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A Certain Family of Infinite Series Associated with Digamma Functions

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The sums of several interesting infinite series were recently expressed in terms of the Psi (or Digamma) functions. The object of this paper is to present a systematic account of these (and of numerous similar or more general) series whose sums can be found in the literature in various equivalent forms. Some relevant unifications and further generalizations are also indicated.

1. INTRODUCTION

We begin by recalling a well-known (rather classical) result, which gives the sum of an infinite series in terms of the Psi (or Digamma) function

$$\psi(z)=\frac{d}{dz}\{\log \Gamma(z)\}=\frac{\Gamma'(z)}{\Gamma(z)}, \tag{1.1}$$

in the form

$$\sum_{n=1}^{\infty}\frac{(\nu)_n}{n(\lambda)_n}=\psi(\lambda)-\psi(\lambda-\nu) \qquad (\operatorname{Re}(\lambda-\nu)>0;\, \lambda\neq 0,\, -1,\, -2,\, ...), \tag{1.2}$$

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where $(\lambda)_n$ is the familiar Pochhammer symbol defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & \text{if } n=0,\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n\in\mathbb{N}=\{1,2,3,...\}.\end{cases} \tag{1.3}$$

The summation formula (1.2) and its obvious special cases were revived, in recent years, as illustrations emphasizing the usefulness of fractional calculus in evaluating infinite sums. For a detailed historical account of (1.2), and of its various consequences and generalizations, see one of the latest works on the subject by Nishimoto and Srivastava [9].

In terms of a generalized hypergeometric ${}_rF_s$ series defined by

$${}_rF_s\left[\begin{matrix}\alpha_1,...,\alpha_r;\\ \beta_1,...,\beta_s;\end{matrix}\, z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\frac{z^n}{n!}$$

$$\left(r\leqslant s,\ |z|<\infty;\ r=s+1,\ |z|<1;\ r=s+1,\ |z|=1, \text{ and } \operatorname{Re}\left(\sum_{j=1}^{s}\beta_j-\sum_{j=1}^{r}\alpha_j\right)>0\right), \tag{1.4}$$

provided that no zeros appear in the denominator, the summation formula (1.2) can be rewritten at once as

$${}_3F_2\left[\begin{matrix}\nu+1,1,1;\\ \lambda+1,2;\end{matrix}\, 1\right]=\begin{cases}\dfrac{\lambda}{\nu}\{\psi(\lambda)-\psi(\lambda-\nu)\} & (\nu\neq 0;\ \operatorname{Re}(\lambda-\nu)>0)\\ \lambda\psi'(\lambda) & (\nu=0;\ \operatorname{Re}(\lambda)>0),\end{cases} \tag{1.5}$$

which was proven, by using a simple technique involving l'Hôpital's theorem on limits, by Luke [7, p. 111]. Making use of essentially the same technique, Kalla and Al-Saqabi [5] gave an alternative proof of the equivalent result (1.2) *without* applying the operators of fractional calculus. More importantly, they recorded the following additional consequences of the same technique[1] [5, p. 16, Sect. 3]:

$$\sum_{n=1}^{\infty}\frac{(-1)^n(a)_n}{n(1+a)_n}=\psi\left(1+\frac{1}{2}a\right)-\psi(1+a); \tag{1.6}$$

$$\sum_{n=1}^{\infty}\frac{(a)_n}{n(1/2+(1/2)a)_n 2^{n-1}}=\psi\left(\frac{1}{2}+\frac{1}{2}a\right)-\psi\left(\frac{1}{2}\right); \tag{1.7}$$

[1] Formula (1.7) is the *corrected* version of the corresponding result in Kalla and Al-Saqabi [5, p. 16, Eq. 3(b)].

