$$\sum_{j=1}^m \|\varphi_j\|_g^2 \geq \lambda_1 + \cdots + \lambda_m \geq c_1\lambda_1 m^2.$$  (3.32)

The constant $c_1$ converge to the constant $c'_{\lambda}$ of (3.32) of Chapter VI in [28] as $g$ goes to 1 in $W^{1,\infty}(\Omega)$.

Another method to estimate the lower bound of the eigenvalues can be obtained from Remark 2.2,

$$\sum_{j=1}^m \|\varphi_j\|_g^2 \geq \lambda_1 + \cdots + \lambda_m \geq \frac{m_0}{M_0} (\lambda'_1 + \cdots + \lambda'_m) \geq \frac{m_0}{M_0} c'_\lambda \lambda'_1 m^2.$$  (3.17)
where the constant \( c'_1 \) is from (3.32) of Chapter VI in [28] and \( \lambda'_j \) are eigenvalues of the Stokes operator.

So, (3.15) and (3.17) allows us to majorise \( \text{Tr} F'(u) \circ Q_m \) as follows:

\[
\text{Tr} F'(u(\tau)) \circ Q_m(\tau) \leq -\frac{vm_0}{2M_0} \left( 1 - \frac{2|\nabla g|_\infty}{m_0\lambda'_1^{1/2}} \right) c'_1 \lambda'_1 m^2 + \frac{c'_2 d_4}{2vm_0} \|u\|_g^2.
\]

Hence

\[
\frac{1}{t} \int_0^t \text{Tr} F'(u(\tau)) \circ Q_m(\tau) \, d\tau \leq -\frac{vm_0}{2M_0} \gamma_1 c'_1 \lambda'_1 m^2 + \frac{c'_2 d_4}{2vm_0} \frac{1}{t} \int_0^t \|u(\tau)\|_g^2 \, d\tau,
\]

where \( \gamma_1 = 1 - \frac{2|\nabla g|_\infty}{m_0\lambda'_1^{1/2}} > 0 \).

Now we define

\[
q_m(t) = \sup_{u_0 \in A_g} \sup_{\xi_j \in H_g, \|\xi_j\| \leq 1, j=1,\ldots,m} \left( \frac{1}{t} \int_0^t \text{Tr} F'(u(\tau)) \circ Q_m(\tau) \, d\tau \right),
\]

\[
q_m(t) \leq -\frac{vm_0}{2M_0} \gamma_1 c'_1 \lambda'_1 m^2 + \frac{c'_2 d_4}{2vm_0} \sup_{u_0 \in A_g} \frac{1}{t} \int_0^t \|u(\tau)\|_g^2 \, d\tau,
\]

\[
q_m = \limsup_{t \to \infty} q_m(t) \leq -\kappa_1 m^2 + \kappa_2,
\]

where the set \( A_g \) is the global attractor of 2D \( g \)-Navier–Stokes equations and

\[
\kappa_1 = \frac{vm_0}{2M_0} \gamma_1 c'_1 \lambda'_1, \quad \kappa_2 = \frac{c'_2 d_4}{2m_0} \frac{\varepsilon}{v^2 \lambda_1},
\]

\[
\varepsilon = v \lambda_1 \limsup_{t \to \infty} \sup_{u_0 \in A_g} \frac{1}{t} \int_0^t \|u(\tau)\|_g^2 \, d\tau.
\]

Then, by Proposition 3.1 we have the following theorem.

**Theorem 3.4.** We consider the dynamical system associated with the two-dimensional \( g \)-Navier–Stokes equations when \( |\nabla g|_\infty < \pi^2 m_0^3 / M_0 \). Then for the global attractor \( A_g \), we have

\[
\dim_F A_g \leq \sqrt{\frac{\kappa_2}{\kappa_1}},
\]

where

\[
\kappa_1 = \frac{vm_0}{2M_0} \gamma_1 c'_1 \lambda'_1, \quad \kappa_2 = \frac{c'_2 d_4}{2m_0} \frac{\varepsilon}{v^2 \lambda_1},
\]

\[
\varepsilon = v \lambda_1 \limsup_{t \to \infty} \sup_{u_0 \in A_g} \frac{1}{t} \int_0^t \|u(\tau)\|_g^2 \, d\tau.
\]
\[ \varepsilon = \nu \lambda_1 \limsup_{t \to \infty} \sup_{u_0 \in A_g} \frac{1}{t} \int_0^t \| u(\tau) \|_g^2 \, d\tau, \]

\[ d_4 = \frac{\| \nabla g \|_\infty^2}{4m_0} + \| \nabla g \|_\infty + M_0, \quad \gamma_1 = 1 - \frac{2\| \nabla g \|_\infty}{m_0 \lambda_1^{1/2}}, \]

\[ \lambda_1 = \text{the first eigenvalue of the } g-\text{Stokes operator}, \]

\[ \lambda'_1 = \text{the first eigenvalue of the Stokes operator}, \]

\[ c'_1 \text{ and } c'_2 \text{ are from (3.32) and (3.29) of Chapter VI in [28].} \]

