Mixed equilibrium problems and optimization problems

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\textbf{A B S T R A C T}

In this paper, we introduce and analyze a new hybrid iterative algorithm for finding a common element of the set of solutions of mixed equilibrium problems and the set of fixed points of an infinite family of nonexpansive mappings. Furthermore, we prove some strong convergence theorems for the hybrid iterative algorithm under some mild conditions. We also discuss some special cases. Results obtained in this paper improve the previously known results in this area.

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1. Introduction

Equilibrium problems which were introduced by Blum and Oettli [1] and Noor and Oettli [2] in 1994 have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. It has been shown [1,2] that equilibrium problems include variational inequalities, fixed point, Nash equilibrium and game theory as special cases. Hence collectively, equilibrium problems cover a vast range of applications. Due to the nature of the equilibrium problems, it is not possible to extend the projection and its variant forms for solving equilibrium problems. To overcome this drawback, one usually uses the auxiliary principle technique. The main and basic idea in this technique is to consider an auxiliary equilibrium problem related to the original problem and then show that the solution of the auxiliary problems is a solution of the original problem. This technique has been used to suggest and analyze a number of iterative methods for solving various classes of equilibrium problems and variational inequalities, see [16,25–29] and the references therein.

Related to the equilibrium problems, we also have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the equilibrium problems and a set of the fixed points of finitely many nonexpansive mappings, see [6,8,11–17,30,32–45] and the references therein. In this paper, we suggest and analyze a hybrid iterative method for finding a common element of a set of the solutions of mixed equilibrium problems and a set of fixed points of an infinite family of nonexpansive mappings...
nonexpansive mappings. We also study the strong convergence of the iterative method under suitable conditions. Several special cases are also discussed.

Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\varphi : C \rightarrow \mathbb{R}$ be a real-valued function and $\Theta : C \times C \rightarrow \mathbb{R}$ be an equilibrium bifunction, i.e., $\Theta(u, u) = 0$ for each $u \in C$. The mixed equilibrium problem (for short, MEP) is to find $x^* \in C$ such that

$$\text{MEP: } \Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C.$$ 

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (for short, EP), which is to find $x^* \in C$ such that

$$\text{EP: } \Theta(x^*, y) \geq 0, \quad \forall y \in C.$$ 

Denote the set of solutions of MEP by $\Omega$. The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems and the equilibrium problems as special cases; see, for example, [1–5]. Some methods have been proposed to solve the MEP and EP, see, for example, [5–17,25–31].

First we recall some important results as follows.

Recall that a mapping $f : C \rightarrow C$ is called contractive if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$, $\forall x, y \in C$. A mapping $T : C \rightarrow H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. Recall also that a mapping $T : C \rightarrow H$ is called firmly nonexpansive if $\|Tx - Ty\|^2 \leq (Tx - Ty, x - y)$ for all $x, y \in C$. Denote the set of fixed points of $T$ by $F(T)$.

Let $A$ be a strongly positive bounded linear operator on $H$, that is,

$$\langle Ax, x \rangle \geq 0 \quad \text{for all } x \in H.$$ 

Now we consider the following optimization problem (for short, OP):

$$\text{OP: } \min_{x \in \hat{F}} \frac{\mu}{2} \langle Ax, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x),$$

where $\hat{F} = \bigcap_{i=1}^{\infty} C_i$, $C_1, C_2, \ldots$ are infinitely many closed convex subsets of $H$ such that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, $A$ is a strongly positive bounded linear operator on $H$ and $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$). This kind of optimization problems has been studied extensively by many authors, see, for example, Bauschke and Borwein [18], Combettes [19], Deutsch and Yamada [20] and Xu [21] when $\hat{F} = \bigcap_{i=1}^{N} C_i$ and $h(x) = \langle x, b \rangle$.

2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $C$ be a nonempty closed convex subset of $H$. Then, for any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|$$

for all $y \in C$. Such a $P_C$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is nonexpansive. Furthermore, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \iff \langle x - x^*, x^* - y \rangle \geq 0 \quad \text{for all } y \in C.$$ 

In this paper, for solving the mixed equilibrium problems for an equilibrium bifunction $\Theta : C \times C \rightarrow \mathbb{R}$, we assume that $\Theta$ satisfies the following conditions:

(H1) $\Theta$ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;
(H2) for each fixed $y \in C$, $x \mapsto \Theta(x, y)$ is concave and upper semicontinuous;
(H3) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex.

We also recall some well-known concepts from convex analysis.