$$\sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n(a-m)_n (1+b)_n} = \frac{\Gamma(1+b-a+m)\,\Gamma(1-a)\,\Gamma(1+b)}{\Gamma(1+b-a)\,\Gamma(1-a+m)\,\Gamma(1+b)} \{\psi(1+b)+\gamma\}, \tag{1.8}$$

where (as claimed in [5, p. 16]) $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\gamma = -\psi(1)$ is the Euler–Mascheroni constant;

$$\sum_{n=1}^{\infty} \frac{2(a)_n (b)_n}{n(1/2+(1/2)a)_n (2b)_n} = \psi\left(\frac{1}{2}+\frac{1}{2}a\right) + \psi\left(\frac{1}{2}+b\right) - \psi\left(\frac{1}{2}-\frac{1}{2}a+b\right) - \psi\left(\frac{1}{2}\right) \quad (\mathrm{Re}(2b-a) > -1); \tag{1.9}$$

$$\sum_{n=1}^{m} \frac{(-m)_n (a)_n}{n(b)_n (1+a-b-m)_n} = \psi(b-a+m) + \psi(b) - \psi(b+m) - \psi(b-a) \qquad (m \in \mathbb{N}_0); \tag{1.10}$$

$$\sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n(a-m)_n n!} = \psi(1-a+m) - \psi(1-a) - \psi(1-b) - \gamma \quad (\mathrm{Re}(b) < 1-m;\, m \in \mathbb{N}_0); \tag{1.11}$$

$$\sum_{n=1}^{\infty} \frac{(a)_n (b)_n (c)_n}{n(a+1)_n (b+1)_n (c-1)_n} = \frac{b(c-a-1)}{(c-1)(b-a)} \{\psi(1+a)+\gamma\} - \frac{a(c-b-1)}{(c-1)(b-a)} \{\psi(1+b)+\gamma\} \tag{1.12}$$

or, equivalently,

$$\sum_{n=1}^{\infty} \frac{c+n-1}{n(a+n)(b+n)} = \frac{c-a-1}{a(b-a)} \psi(1+a) - \frac{c-b-1}{b(b-a)} \psi(1+b) + \frac{(c-1)\gamma}{ab}; \tag{1.13}$$

$$\sum_{n=1}^{\infty} \frac{(a)_n (1+(1/2)a)_n (b)_n (c)_n}{n((1/2)a)_n (1+a)_n (1+a-b)_n (1+a-c)_n}$$
$$= \psi(1+a-b) + \psi(1+a-c) - \psi(1+a) - \psi(1+a-b-c)$$
$$(\mathrm{Re}(a-b-c) > -1), \tag{1.14}$$

and

$$\sum_{n=1}^{\infty} \frac{(a+m)_n (b)_n}{n(a)_n (1+b)_n} = \frac{\Gamma(a-b+m)\,\Gamma(a)}{\Gamma(a+m)\,\Gamma(a-b)} \{\psi(1+b)+\gamma\}$$
$$(m \in \mathbb{N}_0;\ \text{see [5, p. 17]}), \tag{1.15}$$

provided, in *each* case, that no zeros appear in the denominators.

In this paper we aim at presenting a systematic account of each of the series (1.6) to (1.15), and of numerous other series related to them. We also consider some relevant unifications and further generalizations of many of these results.

2. Validity and Novelty of the Summation Formulas (1.6) to (1.15)

First of all, a closer look at the right-hand side of (1.8) will reduce it readily to the form

$$\sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n(a-m)_n (1+b)_n}$$
$$= \frac{\Gamma(1+b-a+m)\,\Gamma(1-a)}{\Gamma(1+b-a)\,\Gamma(1-a+m)} \{\psi(1+b)+\gamma\} \qquad (m \in \mathbb{N}_0). \tag{2.1}$$

Replacing a by $a+m$, and noting that

$$\frac{\Gamma(1+b-a)\,\Gamma(1-a-m)}{\Gamma(1+b-a-m)\,\Gamma(1-a)} = \frac{(1-a)_{-m}}{(1+b-a)_{-m}} = \frac{(a-b)_m}{(a)_m}$$
$$= \frac{\Gamma(a-b+m)\,\Gamma(a)}{\Gamma(a+m)\,\Gamma(a-b)} \qquad (m \in \mathbb{N}_0),$$

