A Geometric Theory of Ordinary First Order Variational Problems in Fibered Manifolds. II. Invariance

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1. Introduction

The purpose of this paper is to study the subject, 1-parameter groups of symmetry transformations of the first order variational problems in fibred manifolds.

We suppose that we are given a fibered manifold \( \pi: Y \to X \) with \( n \)-dimensional, orientable base space \( X \). The space of all \( r \)-jets of local sections of \( \pi \) will be denoted by \( J^r Y \). It can be regarded as a fibered manifold over \( J^s Y \), \( 0 \leq s < r \) \((J^0 Y = Y)\), and also over \( X \); we shall write \( \pi_{rs} \) and \( \pi_r \) for the corresponding natural projections. As usual \( J^r \) will denote the \( r \)-jet extension map. If \( \Omega \) is a subset of \( X \) we write \( \Gamma_{\Omega}(\pi) \) for the set of all local sections of \( \pi \) defined on a neighborhood of \( \Omega \) (not necessarily the same for all sections). Let \( R \) denote the field of real numbers.

Suppose that we are given a \( \pi_r \)-horizontal \( n \)-form \( \lambda \) (a first order Lagrangian form on \( \pi \)). To any compact, \( n \)-dimensional submanifold \( \Omega \) of \( X \) is then associated a function

\[
\Gamma_{\Omega}(\pi) \ni \gamma \to \lambda_{\Omega}(\gamma) = \int_{\Omega} j^1 \gamma^* \lambda \in R,
\]

defining a first order variational problem on \( \pi \). To the study of the critical points of this function (or, more precisely, of the collection of the functions \( \lambda_\Omega \) labeled by \( \Omega \)) we devoted our previous work [1], and now we wish to complete the considerations by a description of its invariance properties under local automorphisms of the fibered manifold \( \pi \). It is undoubtedly associated with the geometric nature of the matter that one does not need any additional structure in the space \( \Gamma_{\Omega}(\pi) \) of sections for this, and all the reasonings can be made in terms of local 1-parameter groups and their generators.

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The approach we follow is closely related to the work of Trautman [2, 3] and our previous papers [4, 5]. Trautman first applied modern geometric methods in the theory of the invariant variational problems and deserved well of deeper understanding of the geometry of these problems. In Section 2 we explain his classification of the symmetry transformations, with an intrinsic definition of the so called generalized invariant transformations studied by the author in [5]. Section 3 contains a well-known result—the necessary and sufficient conditions for a vector field to generate invariant transformations of \( \lambda_{\Omega} \), and is presented essentially for the sake of completeness (see [2-4]). Section 4 contains a description of the generalized invariant transformations (see [2-5]), and we discuss here the structure of the set of all generators of such transformations, in a way slightly different from Trautman's original description [2-3]. We show that this set has a natural structure of the Lie algebra. Finally, Section 5 is devoted to the study of a wide class of local automorphisms of \( \pi \), the symmetry transformations of a first order variational problem. We formulate a problem, very natural in our geometric terminology concerning basic notions of the calculus of variations, of finding of those critical points of a variational problem (or the functional \( \lambda_{\Omega} \)) for which a prescribed vector field generates a 1-parameter group of the symmetry transformations. This problem is easy to generalize to any set of vector fields, and it seems the problem of the symmetry transformations has not yet been posed in this manner. We obtain its solution in terms of a system of partial differential equations similar to the Euler-Lagrange equations (with a new Lagrangian), or, equivalently, in terms of the Lie derivative of the Euler form (see [1]). If we want to stress some other aspects of the problem in question we say that its solution describes all vector fields generating symmetry transformations of the critical point.

For background definitions of this paper we refer to [1]. We suppose throughout that our manifolds are real and finite-dimensional. All our considerations are in the category \( \mathcal{C}^\infty \).

2. Classes of Symmetry Transformations

Let \( \lambda \) be a first order Lagrangian form on \( \pi \), \( E(\lambda) \) the Euler form associated to \( \lambda \). Recall that if \( \rho \) is a differential form on a manifold \( M \) and \( \alpha \) an isomorphism of \( M \), then we say that \( \alpha \) leaves \( \rho \) invariant provided \( \alpha^\ast \rho = \rho \). In this sense isomorphisms leaving \( \lambda \) and \( E(\lambda) \) invariant can be considered. However, to apply this definition to the jet spaces and the jet prolongations of isomorphisms of a fibered manifold, it is reasonable to modify it a little.