A differentiable function $K : C \rightarrow \mathbb{R}$ on a convex set $C$ is called:

(i) convex, if

$$K(y) - K(x) \geq \langle K'(x), y - x \rangle, \quad \forall x, y \in C,$$

where $K'$ is the Frechet derivative of $K$ at $x$;

(ii) strongly convex, if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), y - x \rangle \geq (\sigma/2)\|x - y\|^2, \quad \forall x, y \in C.$$
Let \( \varphi : C \to \mathbb{R} \) be a real-valued function and \( \Theta : C \times C \to \mathbb{R} \) be an equilibrium bifunction. Let \( r \) be a positive number. For a given point \( x \in C \), we consider the problem of finding \( y \in C \) such that
\[
\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), z - y \rangle \geq 0, \quad \forall z \in C,
\]
which is known as the auxiliary mixed equilibrium problem. Let \( S_r : C \to C \) be the mapping such that for each \( x \in C \), \( S_r(x) \) is the solution set of the auxiliary problem MEP, i.e., \( \forall x \in C \),
\[
S_r(x) = \left\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), z - y \rangle \geq 0, \quad \forall z \in C \right\}.
\]

Using the technique of Ceng and Yao [16], one can easily prove the following result.

**Lemma 2.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( \varphi : C \to \mathbb{R} \) be a lower semicontinuous and convex functional. Let \( \Theta : C \times C \to \mathbb{R} \) be an equilibrium bifunction satisfying conditions (H1)–(H3). Assume that

(i) \( K : C \to \mathbb{R} \) is strongly convex with constant \( \sigma > 0 \) and its derivative \( K' \) is sequentially continuous from the weak topology to the strong topology;

(ii) for each \( x \in C \), there exist a bounded subset \( D_x \subset C \) and \( z_x \in C \) such that for any \( y \in C \setminus D_x \),
\[
\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r} \langle K'(y) - K'(x), z_x - y \rangle < 0.
\]

Then there hold the following:

(i) \( S_r \) is single-valued;

(ii) \( S_r \) is nonexpansive if \( K' \) is Lipschitz continuous with constant \( \nu > 0 \) and
\[
\langle K'(x_1) - K'(x_2), u_1 - u_2 \rangle \geq \nu \langle K'(u_1) - K'(u_2), u_1 - u_2 \rangle, \quad \forall (x_1, x_2) \in C \times C,
\]
where \( u_i = S_r(x_i) \) for \( i = 1, 2 \);

(iii) \( F(S_r) = \Omega \);

(iv) \( \Omega \) is closed and convex.

We also need the following lemmas for proving our main results.

**Lemma 2.2.** (See [22].) Let \( \{x_n\} \) and \( \{z_n\} \) be bounded sequences in a Banach space \( X \) and let \( \{\beta_n\} \) be a sequence in \( (0, 1) \) with \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). Suppose
\[
x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n
\]
for all integers \( n \geq 0 \) and
\[
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]
Then, \( \lim_{n \to \infty} \|z_n - x_n\| = 0 \).

**Lemma 2.3.** (See [24].) Assume \( \{a_n\} \) is a sequence of nonnegative real numbers such that
\[
a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,
\]
where \( \{\gamma_n\} \) is a sequence in \( (0, 1) \) and \( \{\delta_n\} \) is a sequence such that

(1) \( \sum_{n=1}^{\infty} \gamma_n = \infty \);

(2) \( \limsup_{n \to \infty} \delta_n/\gamma_n \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 2.4.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \), and \( g : C \to \mathbb{R} \cup \{\infty\} \) be a proper lower-semicontinuous differentiable convex function. If \( x^* \) is a solution to the minimization problem
\[
g(x^*) = \inf_{x \in C} g(x),
\]
then
\[
\langle g'(x), x - x^* \rangle \geq 0, \quad x \in C.
\]
In particular, if \( x^* \) solves problem OP, then
\[
\langle u + (\gamma f - (I + \mu A))x^*, x - x^* \rangle \leq 0.
\]

**Proof.** Since \( C \) is convex, \( x^* + t(x - x^*) \in C \) for all \( x \in C \) and \( 0 < t < 1 \). Hence
\[
\lim_{t \to 0^+} \frac{g(x^* + t(x - x^*)) - g(x^*)}{t} = \langle g'(x^*), x - x^* \rangle \geq 0.
\]
In particular, if
\[
g(x) = \frac{\mu}{2} \langle Ax, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x) = \frac{1}{2} \| (I + \mu A)x, x \rangle - \langle x, u \rangle + \frac{1}{2} \|u\|^2 - h(x),
\]
then
\[
g'(x) = (I + \mu A)x - u - \gamma f(x).
\]
Consequently, we obtain
\[
\langle u + (\gamma f - (I + \mu A))x^*, x - x^* \rangle \leq 0.
\]
This completes the proof. \( \square \)