(2.1) immediately implies the summation formula (1.15). On the other hand, (1.15) with a replaced by $a-m$ ($m \in \mathbb{N}_0$) similarly yields the summation formula (1.8). Thus the two summation formulas (1.8) [or (2.1)] and (1.15) are the same result written in two seemingly different forms. More importantly, the series occurring on the left-hand sides of (1.8), (2.1), and (1.15) diverge for *each* $m \in \mathbb{N}$. It follows that the summation formulas (1.8), (2.1), and (1.15) are valid only when $m=0$, in which case each of these results reduces to the form

$$\sum_{n=1}^{\infty} \frac{(b)_n}{n(1+b)_n} = \psi(1+b)+\gamma \tag{2.2}$$

or, equivalently,

$$\sum_{n=1}^{\infty} \frac{1}{n(b+n)} = \frac{1}{b}\{\psi(1+b)+\gamma\}. \tag{2.3}$$

Formula (2.2) or (2.3) is a widely recorded special case of the well-known result (1.2) when

$$v=b \qquad \text{and} \qquad \lambda = 1+b,$$

since $\psi(1)=-\gamma$. Consequently, the corrected (convergent) version (2.2) of the summation formulas (1.8), (2.1), and (1.15) is an obvious consequence of the familiar result (1.2).

Next we turn to the summation formula (1.12). Just as the transition from (2.2) to (2.3), the equivalent form (1.13) follows readily from (1.12) when we apply the definition (1.3) to Pochhammer quotients like

$$\frac{(\lambda)_n}{(\lambda+1)_n} = \frac{\lambda}{\lambda+n} \qquad (n \in \mathbb{N}_0). \tag{2.4}$$

Thus it would suffice to consider (1.13). Indeed, observing that

$$\frac{c+n-1}{(a+n)(b+n)} = \frac{c-a-1}{b-a}\,\frac{1}{a+n} - \frac{c-b-1}{b-a}\,\frac{1}{b+n}, \tag{2.5}$$

the left-hand side of (1.13) can be rewritten in the form

$$\sum_{n=1}^{\infty} \frac{c+n-1}{n(a+n)(b+n)} = \frac{c-a-1}{b-a} \sum_{n=1}^{\infty} \frac{1}{n(a+n)} - \frac{c-b-1}{b-a} \sum_{n=1}^{\infty} \frac{1}{n(b+n)}. \tag{2.6}$$

If we now apply (2.3) to sum each series on the right-hand side of (2.6), we are led at once to the summation formula (1.13). Since (2.3) is a (widely recorded) special case of the well-known result (1.2), the summation formula (1.12) or (1.13) may be looked upon as being an interesting corollary of (1.2) itself.

In view of the elementary identity (2.4), the summation formula (1.6) assumes the form

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(a+n)} = \frac{1}{a}\left\{\psi\left(1+\frac{1}{2}a\right) - \psi(1+a)\right\}, \tag{2.7}$$

which is essentially the same as the known result[2] (cf. [3, p. 102, Eq. (6.1.14)])

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(\lambda+\mu n)} = \frac{1}{\lambda}\left\{\beta\left(1+\frac{\lambda}{\mu}\right) - \ln 2\right\}, \tag{2.8}$$

where (and in what follows) $\beta(z)$ is defined by

$$\beta(z) = \frac{1}{2}\left\{\psi\left(\frac{1}{2}+\frac{1}{2}z\right) - \psi\left(\frac{1}{2}z\right)\right\} = \sum_{N=0}^{\infty} \frac{(-1)^N}{z+N}. \tag{2.9}$$

As a matter of fact, since

$$\psi(2z) = \frac{1}{2}\left\{\psi(z) + \psi\left(z+\frac{1}{2}\right)\right\} + \ln 2, \tag{2.10}$$

(2.8) with $\lambda = a$ and $\mu = 1$ yields (2.7), and (2.7) with $a = \lambda/\mu$ leads to (2.8).