**Definition 1.** Let \((\alpha, \omega_0)\) be a local automorphism of a fibered manifold \( \pi \), and denote by \( V \) the domain of \( \alpha \). Then:
1. \( \alpha \) is called an **invariant transformation** of \( \lambda \), if
\[
j^3 \alpha^* \lambda = \lambda \quad \text{on} \quad \pi_{10}^{-1}(V).
\]

2. \( \alpha \) is called a **generalized invariant transformation** of \( \lambda \), if
\[
j^5 \alpha^* E(\lambda) = E(\lambda) \quad \text{on} \quad \pi_{20}^{-1}(V).
\]

3. \( \alpha \) is called a **symmetry transformation** of the pair \((\lambda, \gamma)\), where \( \gamma \) is a section of \( \pi \) satisfying \( E(\lambda) \circ j^2 \gamma = 0 \), if
\[
E(\lambda) \circ j^3 \alpha \circ j^2 \gamma = 0 \quad \text{on} \quad \pi_{20}^{-1}(V) \cap j^2 \gamma(X).
\]

The classes of invariant, generalized invariant, and symmetry transformations are connected with each other. This is described by the next proposition.

**Proposition 1.** Let \( \lambda \) be a first order Lagrangian form on \( \pi \), \((\alpha, \alpha_0)\) a local automorphism of \( \pi \), and \( \gamma \) a section of \( \pi \) such that \( E(\lambda) \circ j^2 \gamma = 0 \). Then:

1. If \( \alpha \) is an invariant transformation of \( \lambda \) then it is a generalized invariant transformation of \( \lambda \).

2. If \( \alpha \) is a generalized invariant transformation of \( \lambda \) then it is a symmetry transformation of \((\lambda, \gamma)\), for any section \( \gamma \in \Gamma_x(\pi) \) satisfying \( E(\lambda) \circ j^2 \gamma = 0 \).

**Proof.** Both parts of the statement follow from [1, Proposition 12].

### 3. Invariant Transformations

Let \( \pi \) and \( \lambda \) be the same as before. Choose a volume element \( \omega \) on \( X \), and define a first order Lagrangian function \( \lambda \) on \( \pi \) by
\[
L_{\pi_1}^* \omega = \lambda.
\]

Let \( \Sigma \) be a \( \pi \)-projectable vector field on \( Y \), \( \xi \) its \( \pi \)-projection, and introduce a function on \( X \), \( \text{div} \xi \), by the formula
\[
\theta(\xi) \omega = \text{div} \xi \cdot \omega.
\]

Then we have the following simple characterization of the generators of invariant transformations.
Theorem 1. The following three conditions are equivalent:

1. $\mathcal{E}$ generates invariant transformations of $\lambda$.
2. The relation $\theta(j^{1}\mathcal{E}) \lambda = 0$ holds.
3. The relation $\langle dL, j^{1}\mathcal{E} \rangle + L \operatorname{div} \xi = 0$ holds.

Proof. The conditions 1 and 2 are obviously equivalent. Since the identity

$$\theta(j^{1}\mathcal{E}) \lambda = (\langle dL, j^{1}\mathcal{E} \rangle + L \cdot \operatorname{div} \xi) \cdot \pi^{*} \omega$$

holds, the second and the third conditions are also equivalent, which finishes the proof.

We note the the condition 3 for the generators of 1-parameter groups of invariant transformations is known as the Noether equation [3]. The set of all generators of invariant transformations of a Lagrangian form $\lambda$ with the natural structure of a real vector space and the Lie bracket, is a Lie algebra.

4. Generalized Invariant Transformations

With the notation of Section 3 we have Theorem 2.

Theorem 2. The following three conditions are equivalent:

1. $\mathcal{E}$ generates generalized invariant transformations of $\lambda$.
2. The relation $E(\theta(j^{1}\mathcal{E}) \lambda) = 0$ holds.
3. There exists a unique $n$-form $\rho$ on $Y$ such that

$$\theta(j^{1}\mathcal{E}) \lambda = \mathcal{H}(\rho) \quad \text{and} \quad d\rho = 0.$$

Proof. Using [1, Proposition 12] and Definition 1 we see that 2 follows from 1. Using [1, Theorem 3; 3] we obtain that 2 implies 3. Assume now that the third condition is satisfied. Then [1, Theorem 3; 3] gives

$$E(\theta(j^{1}\mathcal{E}) \lambda) = E(\mathcal{H}(\rho)) = 0$$

which leads, by [1, Proposition 12], to the equality $\theta(j^{1}\mathcal{E}) E(\lambda) = 0$. This completes the proof.

The set of all generators of the generalized invariant transformations of a first order Lagrangian form has the structure of a Lie algebra, with the usual bracket operation.