### 3.2. Dirichlet boundary conditions

In this section, we will estimate the dimension of the global attractor by two different methods. First method is to estimate by adopting the method of Chepyzhov and Ilyin [2,3] directly into the \( g \)-Navier–Stokes equations. Second method is to estimate by adopting the method of Chepyzhov and Ilyin [2,3] into the following equations which obtained by taking the projection \( P_1 \) of Eq. (2.3),

\[ P_1 \frac{du}{dt} + P_1 P_g (-\Delta u) + P_1 P_g (u \cdot \nabla) u = P_1 P_g f, \quad (3.18) \]

where \( P_1 \) is the Leray projection for \( g = 1 \). Then, the semiflow generated by \( w = P_1 u(t) \) of solution of (3.18) has an invariant set \( X = P_1 A_g \) by the property of the global attractor \( A_g \) of Eq. (2.3). Moreover, we will note that \( \dim_F X \) in the space \( H_1 \) is bigger than \( \dim_F A_g \) in the space \( H_g \), where \( H_1 = H_g \) for \( g = 1 \). Therefore, we will find the upper bound of the dimension of \( A_g \) by calculating the upper bound of the dimension of \( X \).

**Method I.** First, by using the method of Ilyin [15], we will estimate the lower bound of the eigenvalues of the \( g \)-Stokes operator.

We consider the linear \( g \)-Stokes problem: find \( u \) and \( p \) satisfying

\[-\frac{1}{g} (\nabla \cdot g \nabla) u + \nabla p = f \quad \text{in } \Omega \subset \mathbb{R}^2, \]

\[ \nabla \cdot g u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \]

which has the following weak form: for \( f \in V'_g \) find \( u \in V_g \) such that

\[ ((u, v))_g = (f, v)_{V'_g \times V_g}, \quad v \in V_g, \quad (3.19) \]

where \( ((u, v))_g = \sum_{i,j=1}^2 \int_\Omega \frac{\partial u_l}{\partial x_i} \frac{\partial v_j}{\partial x_i} g \, dx \). Then the problem (3.19) has a unique solution (see [25]). Moreover, the \( g \)-Stokes operator has the basis in \( H_g \) of orthogonal eigenfunctions \( \{ u_k \}_{k=1}^\infty \),

\[-\frac{1}{g} (\nabla \cdot g \nabla) u_k + \nabla p_k = \lambda_k u_k, \quad \nabla \cdot g u_k = 0 \quad (3.20) \]
with the corresponding eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to \infty$. Take the scalar product in $H_g$ of Eq. (3.20) and $u_k$, and integrate by parts. This gives

$$\int_{\Omega} \nabla u_k \cdot \nabla u_k g \, dx = \lambda_k. \quad (3.21)$$

Then, we obtain the following lower bound of the eigenvalues of the $g$-Stokes operator.

**Lemma 3.5.** We denote $\int_{\Omega} g \, dx = \bar{g}$. Then

$$\sum_{k=1}^{m} \lambda_k \geq \frac{\pi m^2 m_0^2}{M_0 \bar{g}} - \frac{4m|\nabla g|_{\infty}^2}{m_0 M_0}. \quad (3.22)$$

**Proof.** We denote by $e_k(\xi)$ the Fourier transform of the vector $u_k(x)$,

$$e_k(\xi) = \int_{\Omega} e^{-i\xi x} u_k(x) g(x) \, dx,$$

and denote by

$$f(\xi) = \sum_{k=1}^{m} |e_k(\xi)|^2.$$

Let $a$ be a constant real vector and denote

$$d_k(\xi) = a \cdot e_k(\xi).$$

Then, we obtain

$$0 \leq \int_{\Omega} \left( ae^{-i\xi x} - \sum_{k=1}^{m} d_k(\xi) u_k(x) \right) \cdot \left( ae^{i\xi x} + \sum_{n=1}^{m} d_n(\xi)^* u_n(x) \right) g(x) \, dx$$

$$= |a|^2 \bar{g} - \sum_{k=1}^{m} |d_k(\xi)|^2$$

by orthonormality of $u_k$ in $H$.