Making use of the hypergeometric notation (1.4), the summation formula (1.6) becomes

$${}_3F_2\left[\begin{matrix} 1, 1, 1+a; \\ 2, 2+a; \end{matrix} \; -1\right] = \frac{1+a}{a}\left\{\psi(1+a) - \psi\left(1+\frac{1}{2}a\right)\right\} \tag{2.11}$$

or, equivalently,

$${}_3F_2\left[\begin{matrix} 1, 1, a; \\ 2, 1+a; \end{matrix} \; -1\right] = \frac{a}{a-1}\{\beta(a) - \ln 2\}, \tag{2.12}$$

which was recorded, for example, by Prudnikov *et al.* [10, p. 547, Entry 7.4.5.16].

The following hypergeometric forms of the summation formulas (1.9) and (1.10) can easily be written from the definition (1.4):

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} 1, 1, 1+a, 1+b; \\ 2, \frac{1}{2}(3+a), 1+2b; \end{matrix} \; 1\right] \\ &= \frac{a+1}{2a}\left\{\psi\left(\frac{1}{2}+\frac{1}{2}a\right) + \psi\left(\frac{1}{2}+b\right) - \psi\left(\frac{1}{2}-\frac{1}{2}a+b\right) - \psi\left(\frac{1}{2}\right)\right\} \\ &\qquad (\mathrm{Re}(2b-a) > -1); \end{aligned} \tag{2.13}$$

[2] Formula (2.8) appears *erroneously* in Hansen [3, p. 102, Eq. (6.1.14)] where $\beta(y/x)$ should be corrected to read $\beta((y/x)+1)$.

$$
{}_4F_3\left[\begin{matrix} 1, 1, 1+a, 1-m; \\ 2, 1+b, 2+a-b-m; \end{matrix} \; 1\right]
= \frac{b(1+a-b-m)}{ma}\{\psi(b-a)+\psi(b+m)-\psi(b)-\psi(b-a+m)\}
$$

$$
(m \in \mathbb{N}). \tag{2.14}
$$

Replacing a, b, and m by $a-1$, $b-1$, and $m+1$, respectively, and applying the relationship

$$
\psi(z) = \psi(1-z) - \pi \cot \pi z, \tag{2.15}
$$

this last result (2.14) can be rewritten as

$$
{}_4F_3\left[\begin{matrix} 1, 1, a, -m; \\ 2, b, 1+a-b-m; \end{matrix} \; 1\right] = \frac{(b-1)(a-b-m)}{(m+1)(a-1)}\{\psi(b+m)+\psi(1+a-b)
-\psi(b-1)-\psi(a-b-m)\}
$$

$$
(m \in \mathbb{N}_0). \tag{2.16}
$$

Each of the hypergeometric summation formulas (2.13), (2.14), and (2.16) is a known result. Formula (2.13) was given by Lavoie [6, p. 272], and (2.14) in the equivalent form (2.16) is recorded, for example, by Prudnikov *et al.* [10, p. 556, Entry 7.5.3.43]. As a matter of fact, (2.16) with $a=\lambda+m$ $(m \in \mathbb{N}_0)$ was given earlier by Luke [8, p. 167, Eq. 5.2(21)].

Formula (1.7) is a limiting case of (1.9) when $b \to \infty$. Equivalently, the hypergeometric summation formula

$$
{}_3F_2\left[\begin{matrix} 1, 1, 1+a; \\ 2, \frac{1}{2}(3+a); \end{matrix} \; \frac{1}{2}\right] = \frac{a+1}{2a}\left\{\psi\left(\frac{1}{2}+\frac{1}{2}a\right)-\psi\left(\frac{1}{2}\right)\right\} \tag{2.17}
$$

follows immediately from the known result (2.13) upon letting $b \to \infty$.

The summation formula (1.11) [with a replaced trivially by $a+m$ $(m \in \mathbb{N}_0)$] was recorded by Hansen [3, p. 131, Eq. (6.6.94)].