**Lemma 2.5.** (See [24]) In a real Hilbert space \( H \), there holds the inequality
\[
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle, \quad \forall x, y \in H.
\]

3. Iterative scheme and strong convergence

In this section, we first introduce our iterative scheme. Consequently, we will establish strong convergence theorems for this iteration scheme. To be more specific, let \( T_1, T_2, \ldots \) be infinitely many mappings of \( C \) into \( H \) and let \( \lambda_1, \lambda_2, \ldots \) be real numbers such that \( 0 \leq \lambda_i \leq 1 \) for every \( i \in N \). For any \( n \in N \), define a mapping \( W_n \) of \( C \) into \( H \) as follows:
\[
\begin{align*}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\
W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I.
\end{align*}
\]
Such a \( W_n \) is usually called the \( W \)-mapping generated by \( T_n, T_{n-1}, \ldots, T_1 \) and \( \lambda_n, \lambda_{n-1}, \ldots, \lambda_1 \).

We have the following crucial Lemmas 3.1 and 3.2 concerning \( W_n \). The readers are referred to [23]. Now we only need the following similar version in Hilbert spaces.

**Lemma 3.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T_1, T_2, \ldots \) be nonexpansive mappings of \( C \) into \( H \) such that \( \bigcap_{i=1}^{\infty} F(T_i) \) is nonempty, and let \( \lambda_1, \lambda_2, \ldots \) be real numbers such that \( 0 < \lambda_i \leq b < 1 \) for any \( i \in N \). Then, for every \( x \in C \) and \( k \in N \), the limit \( \lim_{n \to \infty} U_{n,k}x \) exists.

**Lemma 3.2.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T_1, T_2, \ldots \) be nonexpansive mappings of \( C \) into \( H \) such that \( \bigcap_{i=1}^{\infty} F(T_i) \) is nonempty, and let \( \lambda_1, \lambda_2, \ldots \) be real numbers such that \( 0 < \lambda_i \leq b < 1 \) for any \( i \in N \). Then, \( F(W) = \bigcap_{i=1}^{\infty} F(T_i) \).

The following remark, which is due to Yao, Liou and Yao, [14], is important to prove our main results.
Theorem 3.1. that Algorithm 3.1. given Step 1. Proof. We divide the proof into several steps. Throughout this paper, we will assume that $0 < \lambda_i \leq b < 1$ for every $i \in \mathbb{N}$.

Now we introduce the following iteration algorithm.

Algorithm 3.1. Let $\mu > 0$, $\gamma > 0$, $r > 0$ be three constants. Let $f$ be a contraction of $H$ into itself with coefficient $\alpha \in (0, 1)$ and let $A$ be a strongly positive bounded linear operator on $H$ with coefficient $\gamma > 0$ such that $0 < \gamma < (1 + \mu)/\alpha$. For given $x_0 \in H$ arbitrarily and fixed $u \in H$, suppose the sequences $\{x_n\}$ and $\{y_n\}$ are generated iteratively by

\[
\begin{align*}
  y_n &= S_t r_n; \\
  x_{n+1} &= \alpha_n (u + \gamma f(x_n)) + \beta_n x_n + \left(1 - \beta_n\right) I - \alpha_n (I + \mu A) W_n y_n, \quad \forall n \geq 1,
\end{align*}
\]

Remark 3.1. Using Lemma 3.1, one can define mapping $W$ of $C$ into $H$ as: $Wx = \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_n x$, for every $x \in C$. If $\{x_n\}$ is a bounded sequence in $C$, then we have

\[
\lim_{n \to \infty} \|W x_0 - W_n x_0\| = 0.
\]

Now we study the strong convergence of the hybrid iterative method (2) and this is the main task of this paper.