Substituting in (3.23) $a = e_j$, where $e_j$ are unit coordinate vectors in $R^2$ and summing up the resulting inequalities, we obtain

$$f(\xi) = \sum_{k=1}^{m} |e_k(\xi)|^2 \leq 2 \sum_{k=1}^{m} |d_k(\xi)|^2 \leq 2 \bar{g}.$$  

We also have

$$\xi \cdot e_k(\xi) = \xi \cdot \int_{\Omega} e^{-i\xi x} u_k(x) g(x) \, dx$$

$$= i \int_{\Omega} \nabla_x (e^{-i\xi x}) \cdot u_k(x) g(x) \, dx = 0, \quad (3.24)$$
because \( \text{div}(g u_k) = 0 \) and \( u_k|_{\partial \Omega} = 0 \). Since \( \text{div}(g u_k) = 0 \) and \( u_k|_{\partial \Omega} = 0 \), we obtain
\[
\xi \times e_k(\xi) = i \int_{\Omega} \nabla_x (e^{-i\xi x}) \times u_k(x) g(x) \, dx = -i \int_{\Omega} e^{-i\xi x} \cdot \text{rot}(g u_k) \, dx,
\]
where \( \times \) denotes the vector product.

Thus, by (3.21) and the Plancherel theorem, we have
\[
\| \xi \times e_k(\xi) \|^2_{L^2} = 4\pi^2 \| \text{rot}(g u_k(x)) \|^2_{L^2} \leq 4\pi^2 \left( M_0 \| u_k \|^2_{g} + \frac{4\| \nabla g \|^2_{\infty}}{m_0} \right) \leq 4\pi^2 \left( M_0 \lambda_k + \frac{4\| \nabla g \|^2_{\infty}}{m_0} \right).
\] (3.25)

Finally, by (3.24), we have
\[
|\xi \times e_k(\xi)|^2 = |\xi|^2 |e_k(\xi)|^2.
\]
Hence the function \( f(\xi) \) satisfies
\[
0 \leq f(\xi) \leq 2\bar{g} = M_1
\] (3.26)
and
\[
\int |\xi|^2 f(\xi) \, d\xi = \int |\xi|^2 \sum_{k=1}^{m} |e_k(\xi)|^2 \, d\xi 
\leq 4\pi^2 \left( M_0 \sum_{k=1}^{m} \lambda_k + \frac{4m\| \nabla g \|^2_{\infty}}{m_0} \right) = M_2.
\] (3.27)

**Lemma 3.6** [19]. *Let a function \( f(\xi), f: R^2 \to R \) satisfy (3.26) and (3.27). Then we have*
\[
\int f(\xi) \, d\xi \leq (2M_1 M_2 \pi)^{1/2}.
\] (3.28)

Applying lemma to the function \( f(\xi) = \sum_{k=1}^{m} |e_k(\xi)|^2 \) and using the Plancherel theorem once again, we obtain
\[
\int |e_k(\xi)|^2 \, d\xi = 4\pi^2 \int_{\Omega} |g u_k(x)|^2 \, dx \geq 4\pi^2 m_0,
\]
because \( \int_{\Omega} u_k \cdot u_k g \, dx = 1 \). So, we have
\[
\int f(\xi) \, d\xi = \int \sum_{k=1}^{m} |e_k(\xi)|^2 \, d\xi \geq 4\pi^2 mm_0.
\] (3.29)

Hence, by (3.28) and (3.29) we have
\[
4\pi^2 mm_0 \leq \int f(\xi) \, d\xi \leq \left( 16\pi^3 \bar{g} \left( M_0 \sum_{k=1}^{m} \lambda_k + \frac{4m\| \nabla g \|^2_{\infty}}{m_0} \right) \right)^{1/2}.
\]
Therefore, we obtain
\[ \sum_{k=1}^{m} \lambda_k \geq \frac{\pi m^2 \bar{m}_0^2}{M_0 \bar{g}} - \frac{4m|\nabla g|^2_{\infty}}{m_0 M_0} \]
which completes the proof. \( \Box \)

For the Dirichlet boundary conditions, to compare with the results of [2,3], we follow the method of [2,3], we will use the similar estimate with the one in [3] for nonlinear term. So instead of (3.30) we will use the following:
\[ \sum_{j=1}^{m} b_{g}(\varphi_j, u, \varphi_j) = \int_{\Omega} \sum_{j=1}^{m} \sum_{i,k=1}^{2} \varphi_{ji}(x) D_i u_k(x) \varphi_{jk}(x) g(x) \, dx \leq \frac{1}{\sqrt{2}} \int_{\Omega} |\nabla u(x)| |\rho(x)| \, dx \tag{3.30} \]
which implies
\[ \left| \sum_{j=1}^{m} b_{g}(\varphi_j, u, \varphi_j) \right| \leq \frac{1}{\sqrt{2}} \|u\| |\rho|, \]
where \( \rho(x) = \sum_{j=1}^{m} |\sqrt{g(x)} \varphi_j(x)|^2 \). As a result, we will have the following inequality instead of (3.13):
\[ \text{Tr} F'(u(\tau)) \circ Q_m(\tau) \leq -\nu \left( 1 - \frac{|\nabla g|_{\infty}}{m_0 \lambda_1^{1/2}} \right) \sum_{j=1}^{m} \|\varphi_j\|^2_{g} + \frac{1}{\sqrt{2}} \|u\| |\rho|. \tag{3.31} \]

Now, we introduce very useful lemma on estimates for orthonormal families of functions is proved in [18] and also used in [3].