We conclude this section by remarking that the summation formula (1.14) (of Kalla and Al-Saqabi [5]) does not seem to have been noticed earlier. Indeed, in view of the elementary identity (2.4), we can rewrite (1.14) in the form

$$
\sum_{n=1}^{\infty} \frac{a+2n}{n(a+n)} \frac{(b)_n (c)_n}{(1+a-b)_n (1+a-c)_n}
= \psi(1+a-b)+\psi(1+a-c)-\psi(1+a)
-\psi(1+a-b-c) \qquad (\mathrm{Re}(a-b-c) > -1). \tag{2.18}
$$

3. Unifications and Generalizations

Suppose that $P(x)$ is a polynomial in x of degree $\leqslant r-2$, and let

$$\frac{xP(x)}{(\lambda_1+\mu_1 x)\cdots(\lambda_r+\mu_r x)}=\sum_{k=1}^{r}\frac{A_k}{\lambda_k+\mu_k x}, \tag{3.1}$$

where (and in what follows) the parameters $\lambda_1, ..., \lambda_r$ are constrained by

$$\lambda_j\mu_k\neq\lambda_k\mu_j \qquad (j\neq k; j, k=1, ..., r). \tag{3.2}$$

Then, clearly,

$$A_k=-\lambda_k\mu_k^{r-2}P\left(-\frac{\lambda_k}{\mu_k}\right)\prod_{\substack{j=1\\ j\neq k}}^{r}(\lambda_j\mu_k-\lambda_k\mu_j)^{-1} \qquad (k=1, ..., r) \tag{3.3}$$

and

$$\sum_{k=1}^{r}A_k\prod_{\substack{j=1\\ j\neq k}}^{r}\lambda_j=(\lambda_1\cdots\lambda_r)\sum_{k=1}^{r}\frac{A_k}{\lambda_k}=0. \tag{3.4}$$

Making use of (3.1), (3.4), and the summation formula (2.3), it is easy to derive an interesting unification (and generalization) of numerous results like (1.13), (2.2), and (2.3) in the form

$$\sum_{n=1}^{\infty}\frac{P(n)}{(\lambda_1+\mu_1 n)\cdots(\lambda_r+\mu_r n)}$$
$$=-\sum_{k=1}^{r}\mu_k^{r-2}P\left(-\frac{\lambda_k}{\mu_k}\right)\psi\left(1+\frac{\lambda_k}{\mu_k}\right)\prod_{\substack{j=1\\ j\neq k}}^{r}(\lambda_j\mu_k-\lambda_k\mu_j)^{-1}, \tag{3.5}$$

provided that each of the inequalities in (3.2) holds true.

In precisely the same manner, we can show for a polynomial $Q(x)$ in x of degree $\leqslant r-1$ that

$$\sum_{n=1}^{\infty}\frac{(-1)^n Q(n)}{(\lambda_1+\mu_1 n)\cdots(\lambda_r+\mu_r n)}$$
$$=-\sum_{k=1}^{r}\mu_k^{r-2}Q\left(-\frac{\lambda_k}{\mu_k}\right)\beta\left(1+\frac{\lambda_k}{\mu_k}\right)\prod_{\substack{j=1\\ j\neq k}}^{r}(\lambda_j\mu_k-\lambda_k\mu_j)^{-1}, \tag{3.6}$$

where $\beta(z)$ is defined by (2.9), and each of the inequalities in (3.2) is assumed to hold true.

The summation formulas (3.5) and (3.6) can be suitably specialized to yield scores of hitherto scattered results. If, for example, we set

$$r=3 \qquad \text{and} \qquad P(x)=p+qx$$

in (3.5), we shall obtain an interesting generalization of (1.13) in the form

$$\sum_{n=1}^{\infty} \frac{p+qn}{(\lambda+\mu n)(\rho+\sigma n)(\xi+\eta n)}$$
$$=\frac{p\mu-q\lambda}{(\xi\mu-\eta\lambda)(\lambda\sigma-\mu\rho)}\psi\left(1+\frac{\lambda}{\mu}\right)+\frac{p\sigma-q\rho}{(\lambda\sigma-\mu\rho)(\rho\eta-\sigma\xi)}\psi\left(1+\frac{\rho}{\sigma}\right)$$
$$+\frac{p\eta-q\xi}{(\rho\eta-\sigma\xi)(\xi\mu-\eta\lambda)}\psi\left(1+\frac{\xi}{\eta}\right), \tag{3.7}$$

provided that λ/μ, ρ/σ, and ξ/η are unequal. Indeed, since $\psi(1)=-\gamma$, (3.7) does yield (1.13) when we set

$$p=c-1,\ q=1,\ \lambda=0,\ \mu=1,\ \rho=a,\ \sigma=1,\ \xi=b, \text{ and } \eta=1.$$