Theorem 3.1. Let $H$ be a real Hilbert space. Let $\varphi : H \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : H \times H \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions (H1)-(H3) and let $T_1, T_2, \ldots$ be an infinite family of nonexpansive mappings on $H$ such that $\cap_{n=1}^{\infty} F(T_n) \cap \Omega \neq \emptyset$. Let $f$ be a contraction of $H$ into itself with coefficient $\alpha \in (0, 1)$, and let $A$ be a strongly positive bounded linear operator on $H$ with coefficient $\gamma > 0$ and $0 < \gamma < (1 + \mu)/\alpha$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Assume that:

(i) $K : H \to \mathbb{R}$ is strongly convex with constant $\sigma > 0$ and its derivative $K'$ is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$;

(ii) for each $x \in H$, there exist a bounded subset $D_x \subset H$ and $z_x \in H$ such that, for any $y \notin D_x$,

\[
\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{\nu} K'(y) - K'(x), z_x - y < 0;
\]

(iii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

Given $x_0 \in H$ arbitrarily, then the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by (2) converge strongly to $x^* \in \cap_{n=1}^{\infty} F(T_n) \cap \Omega$ which solves OPI below provided $S_t$ is firmly nonexpansive,

\[
\min_{x \in \cap_{n=1}^{\infty} F(T_n) \cap \Omega} \frac{\mu}{2} \langle A x, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x).
\]

Proof. We divide the proof into several steps.

Step 1. First, we prove that $\{x_n\}$ is bounded. Note that from the control conditions (iii), we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)(1 + \mu \|A\|)^{-1}$. Recall that a standard result in functional analysis is that if $A$ is linear bounded self-adjoint operator on $H$, then

\[
\|A\| = \sup \left\{ \| \langle A u, u \rangle \| : u \in H, \|u\| = 1 \right\}.
\]

Observe that

\[
\left\| (1 - \beta_n) I - \alpha_n (I + \mu A) \right\| u, u \rangle = 1 - \beta_n - \alpha_n - \alpha_n \mu \langle A u, u \rangle \geq 1 - \beta_n - \alpha_n - \alpha_n \mu \|A\| \geq 0,
\]

that is to say $(1 - \beta_n) I - \alpha_n (I + \mu A)$ is positive. It follows that

\[
\left\| (1 - \beta_n) I - \alpha_n (I + \mu A) \right\| = \sup \left\{ \left\| (1 - \beta_n) I - \alpha_n (I + \mu A) \right\| u, u \rangle : u \in H, \|u\| = 1 \right\} = \sup \left\{ 1 - \beta_n - \alpha_n - \alpha_n \mu \langle A u, u \rangle : u \in H, \|u\| = 1 \right\} \leq 1 - \beta_n - \alpha_n - \alpha_n \mu \gamma.
\]
Let \( p \in \Gamma = \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \). Then, we have
\[
\| y_n - p \| = \| S_r y_n - S_r p \| \leq \| x_0 - p \|.
\] (3)

Next prove that \( \{ x_n \} \) and \( \{ y_n \} \) are all bounded. Indeed, from Lemma 3.2 we have \( p \in F(W_n) \). Set \( \bar{A} = (I + \mu A) \). Then from (2) and (3), we obtain
\[
\| x_{n+1} - p \| = \| \alpha_n (u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \bar{A}) W_n y_n - p \|
\]
\[
= \| \alpha_n u + \alpha_n (\gamma f(x_n) - \bar{A} p) + \beta_n (x_n - p) + ((1 - \beta_n)I - \alpha_n \bar{A}) (W_n y_n - p) \|
\]
\[
\leq (1 - \beta_n - \alpha_n - \alpha_n \mu \bar{\gamma}) \| y_n - p \| + \beta_n \| x_n - p \| + \alpha_n \| u \| + \alpha_n \| \gamma f(x_n) - \bar{A} p \|
\]
\[
\leq (1 - \alpha_n - \alpha_n \mu \bar{\gamma}) \| y_n - p \| + \alpha_n \| u \| + \alpha_n \| \gamma f(x_n) - f(p) \| + \alpha_n \| \gamma f(p) - \bar{A} p \|
\]
\[
\leq (1 - \alpha_n - \alpha_n \mu \bar{\gamma}) \| y_n - p \| + \alpha_n \| u \| + \alpha_n \| \gamma f(x_n) - f(p) \| + \alpha_n \| \gamma f(p) - \bar{A} p \|
\]
\[
= \left[ 1 - (1 + \mu \bar{\gamma}) - \gamma \alpha \right] \| x_n - p \| + \alpha_n \| \gamma f(p) - \bar{A} p \| + \| u \|
\]
\[
= \frac{1}{(1 + \mu \bar{\gamma}) - \gamma \alpha} \max \{ \| x_0 - p \|, \| y f(p) - \bar{A} p \|, \| u \| \}. \tag{4}
\]

It follows from (4) by induction that
\[
\| x_n - p \| \leq \max \left\{ \| x_0 - p \|, \frac{\| y f(p) - \bar{A} p \| + \| u \|}{(1 + \mu \bar{\gamma}) - \gamma \alpha} \right\}, \quad n \geq 0.
\]

Therefore \( \{ x_n \} \) is bounded. We also obtain that \( \{ y_n \}, \{ W_n x_n \}, \{ W_n y_n \} \) and \( \{ f(x_n) \} \) are all bounded.