**Lemma 3.7.** Suppose that vector functions \( \xi_1, \ldots, \xi_m \in H^1_0(\Omega) \) make up an orthonormal family in \( L^2(\Omega) \), \( \Omega \subset \mathbb{R}^2 \):
\[ \int_{\Omega} \xi_i(x) \cdot \xi_j(x) \, dx = \delta_{ij}. \]
Then the following inequality holds:
\[ \int_{\Omega} \zeta(x)^2 \, dx \leq \frac{2}{\pi} \sum_{j=1}^{m} |\nabla \xi_j|^2, \]
where \( \zeta(x) = \sum_{j=1}^{m} |\xi_j(x)|^2 \).

So, by Lemma 3.7, we obtain
\[ |\rho|^2 \leq \frac{2}{\pi} \sum_{j=1}^{m} |\nabla(\sqrt{g}\varphi_j)|^2 \leq \frac{2}{\pi} \sum_{j=1}^{m} \left( \frac{|\nabla g|_\infty^2}{4m_0} + |\nabla g| + M \right) |\nabla \varphi_j|^2 \]
\[ \leq \frac{2d_4}{\pi} \sum_{j=1}^{m} \|\varphi_j\|^2 \leq \frac{2d_4}{m_0\pi} \sum_{j=1}^{m} \|\varphi_j\|^2_g, \]  
\tag{3.32}

where \( d_4 = \frac{|\nabla g|_\infty^2}{4m_0} + |\nabla g|_\infty + M \).

Also, by Lemma 3.5, we have
\[ \sum_{j=1}^{m} \|\varphi_j\|^2_g \geq \sum_{j=1}^{m} \lambda_j \geq \frac{\pi m^2 m_0^2}{M_0 \bar{g}} - \frac{4m|\nabla g|_\infty^2}{m_0 M_0}. \]
\tag{3.33}

Thus, by (3.31)–(3.33) we have
\[ \text{Tr } F'(u(\tau)) \circ Q_m(\tau) \]
\[ \leq -\nu \left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) \sum_{j=1}^{m} \|\varphi_j\|^2_g + \frac{1}{\sqrt{2}} \|u\| \|\rho\| \]
\[ \leq -\nu \left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) \sum_{j=1}^{m} \|\varphi_j\|^2_g + \frac{\sqrt{d_4}}{\sqrt{m_0 \pi}} \|u\| \left( \sum_{j=1}^{m} \|\varphi_j\|^2_g \right)^{1/2} \]
\[ \leq (\text{with the Schwarz inequality}) \]
\[ \leq -\nu \left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) \sum_{j=1}^{m} \|\varphi_j\|^2_g + \frac{d_4}{2vm_0\pi} \|u\|^2 \]
\[ \leq -\nu \left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) \left( \frac{\pi m^2 m_0^2}{M_0 \bar{g}} - \frac{4m|\nabla g|_\infty^2}{m_0 M_0} \right) + \frac{d_4}{2vm_0^2 \pi} \|u\|^2_g \]

which implies by (3.6)
\[ q_m = \limsup_{t \to \infty} q_m(t) \leq -\nu \left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) \left( \frac{\pi m^2 m_0^2}{M_0 \bar{g}} - \frac{4m|\nabla g|_\infty^2}{m_0 M_0} \right) \]
\[ + \frac{d_4}{2vm_0^2 \pi} \frac{|f|^2}{\lambda_1 v^2} \left(1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right)^{-1}. \]

Therefore, by Proposition 3.1 we have the following theorem.

**Theorem 3.8.** We consider the dynamical system associated with the two-dimensional \( g \)-Navier–Stokes equations with Dirichlet boundary conditions when \( |\nabla g|_\infty^2 \leq cm_0^3/4M_0 \)

where \( c \) is the Poincaré constant on \( \Omega \). Then for the global attractor \( \mathcal{A}_g \), we have
\[ \dim_F \mathcal{A}_g \leq \frac{-b - \sqrt{b^2 - 4ac}}{a}, \]  
\tag{3.34}

where
\[ a = -\frac{\nu \pi m_0^2}{2M_0 \bar{g}} \left( 1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \right), \]
\[ b = \frac{\nu |\nabla g|_\infty^2}{m_0 M_0} \left( 1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \right), \]
\[ c = \frac{d_4}{2\nu m_0 \pi} \frac{\|f\|_2}{\lambda_1 v^2} \left( 1 - \frac{2|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \right)^{-1}. \]

**Method II.** We note that for \( g = 1 \), the 2D \( g \)-Navier–Stokes equations are equal to the Navier–Stokes equations. Also, for the case of \( g = 1 \), we denote the functional spaces \( H_g = H_1 \), \( V_g = V_1 \), and the operators \( A_g = A_1 \), \( P_g = P_1 \). Since \( H_g \subset L^2(\Omega) \), by applying \( P_1 : H_g(\subset L^2(\Omega)) \rightarrow H_1 \), we can write, for any \( u \in H_g \), \( u = w + \nabla r \) for some \( w \in H_1 \) and \( \nabla r \in Q \), where the space \( Q \) does not depend on the function \( g(x) \). Then we can have the useful following lemmas.