On the other hand, (3.7) with

$$p=\xi \qquad \text{and} \qquad q=\eta$$

immediately reduces to a result given, for example, by Bromwich [1, p. 522, Example 42] and Hansen [3, p. 103, Eq. (6.1.18)]. As a matter of fact, Hansen [3, p. 106, Eq. (6.1.65); p. 108, Eq. (6.1.89)] records two other special cases of the summation formula (3.7) when

$$\text{(i) } p=1 \quad \text{and} \quad q=0; \qquad \text{(ii) } p=-1 \quad \text{and} \quad q=1.$$

A summation formula, analogous to (3.7), would follow from (3.6) upon setting

$$r=3 \qquad \text{and} \qquad Q(x)=(p+qx)(u+vx).$$

We thus obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n (p+qn)(u+vn)}{(\lambda+\mu n)(\rho+\sigma n)(\xi+\eta n)}$$
$$=\frac{(p\mu-q\lambda)(u\mu-v\lambda)}{\mu(\xi\mu-\eta\lambda)(\lambda\sigma-\mu\rho)}\beta\left(1+\frac{\lambda}{\mu}\right)+\frac{(p\sigma-q\rho)(u\sigma-v\rho)}{\sigma(\lambda\sigma-\mu\rho)(\rho\eta-\sigma\xi)}\beta\left(1+\frac{\rho}{\sigma}\right)$$
$$+\frac{(p\eta-q\xi)(u\eta-v\xi)}{\eta(\rho\eta-\sigma\xi)(\xi\mu-\eta\lambda)}\beta\left(1+\frac{\xi}{\eta}\right), \tag{3.8}$$

which, for $u=1$ and $v=0$, provides the aforementioned analog of (3.7).

Among the various special cases of the summation formula (3.8), recorded in the literature, we mention only the following three:

(i) $u=\xi$, $v=\eta$, $p=1$, and $q=0$;

(ii) $u=\xi$, $v=\eta$, $p=-1$, and $q=1$;

(iii) $u=1$, $v=0$, $p=-1$, and $q=1$,

given in Hansen [3, p. 103, Eq. (6.1.23); p. 105, Eq. (6.1.53); p. 108, Eq. (6.1.91)]. In fact, the first one of these known special cases [3, p. 103, Eq. (6.1.23)] contains (1.6), (2.7), and (2.8) as its obvious further particular cases.

For positive integer values of the parameter quotients λ_k/μ_k $(k=1, ..., r)$, the general results (3.5) and (3.6) were given by Chrystal [2, p. 248, Eq. (9); p. 253, Exercise 38] and Jolley [4, p. 218, Entries 1107 and 1108]. On the other hand, Hansen [3, p. 115, Eq. (6.1.193); p. 122, Eqs. (6.3.56) and (6.3.58)] recorded the special cases of (3.5) and (3.6) when

$$P(x)=(x-1)^s \qquad (s=0, 1, 2, ..., r-2)$$

and

$$Q(x)=(x-1)^s \qquad (s=0, 1, 2, ..., r-1),$$

respectively, and Prudnikov *et al.* [10, p. 573, Entry 7.10.2.3] gave the generalized hypergeometric forms of (3.5) and (3.6) when

$$P(x)=Q(x)=1.$$

Finally, we recall a unification (and generalization) of numerous results including, for example, (1.6), (1.10), (1.14), (2.7), (2.8), (2.11), (2.12), (2.14), (2.16), and (2.18) in the form

$$\begin{aligned}\sum_{n=1}^{m} \frac{a+2n}{n(a+n)} \frac{(b)_n\,(c)_n\,(1+2a-b-c+m)_n\,(-m)_n}{(1+a-b)_n\,(1+a-c)_n\,(b+c-a-m)_n\,(1+a+m)_n} &= \psi(1+a-b)+\psi(1+a-c)+\psi(1+a+m)+\psi(1+a-b-c+m)\\ &\quad -\psi(1+a)-\psi(1+a-b-c)-\psi(1+a-b+m)\\ &\quad -\psi(1+a-c+m) \qquad (m\in\mathbb{N}_0), \end{aligned} \tag{3.9}$$

which was derived recently by Srivastava [12] as an interesting consequence of Dougall's theorem [11, p. 244, Eq. (III.14)].