Proposition 2. If some $\pi$-projectable vector fields $\mathcal{E}_1$ and $\mathcal{E}_2$ on $Y$ generate generalized invariant transformations of a first order Lagrangian form $\lambda$ on $\pi$, 

then the Lie bracket \([\mathcal{E}_1, \mathcal{E}_2]\) also generates generalized invariant transformations of \(\lambda\).

**Proof.** Assume that \(\pi\)-projectable vector fields \(\mathcal{E}_1\) and \(\mathcal{E}_2\) generate generalized invariant transformations of \(\lambda\). Then, by Theorem 2, there are some forms, \(\rho_1\) and \(\rho_2\), defined on \(Y\), such that \(\theta(j^1\mathcal{E}_i)\lambda = \mathcal{A}(\rho_i)\), and \(d\rho_i = 0\), \(i = 1, 2\). Using [1, Proposition 2] and the formula

\[
\theta([j^1\mathcal{E}_1, j^1\mathcal{E}_2]) \lambda = \theta(j^1\mathcal{E}_2) \theta(j^1\mathcal{E}_1) \lambda - \theta(j^1\mathcal{E}_1) \theta(j^1\mathcal{E}_2) \lambda,
\]

we obtain

\[
\theta(j^1[\mathcal{E}_1, \mathcal{E}_2]) \lambda = \theta(j^1\mathcal{E}_2) \mathcal{A}(\rho_1) - \theta(j^1\mathcal{E}_1) \mathcal{A}(\rho_2).
\]

But [1, Proposition 3; 5] gives

\[
\theta(j^1[\mathcal{E}_1, \mathcal{E}_2]) \lambda = \mathcal{A}(\theta(\mathcal{E}_2) \rho_1 - \theta(\mathcal{E}_1) \rho_2).
\]

Evidently \(d(\theta(\mathcal{E}_2) \rho_1 - \theta(\mathcal{E}_1) \rho_2) = 0\) which proves our assertion.

Suppose that we are given a first order Lagrangian form \(\lambda\) on \(\pi\) and a generator of a group of generalized invariant transformations of \(\lambda\). Then the form \(\rho\) from Theorem 2 (uniquely determined by \(\lambda\) and \(\mathcal{E}\)) can explicitly be determined in terms of coordinate functions. This can be done by means of a process of differentiation similar to this one used in the proof of Theorem 3 [1], applied, however, to the form \(\theta(j^1\mathcal{E}) \lambda = \mathcal{A}(\rho)\). Conversely, one can pose the problem of finding of all generators of the generalized invariant transformations. According to Theorem 2, we must consider the condition

\[
\theta(j^1\mathcal{E}) \lambda = \mathcal{A}(\rho)
\]

as the equation for \(\mathcal{E}\), where the closed \(\pi\)-form \(\rho\) must be given. This equation can be locally further simplified. According to the Poincaré lemma, it suffices to consider the equations

\[
\theta(j^1\mathcal{E}) \lambda = \mathcal{A}(d\eta),
\]

where \(\eta\) is any \((\pi - 1)\)-form defined locally on \(Y\). Solutions of these equations, formulated for each such \(\eta\), present the desired vector fields, the generators of the generalized invariant transformations of \(\lambda\).

We note that the conditions 3 from Theorem 2 can be regarded as a generalization of the so called Noether–Bessel–Hagen equation (see [2, 3, 5]).

5. Symmetry Transformations

Let \(\pi: Y \rightarrow X\) be a fibered manifold with orientable base space \(X\), \(\lambda\) a first order Lagrangian form on \(\pi\), and \(\gamma\) a section of \(\pi\) satisfying the Euler–Lagrange equations, \(E(\lambda) \circ j^2\gamma = 0\). Then the following theorem holds.
Theorem 3. A \( \pi \)-projectable vector field \( \mathcal{E} \) on \( Y \) generates symmetry transformations of \( (\lambda, \gamma) \) if and only if

\[
E(\theta(j^1\mathcal{E}) \lambda) \circ j^2\gamma = 0.
\]

Proof. Let \( x \) be any point of \( X \), \( \gamma \) a section of \( \pi \) satisfying \( E(\lambda) \circ j^2\gamma = 0 \), \( \mathcal{E} \) a \( \pi \)-projectable vector field, and \( (\alpha_t, \alpha_0t) \) its 1-parameter group. Consider the point \( j_2\gamma \in T^2Y \) and choose a collection of tangent vectors to \( T^2Y \) at the point \( \mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n \). We have, by [1, Proposition 12]

\[
\langle E(j^2\alpha_t^*\lambda) (j_2\gamma), \mathcal{E}_0 \times \cdots \times \mathcal{E}_n \rangle
= \langle j^2\alpha_t^*E(\lambda) (j_2\gamma), \mathcal{E}_0 \times \cdots \times \mathcal{E}_n \rangle
= \langle E(\lambda) (j^2\alpha_t(j_2\gamma)), Tj^2\alpha_t \cdot \mathcal{E}_0 \times \cdots \times Tj^2\alpha_t \cdot \mathcal{E}_n \rangle.
\]