**Lemma 3.9.** Assume that \( \nabla p \in Q \) and \( p \in H^3(\Omega) \). Then we have
\[
\begin{align*}
P_g \left[ \frac{d}{dt}(\nabla p(t)) \right] &= \frac{d}{dt} P_g(\nabla p(t)) = 0, \\
P_g \left[ -\Delta (\nabla p(t)) \right] &= P_g \left[ \nabla (-\Delta p(t)) \right] = 0, \\
P_g \left[ (\nabla p(t) \cdot \nabla) \nabla p(t) \right] &= P_g \left[ \nabla \left( \frac{1}{2} (\nabla p(t) \cdot \nabla p(t)) \right) \right] = 0.
\end{align*}
\]

One should note that Lemma 3.9 also holds for the constant function \( g = 1 \).

**Lemma 3.10.** For every \( u_1, u_2 \in H_g \), if \( P_1 u_1 = P_1 u_2 \) then \( u_1 = u_2 \). Also, for \( v_1, v_2 \in H_1 \), if \( P_g v_1 = P_g v_2 \) then \( v_1 = v_2 \). In other words, \( P_1 P_g(v) = v \), for \( v \in H_1 \), and \( P_g P_1(u) = u \), for \( u \in H_g \).

**Lemma 3.11.** For \( u \in H_g \), we let
\[ u = w + \nabla r, \quad \text{for } w \in H_1, \ \nabla r \in Q. \]
Then there exist some positive constants \( c_4 = c_4(m_0, M_0, \Omega) \), \( c_5 = c_5(m_0, M_0, \Omega) \) independent on \( u \) and \( w \) such that
\[ |r|_{H^2} \leq c_4 |\nabla g|_\infty |u|, \quad |r|_{H^2} \leq c_5 |\nabla g|_\infty |w|. \]
Moreover, if \( u \in V_g \), then there is a constant \( c_6 = c_6(m_0, M_0, \Omega) \) such that
\[ |r|_{H^3} \leq c_6 |g|_{2,\infty} \| w \|_g, \]
where
\[ |g|_{2,\infty} = \sum_{i=1}^2 |D^i g|_\infty. \]
Lemma 3.12. For \( u \in L^2(\Omega) \) we have \( P_1 P_g u = P_1 u \). As a result, for \( w \in H_1 \) we have \( (P_1 P_g u, w) = (u, w) \).

Recall the evolution equation of the two-dimensional \( g \)-Navier–Stokes equations

\[
\frac{du}{dt} - \nu P_g \Delta u + P_g (u \cdot \nabla) u = f \quad \text{in } H_g.
\] (3.35)

For our purpose, we apply the projection \( P_1 \) to Eq. (3.35) and then, due to Lemmas 3.9 and 3.12, we have

\[
\frac{dw}{dt} - \nu P_1 \Delta w + P_1 (P_g w \cdot \nabla) P_g w = P_1 f.
\] (3.36)

Then we obtain the following lemma.

Lemma 3.13. The set \( X \) is a bounded invariant subset of \( H_1 \) under the semiflow generated by the solutions of (3.36).

Proof. For any \( w_0 \in X \), we can choose an element \( u_0 \in A_g \) such that \( w_0 = P_1 u_0 \). Then there is a solution \( u(t) \) of (3.35) with \( u(0) = u_0 \). Since the set \( A_g \) is the global attractor, \( u(t) \in A_g \) for all \( t \geq 0 \). Therefore, if we let \( w(t) = P_1 u(t) \), then \( w(t) \in X \) for all \( t \geq 0 \).

Let \( w_1 \in X \) with \( w_1 = P_1 u_1, u_1 \in A_g \). Since \( A_g \) is the global attractor, for any \( \tau > 0 \), there is a \( u_0 \in A_g \) and a solution \( u(t) \) such that \( u(0) = u_0 \) and \( u(\tau) = u_1 \). Therefore, if we let \( w(t) = P_1 u(t) \), then \( w(t) \) satisfies Eq. (3.36) such that \( w(0) = P_1 u_0 \in X \) and \( w(\tau) = w_1 \) and hence the set \( X \) is invariant and bounded in \( H_1 \). \( \square \)

Next, we want to estimate the dimension of \( X = P_1 A_g \) in \( H_1 \). From Lemma 3.10 we note that \( P_g X = P_g P_1 A_g = A_g \). So we have that the fractal dimension of \( A_g \) in the space \( L^2(\Omega, g) \) is less than the dimension of \( X \) in the space \( L^2(\Omega, g) \). Also, since the space \( L^2(\Omega, g) \) is equivalent Banach space to the space \( L^2(\Omega) \), by using the Definition 1.4 we note that the dimension of \( X \) in the space \( L^2(\Omega, g) \) is equal to the dimension of \( X \) in the space \( L^2(\Omega) \).