Letting $m\to\infty$, (3.9) readily yields the summation formula (1.14) in its equivalent form (2.18). Moreover, if we replace c on both sides of (3.9) by $1+a-c$, and then let $a \to \infty$, we shall obtain [cf. Eqs. (2.14) and (2.16)]

$$\sum_{n=1}^{m} \frac{(b)_n (-m)_n}{n(c)_n (1+b-c-m)_n} = \psi(c)+\psi(c-b+m)-\psi(c-b)-\psi(c+m) \qquad (m \in \mathbb{N}_0), \tag{3.10}$$

which is essentially the same as the summation formula (1.10). Obviously, in their limiting case when $m \to \infty$, both (3.10) and (1.10) would at once correspond to the well-known summation formula (1.2).

When $c \to -\infty$, the general summation formula (3.9) reduces to

$$\sum_{n=1}^{m} \frac{a+2n}{n(a+n)} \frac{(b)_n (-m)_n}{(1+a-b)_n (1+a+m)_n} = \psi(1+a-b)+\psi(1+a+m)-\psi(1+a) - \psi(1+a-b+m) \qquad (m \in \mathbb{N}_0), \tag{3.11}$$

which, upon letting $m \to \infty$, yields

$$\sum_{n=1}^{\infty} \frac{a+2n}{n(a+n)} \frac{(-1)^n (b)_n}{(1+a-b)_n} = \psi(1+a-b)-\psi(1+a) \qquad (\mathrm{Re}(a-2b) > -2). \tag{3.12}$$

Formula (1.6), as well as its equivalent forms (2.7), (2.8), (2.11), and (2.12), would correspond to the special case of (3.12) when $b=(1/2)a$.

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References

1. T. J. I'A. Bromwich, "An Introduction to the Theory of Infinite Series," 2nd rev. ed., Macmillan & Co., London, 1926.
2. G. Chrystal, "Algebra: An Elementary Text-Book," Part II, 7th ed., Chelsea, New York, 1964.
3. E. R. Hansen, "A Table of Series and Products," Prentice-Hall, Englewood Cliffs, NJ, 1975.
4. L. B. W. Jolley, "Summation of Series," 2nd rev. ed., Dover, New York, 1961.
5. S. L. Kalla and B. Al-Saqabi, Summation of certain infinite series with Digamma functions, *C. R. Acad. Bulgare Sci.* **41**, No. 11 (1988), 15–17.

6. J. L. Lavoie, Some summation formulas for the series ${}_3F_2(1)$, *Math. Comp.* **49** (1987), 269–274.
7. Y. L. Luke, "The Special Functions and Their Approximations," Vol. I, Academic Press, New York/San Francisco/London, 1969.
8. Y. L. Luke, "Mathematical Functions and Their Approximations," Academic Press, New York/San Francisco/London, 1975.
9. K. Nishimoto and H. M. Srivastava, Certain classes of infinite series summable by means of fractional calculus, *J. College Engrg. Nihon Univ. Ser. B* **30** (1989), 97–106.
10. A. P. Prudnikov, Yu. A. Bryčkov, and O. I. Maričev, "Integrals and Series: Supplementary Chapters," Nauka, Moscow, 1986. [In Russian]
11. L. J. Slater, "Generalized Hypergeometric Functions," Cambridge Univ. Press, Cambridge/London/New York, 1966.
12. H. M. Srivastava, A simple algorithm for the evaluation of a class of generalized hypergeometric series, *Studies in Appl. Math.*, in press.