**Step 2.** We show that \( \| x_{n+1} - x_n \| \to 0 \) as \( n \to \infty \).

Define
\[
x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n
\]
for all \( n \geq 0 \).

Observe that from the definition of \( z_n \), we obtain
\[
z_{n+1} - z_n = \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}
\]
\[
= \frac{\alpha_{n+1} (u + \gamma f(x_{n+1})) + ((1 - \beta_{n+1})I - \alpha_{n+1} \bar{A}) W_{n+1} y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n (u + \gamma f(x_n)) + ((1 - \beta_n)I - \alpha_n \bar{A}) W_n y_n}{1 - \beta_n}
\]
\[
= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (u + \gamma f(x_{n+1})) - \frac{\alpha_n}{1 - \beta_n} (u + \gamma f(x_n)) + W_{n+1} y_{n+1} - W_n y_n + \frac{\alpha_n}{1 - \beta_n} W_{n+1} y_{n+1} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \bar{A} W_{n+1} y_{n+1}
\]
\[
= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (u + \gamma f(x_{n+1}) - \bar{A} W_{n+1} y_{n+1}) + \frac{\alpha_n}{1 - \beta_n} [\bar{A} W_n y_n - u - \gamma f(x_n)] + W_{n+1} y_{n+1} - W_n y_n.
\]

It follows that
\[
\| z_{n+1} - z_n \| \leq \| x_{n+1} - x_n \| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \| \gamma f(x_{n+1}) \| + \| \bar{A} W_{n+1} y_{n+1} \|
\]
\[
+ \frac{\alpha_n}{1 - \beta_n} \| \bar{A} W_n y_n \| + \| u \| + \| \gamma f(x_n) \| + W_{n+1} y_{n+1} - W_n y_n
\]
\[
+ \| W_{n+1} y_n - W_n y_n \| - \| x_{n+1} - x_n \|
\]
\[
\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \| \gamma f(x_{n+1}) \| + \| \bar{A} W_{n+1} y_{n+1} \|
\]
\[
+ \frac{\alpha_n}{1 - \beta_n} \| \bar{A} W_n y_n \| + \| u \| + \| \gamma f(x_n) \| + W_{n+1} y_n - W_n y_n
\]
\[
+ \| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|. \tag{5}
\]
From (1), since $W_n$, $T_n$ and $U_{n,i}$ are all nonexpansive, we have

\[
\|W_{n+1}y_n - W_ny_n\| = \|\lambda_1 T_1 U_{n+1,2} y_n - \lambda_1 T_1 U_{n,2} y_n\| \\
\leq \lambda_1 \| U_{n+1,2} y_n - U_{n,2} y_n\| \\
= \lambda_1 \| u + \gamma f(x_n) + (1 - \alpha_n)\bar{A} W_{n+1} y_n\| \\
\leq \alpha_n \frac{\| u + \gamma f(x_n)\|}{1 - \beta_n} + \|W_{n+1} - W_n\| + \|U_{n+1,2} - U_{n,2}\| + M_n \prod_{i=1}^{n} \lambda_i \cdot (8)
\]

where $M$ is a constant such that $\sup_r \{\|U_{n+1,1} y_n\| + \|U_{n,1} y_n\|, n \geq 0\} \leq M$.

On the other hand, from $y_n = S_r x_n$ and $y_{n+1} = S_r x_{n+1}$, from the nonexpansivity of $S_r$ we have

\[
\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|. \tag{7}
\]

Using (6) and (7) in (5), we get

\[
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \alpha_{n+1} \frac{\|u + \gamma f(x_n)\|}{1 - \beta_n} + \|W_{n+1} - W_n\| + \|U_{n+1,2} - U_{n,2}\| + M_n \prod_{i=1}^{n} \lambda_i \cdot (9)
\]

which implies that

\[
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]

Hence by Lemma 2.2, we have

\[
\lim_{n \to \infty} \|z_n - x_n\| = 0.
\]

Consequently

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n)\|z_n - x_n\| = 0. \tag{8}
\]