So we have

\[
\dim_F A_g \ (\text{in } L^2(\Omega, g)) \leq \dim_F X \ (\text{in } L^2(\Omega, g)) = \dim_F X \ (\text{in } L^2(\Omega)).
\]

Therefore, we will estimate the upper bound of the dimension of \( X \) in the space \( L^2(\Omega) \) to estimate the upper bound of the dimension of \( A_g \) in the space \( L^2(\Omega, g) \).

For \( w_0 \in H_1 \), let \( w(t) = S_1(t)w_0, t \geq 0 \), be the semiflow of Eq. (3.36) with \( w(0) = w_0 \). Then the linearized flow around \( w \) is given by

\[
W' = F'_1(w)W,
\] (3.37)

where

\[
F'_1(w)W = \nu P_1 \Delta W - B_1(w, W) - B_1(W, w),
\]

which is equivalent to

\[
W' + \nu A_1 W + B_1(w, W) + B_1(W, w) = 0,
\]

\( W(0) = \xi, \quad \xi \in H_1 \) given,
where \( A_1 w = -P_1 \Delta w = -\Delta w \) for \( w \in D(A_1) \) and \( B_1(z, w) = P_1(P_g z \cdot \nabla)P_g w \) for \( z, w \in V_1 \). Here, we recall the notation

\[
\text{rot} u = \left( -\frac{\partial u_2}{\partial x_1}, \frac{\partial u_1}{\partial x_2} \right),
\]

where \( u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2)) \).

Consider \( m \) solutions \( W_1, \ldots, W_m \) corresponding to the initial data \( \xi_1, \ldots, \xi_m \), with \( \xi_j \in V_1, 1 \leq j \leq m \). With \( w = w(\tau) = S(\tau)w_0, w_0 \in V_1 \), we denote by \( \tilde{Q}_m(\tau) = \tilde{Q}_m(\tau, w_0; \xi_1, \ldots, \xi_m) \) the orthogonal projection in \( V_1 \) onto the space spanned by \( W_1(\tau), \ldots, W_m(\tau) \). For each time \( \tau \), let \( \varphi_j(\tau), j = 1, \ldots, m \), be an orthonormal basis of \( \tilde{Q}_m(\tau)V_1 \). Then, by adopting the method of [3], we obtain

\[
\text{Tr} F'_1(w(\tau)) \circ \tilde{Q}_m(\tau) = \sum_{j=1}^{m} \left( F'_1(w(\tau))\varphi_j(\tau), \varphi_j(\tau) \right)
\]

\[
= -\sum_{j=1}^{m} (vA_1 \varphi_j, \varphi_j) - \sum_{j=1}^{m} (B_1(\varphi_j, w), \varphi_j) - \sum_{j=1}^{m} (B_1(w, \varphi_j), \varphi_j)
\]

\[
= -v \sum_{j=1}^{m} |\text{rot} \varphi_j|^2 - \sum_{j=1}^{m} (B_1(\varphi_j, w), \varphi_j) - \sum_{j=1}^{m} (B_1(w, \varphi_j), \varphi_j). \tag{3.38}
\]

Then, since \( P_g w = w + \nabla r \) and \( P_g \varphi_j = \varphi_j + \nabla r_j, j = 1, \ldots, m \), we first have

\[
\sum_{j=1}^{m} (B_1(w, \varphi_j), \varphi_j) = \sum_{j=1}^{m} (P_1(P_g w \cdot \nabla)P_g \varphi_j, \varphi_j)
\]

\[
= \sum_{j=1}^{m} (P_1(w \cdot \nabla)\varphi_j, \varphi_j) + \sum_{j=1}^{m} (P_1(\nabla r \cdot \nabla)\varphi_j, \varphi_j)
\]

\[
+ \sum_{j=1}^{m} (P_1(P_g w \cdot \nabla)\nabla r_j, \varphi_j). \tag{3.39}
\]

First, by Lemma 3.12 we have

\[
(P_1(w \cdot \nabla)\varphi_j, \varphi_j) = ((w \cdot \nabla)\varphi_j, \varphi_j) = 0. \tag{3.40}
\]