Since \( Tj^2\alpha_t \) is a linear isomorphism on each fiber of the tangent bundle space \( T^2Y \), we see that \( E(\lambda) (j^2\alpha_t(j_2\gamma)) = 0 \) if and only if \( E(\lambda) (j^2\alpha_t(j_2\gamma)) = 0 \). This shows that if \( \alpha_t \) is a symmetry transformation, i.e., \( E(\lambda) (j^2\alpha_t(j_2\gamma)) = 0 \), we obtain our assertion by taking the derivative of the curve

\[
t \to E(j^1\alpha_t^*\lambda) (j_2\gamma)
\]

at the point \( t = 0 \). Conversely, if for all \( x \in X \) the equality

\[
E(\theta(j^1\mathcal{E}) \lambda) (j_2\gamma) = 0
\]

holds then

\[
E(j^1\alpha_t^*\lambda) (j_2\gamma) = E(\lambda) (j_2\gamma) = 0,
\]

by our assumption. This finishes the proof.

Let us give a coordinate version of the theorem. Let \((x_1, y_0, z_{i_0}, z_{i_0})\) be some canonical coordinates on \( T^2Y \) (see [1]), \( 1 \leq i \leq k \leq n \), \( 1 \leq \sigma \leq m \), \( n = \dim X \), \( n + m = \dim Y \). Suppose that \( \lambda \) and \( \mathcal{E} \) have the coordinate expressions

\[
\lambda = \mathcal{L} \, dx_1 \wedge \cdots \wedge dx_n, \quad \mathcal{E} = \xi_k D_{1k} + \xi_\sigma D_{2\sigma}.
\]

By means of [1, Proposition 1] we obtain for the corresponding coordinate expression of the Lie derivative \( \theta(j^1\mathcal{E}) \lambda \),

\[
\theta(j^1\mathcal{E}) \lambda = \mathcal{L}_\mathcal{E} \, dx_1 \wedge \cdots \wedge dx_n,
\]

where

\[
\mathcal{L}_\mathcal{E} = D_{1k} \mathcal{L} \cdot \xi_k + D_{2\sigma} \mathcal{L} \cdot \xi_\sigma + D_{z_{i_0}} \mathcal{L} \cdot (dz_{i_0} - z_{i_0} \partial z_{i_0}) + \mathcal{L} \cdot D_{1k} \xi_k.
\]

This gives the following result.
Proposition 3. A \( \pi \)-projectable vector field \( \mathcal{E} \) represented as above generates symmetry transformations of \((\lambda, \gamma)\) if and only if

\[
D_{2\sigma} \mathcal{E} - d_k D_{3\sigma} \mathcal{E} = 0, \quad \text{for all } \sigma = 1, 2, \ldots, m.
\]

Proof. Our assertion follows from [1, Definition 7] and from the coordinate formulas for the Euler–Lagrange expressions (see [1, Section 5]).

Proposition 3 shows that the components of a vector field generating symmetry transformations of \((\lambda, \gamma)\) satisfy certain conditions of differential nature on the section \( \gamma(X) \subset Y \). There exists another interesting interpretation of the obtained result.

Let \( \mathcal{E} \) be any \( \pi \)-projectable vector field, and let us interest in the sections \( \gamma \) of \( \pi \) for which this vector field generates the symmetry transformations of \((\lambda, \gamma)\). Then the components of the vector field \( \mathcal{E} \) can be regarded as known, and it follows from Proposition 3 that the desired section \( \gamma \) must satisfy the system

\[
\begin{align*}
D_{2\sigma} \mathcal{E} - d_k D_{3\sigma} \mathcal{E} &= 0, \\
D_{2\sigma} \mathcal{E} - d_k D_{3\sigma} \mathcal{E} &= 0,
\end{align*}
\]

of partial differential equations. Notice that the second system is nothing but the Euler–Lagrange equations associated to the first order Lagrangian form \( \theta(j^1 \mathcal{E}) \lambda \). The converse of our assertion is evident: If \( \gamma \) satisfies the system then \( \mathcal{E} \), by definition, generates symmetry transformations of \((\lambda, \gamma)\).

Obviously, a similar problem can be stated, if we prescribe a set of \( \pi \)-projectable vector fields instead of the only vector field \( \mathcal{E} \). In other words this means that we are looking for more symmetric extremals. The analogical system of partial differential equations for \( \gamma \) will then be obtained.

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