**Step 3.** $\|W_{n+1} - W_n\| \to 0$ as $n \to \infty$.

Note that $x_{n+1} = \alpha_n (u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n \bar{A})W_n y_n$, then we have

\[
\|x_n - W_n y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n y_n\| \\
\leq \|x_n - x_{n+1}\| + \alpha_n \|u + \gamma f(x_n) - \bar{A} W_n y_n\| + \|x_n - W_n y_n\|.
\]

that is

\[
\|x_n - W_n y_n\| \leq \frac{1}{1 - \beta_n} \|x_n - x_{n+1}\| + \alpha_n \|u + \gamma f(x_n) - \bar{A} W_n y_n\|.
\]

It follows from (iii) and (8) that

\[
\lim_{n \to \infty} \|x_n - W_n y_n\| = 0. \tag{9}
\]

For $p \in \bigcap_{m=1}^{\infty} F(T_m) \cap \Omega$, note that $S_r$ is firmly nonexpansive, then we have

\[
\|y_n - p\|^2 = \|S_r x_n - S_r p\|^2 \\
\leq \langle S_r x_n - S_r p, x_n - p\rangle \\
= \langle y_n - p, x_n - p\rangle \\
= \frac{1}{2} (\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2)
\]

and hence

\[
\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2.
\]
Therefore, from Lemma 2.5 and (2), we have
\[
\|x_{n+1} - p\|^2 = \|\alpha_n (u + \gamma f(x_n) - \bar{A}p) + \beta_n (x_n - W_n y_n) + (I - \alpha_n \bar{A})(W_n y_n - p)\|^2 \\
\leq \|\alpha_n (u + \gamma f(x_n) - \bar{A}p)\|^2 + \beta_n \|x_n - W_n y_n\|^2 + 2\alpha_n \|u + \gamma f(x_n) - \bar{A}p\| \|x_{n+1} - p\| \\
\leq \|\alpha_n (u + \gamma f(x_n) - \bar{A}p)\|^2 + \beta_n \|x_n - W_n y_n\|^2 + 2\alpha_n \|u + \gamma f(x_n) - \bar{A}p\| \|x_{n+1} - p\| \\
\leq (1 - \alpha_n - \alpha_n \bar{A}\bar{p}) \|x_n - p\|^2 + \beta_n \|x_n - W_n y_n\|^2 + 2\alpha_n \|u + \gamma f(x_n) - \bar{A}p\| \|x_{n+1} - p\| \\
+ 2\alpha_n \|u + \gamma f(x_n) - \bar{A}p\| \|x_{n+1} - p\| \\
\leq (1 - \alpha_n - \alpha_n \bar{A}\bar{p}) \|x_n - p\|^2 + \beta_n \|x_n - W_n y_n\|^2 + 2\alpha_n \|u + \gamma f(x_n) - \bar{A}p\| \|x_{n+1} - p\|.
\]
(10)

Then we have
\[
(1 - \alpha_n (1 + \mu)\bar{p})^2 \|x_n - y_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n (1 + \mu)^2 \bar{p}^2 \|x_n - W_n y_n\|^2 \\
+ \beta_n^2 \|x_n - W_n y_n\|^2 + 2\alpha_n \|u + \gamma f(x_n) - \bar{A}p\| \|x_{n+1} - p\| \\
+ 2(1 - \alpha_n - \alpha_n \bar{A}\bar{p}) \|x_n - W_n y_n\|^2 + 2\alpha_n \|u + \gamma f(x_n) - \bar{A}p\| \|x_{n+1} - p\| \\
\leq \|x_n - p\|^2 + \alpha_n (1 + \mu)^2 \bar{p}^2 \|x_n - W_n y_n\|^2 + \beta_n^2 \|x_n - W_n y_n\|^2 \\
+ 2(1 - \alpha_n - \alpha_n \bar{A}\bar{p}) \|x_n - W_n y_n\|^2 + 2\alpha_n \|u + \gamma f(x_n) - \bar{A}p\| \|x_{n+1} - p\|.
\]\n
So, from (8)–(10), we have
\[
\lim_{n \to \infty} \|x_n - y_n\| = 0. \quad (11)
\]

Since
\[
\|W_n y_n - y_n\| \leq \|W_n y_n - x_n\| + \|x_n - y_n\|
\]
from which it follows that
\[
\|W_n y_n - y_n\| \to 0.
\]

Note that
\[
\|W y_n - y_n\| \leq \|W y_n - W_n y_n\| + \|W_n y_n - y_n\|,
\]
this together with Remark 3.1 give \(\lim_{n \to \infty} \|W y_n - y_n\| = 0\).