From now on, we will use \( c > 0 \) as a generic constant which does only depend on the domain \( \Omega \). Due to Lemma 3.11, we have

\[
\left| \sum_{j=1}^{m} (P_1(\nabla r \cdot \nabla)\varphi_j, \varphi_j) \right|
\]

\[
= \sum_{j=1}^{m} \left| (\nabla r \cdot \nabla)\varphi_j, \varphi_j \right| \leq \sum_{j=1}^{m} c|\nabla r|_{H^2} \| \varphi_j \|^2 \leq c|\nabla r|_{H^2} \sum_{j=1}^{m} \| \varphi_j \|^2
\]
\[ \leq c |g|_{2,\infty} |P_g w| \sum_{j=1}^{m} \| \varphi_j \|^2 \leq c |g|_{2,\infty} |P_g w| \sum_{j=1}^{m} |\text{rot} \varphi_j|^2 \tag{3.41} \]

and

\[
\left| \sum_{j=1}^{m} \left( P_1 (P_g w \cdot \nabla) \nabla r_j, \varphi_j \right) \right| \\
= \sum_{j=1}^{m} \left| \left( (P_g w \cdot \nabla) \nabla r_j, \varphi_j \right) \right| \leq \sum_{j=1}^{m} c |P_g w| |r_j|_{H^3} \| \varphi_j \| \\
\leq c |P_g w| \sum_{j=1}^{m} |r_j|_{H^3} \| \varphi_j \| \leq c |g|_{2,\infty} |P_g w| \sum_{j=1}^{m} \| \varphi_j \|^2 \\
\leq c |g|_{2,\infty} |P_g w| \sum_{j=1}^{m} |\text{rot} \varphi_j|^2. \tag{3.42} \]

Secondly, we also have by Lemma 3.12

\[
\sum_{j=1}^{m} \left( B_1 (\varphi_j, w), \varphi_j \right) = \sum_{j=1}^{m} \left( (\varphi_j \cdot \nabla) P_g w, \varphi_j \right) + \left( (\nabla r_j, \nabla) P_g w, \varphi_j \right). \tag{3.43} \]

We have

\[
\left| \sum_{j=1}^{m} \left( P_1 (\varphi_j \cdot \nabla) P_g w, \varphi_j \right) \right| = \sum_{j=1}^{m} \left| (\varphi_j \cdot \nabla) P_g w, \varphi_j \right| \leq \frac{1}{\sqrt{2}} \| P_g w \| |\rho|, 
\]

where \( \rho(x) = \sum_{j=1}^{m} |\varphi_j(x)|^2 \). Then by using Lieb–Thirring inequality (see [3])

\[
|\rho|^2 = \int_{\Omega} \left( \sum_{j=1}^{m} |\varphi_j(x)|^2 \right)^2 \, dx \leq \frac{1}{\pi} \sum_{j=1}^{m} |\text{rot} \varphi_j|^2, 
\]

we obtain

\[
\left| \sum_{j=1}^{m} \left( P_1 (\varphi_j \cdot \nabla) P_g w, \varphi_j \right) \right| \leq \nu \sum_{j=1}^{m} |\text{rot} \varphi_j|^2 + \frac{1}{4\pi \nu} \| P_g w \|^2 \tag{3.44} \]

For the second term in (3.43), we have

\[
\left| \sum_{j=1}^{m} \left( P_1 (\nabla r_j, \nabla) P_g w, \varphi_j \right) \right| \\
= \sum_{j=1}^{m} \left| ((\nabla r_j \cdot \nabla) P_g w, \varphi_j) \right| \leq \sum_{j=1}^{m} c |r_j|_{H^3} \| P_g w \| \| \varphi_j \| \leq c |P_g w| \sum_{j=1}^{m} |r_j|_{H^3} \| \varphi_j \| \\
\leq c |g|_{2,\infty} |P_g w| \sum_{j=1}^{m} \| \varphi_j \|^2 \leq c |g|_{2,\infty} |P_g w| \sum_{j=1}^{m} |\text{rot} \varphi_j|^2. \tag{3.45} \]
Hence, by (3.38)–(3.45) we have

$$\text{Tr} F'(w(\tau)) \circ \tilde{Q}_m(\tau) \leq \left( -\frac{v}{2} + 3c|g|_{2,\infty}|P_g w| \right) \sum_{j=1}^{m} |	ext{rot} \varphi_j|^2 + \frac{1}{4\pi v} \|P_g w\|^2.$$  

We note from Lemma 3.10 that $P_g w = P_g P_1 u = u$, where $u$ is a solution of (3.35). Due to [25] we have some constant $c(\Omega, m_0, M_0)$ such that

$$|P_g w|^2 = |u|^2 \leq \frac{1}{m_0} |u_g|^2 \leq c \quad \text{for all } w \in X = P_1 A_g.$$  