**Step 4.** Next, we show that
\[
\limsup_{n \to \infty} \|u + [\gamma f - (I + \mu A)] x^*, x_n - x^*\| \leq 0,
\]
where \(x^*\) is a solution of OP1.

To show this, we can choose a subsequence \(\{y_{n_j}\}\) of \(\{y_n\}\) such that
\[
\lim_{j \to \infty} \|u + [\gamma f - (I + \mu A)] x^*, y_{n_j} - x^*\| = \limsup_{n \to \infty} \|u + [\gamma f - (I + \mu A)] x^*, y_n - x^*\|.
\]
(12)

Since \(\{y_{n_j}\}\) is bounded, there exists a subsequence \(\{y_{n_{j_k}}\}\) of \(\{y_{n_j}\}\) which converges weakly to \(w\). Without loss of generality, we can assume that \(y_{n_j} \rightharpoonup w\). From \(\|W y_n - y_n\| \to 0\) we obtain \(W y_{n_j} \rightharpoonup w\). By the same argument as in the proof of [16, Theorem 4.1], we can obtain \(w \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega\). Therefore, from Lemma 2.4, (11) and (12), we have
\[
\limsup_{n \to \infty} (u + [\gamma f - (1 + \mu A)]x^*, x_n - x^*) = \limsup_{n \to \infty} (u + [\gamma f - (1 + \mu A)]x^*, y_n - x^*) \\
= \lim_{j \to \infty} (u + [\gamma f - (1 + \mu A)]x^*, y_{nj} - x^*) \\
= (u + [\gamma f - (1 + \mu A)]x^*, w - x^*) \\
\leq 0.
\]

(13)

**Step 5.** Finally, we prove that \(\{x_n\}\) and \(\{y_n\}\) converge strongly to \(x^*\).

From (1), we have

\[
\|x_{n+1} - x^*\|^2 = \|\alpha_n(u + \gamma f(x_n) - \bar{A}x^*) + \beta_n(x_n - x^*) + ((1 - \beta_n)I - \alpha_n\bar{A})(W_ny_n - x^*)\|^2 \\
\leq \|\beta_n(x_n - x^*) + ((1 - \beta_n)I - \alpha_n\bar{A})(W_ny_n - x^*)\|^2 + 2\alpha_n(u + \gamma f(x_n) - \bar{A}x^*, x_{n+1} - x^*) \\
\leq (1 - \beta_n - \alpha_n(1 + \mu)\bar{P})\|y_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + 2\alpha_n'y\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| \\
+ 2\alpha_n(u + \gamma f(x_n) - \bar{A}x^*, x_{n+1} - x^*) \\
\leq (1 - \beta_n(1 + \mu)\bar{P})\|x_n - x^*\|^2 + 2\alpha_n'y\alpha\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \\
+ 2\alpha_n(u + \gamma f(x_n) - \bar{A}x^*, x_{n+1} - x^*),
\]

that is,

\[
\|x_{n+1} - x^*\|^2 \leq 1 - 2\alpha_n(1 + \mu)\bar{P} + \frac{\alpha_n^2(1 + \mu)^2\bar{P}^2 + \alpha_n'y\alpha}{1 - \alpha_n'y\alpha}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \beta_n(1 + \mu)\bar{P}}(u + \gamma f(x_n) - \bar{A}x^*, x_{n+1} - x^*) \\
= \left[1 - \frac{2((1 + \mu)\bar{P} - \gamma\alpha'\alpha_n)}{1 - \alpha_n'y\alpha}\right]\|x_n - x^*\|^2 + \frac{2((1 + \mu)\bar{P} - \gamma\alpha'\alpha_n)}{1 - \alpha_n'y\alpha}\|x_n - x^*\|^2 \\
\times \left\{\frac{1}{2(1 + \mu)\bar{P} - \gamma\alpha'\alpha_n}M_1 + \frac{1}{(1 + \mu)\bar{P} - \gamma\alpha'\alpha_n}u + \gamma f(x_n) - \bar{A}x^*, x_{n+1} - x^*\right\} \\
= (1 - \delta_n)\|x_n - x^*\|^2 + \delta_n\alpha_n,
\]

where \(M_1 = \sup\{\|x_n - x^*\|^2\} : n \geq 1\), \(\delta_n = \frac{2((1 + \mu)\bar{P} - \gamma\alpha'\alpha_n)}{1 - \alpha_n'y\alpha}\) and \(\beta_n = \frac{\alpha_n(1 + \mu)^2\bar{P}^2M_1}{2((1 + \mu)\bar{P} - \gamma\alpha'\alpha_n)} + \frac{1}{(1 + \mu)\bar{P} - \gamma\alpha'\alpha_n}(u + \gamma f(x_n) - \bar{A}x^*, x_{n+1} - x^*)\).

It is easy to see that \(\delta_n \to 0\), \(\sum_{n=1}^{\infty} \delta_n = \infty\) and \(\limsup_{n \to \infty} \beta_n/\delta_n \leq 0\). Hence, by Lemma 2.3, the sequence \(\{x_n\}\) converges strongly to \(x^*\). Consequently, we can obtain that \(\{y_n\}\) also converges strongly to \(x^*\). This completes the proof. \(\square\)

**Corollary 3.1.** Let \(H\) be a real Hilbert space. Let \(\varphi : H \to R\) be a lower semicontinuous and convex functional. Let \(\Theta : H \times H \to R\) be an equilibrium bifunction satisfying conditions (H1)–(H3) such that \(\Sigma \not= \emptyset\). Let \(f\) be a contraction of \(H\) into itself with coefficient \(\alpha \in (0, 1)\), and let \(A\) be a strongly positive bounded linear operator on \(H\) with coefficient \(\beta > 0\) and \(0 < \gamma < (1 + \mu)\bar{P}/\alpha\). Suppose \(\{\alpha_n\}\) and \(\{\beta_n\}\) are two sequences in (1, 2). Assume that:

(i) \(K : H \to R\) is strongly convex with constant \(\sigma > 0\) and its derivative \(K'\) is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant \(\nu > 0\);

(ii) for each \(x \in H\), there exist a bounded subset \(D_x \subset H\) and \(z_x \in H\) such that, for any \(y \not\in D_x\),

\[
\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{\nu}(K'(y) - K'(x), z_x - y) < 0;
\]

(iii) \(\lim_{n \to \infty} \alpha_n = 0\), \(\sum_{n=0}^{\infty} \alpha_n = \infty\) and \(0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1\).

Given \(x_0 \in H\) arbitrarily, let the sequences \(\{x_n\}\) and \(\{y_n\}\) be generated iteratively by

\[
\begin{cases}
  y_n = S_{\gamma f}(x_n) \\
  x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_nx_n + ((1 - \beta_n)I - \alpha_n(1 + \mu A))y_n,
\end{cases}
\]

\(\forall n \geq 1\).
Then \( \{x_n\} \) and \( \{y_n\} \) converge strongly to \( x^* \in \Omega \) which solves OP2 below provided \( S_r \) is firmly nonexpansive,

\[
\text{OP2: } \min_{x \in \Omega} \frac{\mu}{2} \langle Ax, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x).
\]

**Proof.** Put \( T_i x = x \) for all \( i = 1, 2, \ldots, N \) and for all \( x \in H \). Then, the desired result follows from Theorem 3.1. \( \Box \)

**Corollary 3.2.** Let \( H \) be a real Hilbert space. Let \( T_1, T_2, \ldots \) be an infinite family of nonexpansive mappings on \( H \) such that \( \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Let \( f \) be a contraction of \( H \) into itself with coefficient \( \alpha \in (0, 1) \), and let \( A \) be a strongly positive bounded linear operator on \( H \) with coefficient \( \gamma > 0 \) and \( 0 < \gamma < (1 + \mu)\gamma/\alpha \). Suppose \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences in \( (0, 1) \). Assume that:

\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1.
\]

Given \( x_0 \in H \) arbitrarily, let the sequence \( \{x_n\} \) be generated iteratively by

\[
x_{n+1} = \alpha_n (u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n (I + \mu A))W_n x_n, \quad \forall n \geq 1.
\]

Then \( \{x_n\} \) converges strongly to \( x^* \in \bigcap_{i=1}^{\infty} F(T_i) \) which solves OP3,

\[
\text{OP3: } \min_{x \in \bigcap_{i=1}^{\infty} F(T_i)} \frac{\mu}{2} \langle Ax, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x).
\]

**Proof.** Put \( \Theta(x, y) = 0 \), \( \varphi(x) = 0 \) for all \( x, y \in H \) and \( r = 1 \). Take \( K(x) = \frac{\|x\|^2}{2} \) for all \( x, y \in H \). Then we get \( y_n = x_n \) in Theorem 3.1. Therefore the conclusion follows. \( \Box \)

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**References**