Thus, by using the following lower bound for the spectrum of the Stokes operator (see [15])

$$\sum_{j=1}^{m} |	ext{rot} \varphi_j|^2 \geq \lambda_1 + \cdots + \lambda_m \geq \frac{\pi m^2}{|\Omega|},$$  

there exist some constant $c(\Omega, m_0, M_0)$ such that

$$\text{Tr} F'(w(\tau)) \circ \tilde{Q}_m(\tau) \leq \left( -\frac{v}{2} + 3c|g|_{2,\infty} \right) \frac{\pi m^2}{|\Omega|} + \frac{1}{4\pi v} \|P_g w\|^2,$$

(3.46)

when $|g|_{2,\infty}$ is small enough. Since

$$\|P_g w(s)\|^2 = \|u(s)\|^2 \leq \frac{1}{m_0} \|u(s)\|_{g}^2,$$

we have from (3.6)

$$\limsup_{t \to \infty} \sup_{w_0 \in X} \frac{1}{t} \int_0^t \|P_g w(s)\|^2 ds \leq \frac{|f|^2}{\lambda_1 v^2 m_0} \left( 1 - \frac{2|\nabla g|_{\infty}}{m_0 \sqrt{\lambda_1}} \right)^{-1},$$

(3.47)

where $u(s)$ is a solution of Eq. (3.35). Hence, we obtain

$$q_m = \limsup_{t \to \infty} q_m(t) \leq \left( -\frac{v}{2} + 3c|g|_{2,\infty} \right) \frac{\pi m^2}{|\Omega|} + \frac{|f|^2}{4\pi \lambda_1 v^3 m_0} \left( 1 - \frac{2|\nabla g|_{\infty}}{m_0 \sqrt{\lambda_1}} \right)^{-1}.$$  

Therefore, by Proposition 3.1 we have the following theorem.

**Theorem 3.14.** We consider the dynamical system associated with Eq. (3.35) with Dirichlet boundary conditions when $|\nabla g|_{\infty} \leq cm_0^3/4M_0$ where $c$ is the Poincaré constant on $\Omega$. Then for the global attractor $A_g$, we have $c = c(m_0, M_0, \Omega)$ such that

$$\dim F A_g (\text{in } L^2(\Omega, g)) = \dim F X (\text{in } L^2(\Omega)) \leq \frac{\sqrt{|\Omega|} |f|}{\sqrt{2\lambda_1 m_0 \pi v}} \times \frac{1}{\sqrt{v(v - 6c|g|_{2,\infty})}} \times \left( 1 - \frac{2|\nabla g|_{\infty}}{m_0 \sqrt{\lambda_1}} \right)^{-1}.$$  

(3.48)
4. The behavior of the dimensions as $g \to 1$

In this section, we will find that the upper bounds for the dimension of the global attractors $A_g$ of the $g$-NSE converge to the corresponding upper bounds for the global attractor $A_1$ of the NSE as $g \to 1$ in the proper sense.

4.1. Periodic boundary conditions

In Theorem 3.4, we note that the flux $\varepsilon$ of the $g$-Navier–Stokes equations converge to the flux of the Navier–Stokes equation in [28] as $g \to 1$ in $W^{1,\infty}(\Omega)$. Moreover, by Remark 2.2, we note that the first eigenvalue $\lambda_1$ of the $g$-Stokes operator converge to the first eigenvalue of the Stokes operator as $g \to 1$ in $W^{1,\infty}(\Omega)$.

Thus, we can see that the constant $k_1, k_2$ converge to the constant $k_1, k_2$ of (3.35) of Chapter VI in [28]. In other words the upper bound of the dimension of the global attractor of the $g$-Navier–Stokes equations converge well to the upper bound (Temam [28]) of the dimension of the global attractor of the Navier–Stokes equations.

4.2. Dirichlet boundary conditions

In this section, we will see how the dimension behave as the function $g$ converge to the constant 1 in some sense and compare with the upper bound in Chepyzhov and Ilyin [3],

$$\dim_F A_1 \leq \frac{1}{\sqrt{2\pi}} \left( \frac{\lambda_1 |\Omega|^{1/2}}{\lambda_1 v^2} \right) = d_5,$$

where $\lambda'_1$ is the first eigenvalue of the Stokes operator.

For the first method, as $g \to 1$ in $W^{1,\infty}$, we have in Theorem 3.8

$$a \to -\frac{v\pi}{2|\Omega|}, \quad b \to 0, \quad c \to \frac{|f|}{2\pi v^3 \lambda'_1},$$

which implies $\dim_F A_g \to \sqrt{2}d_5$. For the second method, as $g \to 1$ in $W^{2,\infty}$, we obtain easily from Theorem 3.14 that $\dim_F A_g \to d_5$.

Therefore, the first method do not get good estimate, but we have good property for the convergence of the function $g$, when the second method do get good estimate with bad property for the convergence of the function $g$.

References


