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SIMPLICIAL PRESHEAVES

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Introduction

The central organizational theorem of simplicial homotopy theory asserts that the category **S** of simplicial sets has a closed model structure. This means that **S** comes equipped with three classes of morphisms, namely cofibrations (inclusions), fibrations (Kan fibrations) and weak equivalences (maps which induce homotopy equivalences of realizations), which together satisfy Quillen's closed model axioms CM1 to CM5. This theorem is well known and widely used (see [23] and [3]).

One could reasonably ask for such a theorem for the category SShv(**C**) of simplicial sheaves on a Grothendieck site **C**, based on the intuition that any theorem which is true for sets should be true for topoi. Immediately, however, a dilemma presents itself. On the one hand, cohomological considerations, like the Verdier hypercovering theorem, suggest a local theory of Kan fibrations. For example, if $\mathrm{op}|_T$ is the site of open subsets of a topological space T, then a map $f: X \to Y$ of simplicial sheaves should be a local fibration if and only if each map of stalks $f_x: X_x \to Y_x$ is a Kan fibration in the usual sense. On the other hand, monomorphisms surely should be cofibrations, giving a global theory.

The two approaches do, in fact, yield axiomatic homotopy theories for all categories of simplicial sheaves. The local theory for the category of simplicial sheaves on a topological space was constructed by Brown [4]; the corresponding global theory was developed slightly later by Brown and Gersten [5]. The local theory for arbitrary Grothendieck topoi appears in [17]. The global theory in the general setting is a result of Joyal [18]. The global theory of cofibrations is part of a closed model structure on SShv(**C**). The two theories are *distinct,* since it is not true that every local fibration is a global fibration. The Eilenberg-MacLane objects $K(F, n)$ certainly fail to be globally fibrant in general, essentially since sheaf cohomology is non-trivial.

A point that all authors (including myself) seemed to miss up to now is that, in

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the proofs of the results above, it is not so much the ambient topos that is creating the homotopy theory as it is the topology of the underlying site. These proofs may be generalized to produce local and global homotopy theories for simplicial *presheaves* on a Grothendieck site which depend on nothing but the axioms for the site. This paper presents these results. The local theory is given in the first section; the main result there is Theorem 1.13. The global theory appears in the second section, culminating in the proof of Theorem 2.3. The corresponding results for simplicial sheaves appear as corollaries of these theorems. The first two sections may seem lengthy, but the idea was that these results should be presented in a 'user-friendly' fashion.

In retrospect, it has been intuitively obvious for some time that there should be some sort of homotopy theory for simplicial presheaves, such that the canonical map from a simplicial presheaf to its associated simplicial sheaf is a weak equivalence. Evidence for this appears in [17], in fact, and it is achieved by both theories. It follows, in particular, that the associated homotopy categories are equivalent. Thus, the local and global theories, while proveably distinct, describe the same thing, rather like the way that hypercovers and injective resolutions describe sheaf cohomology.

One may recall that a proper description of sheaf cohomology requires both of these points of view. The same is true of the homotopy theory of simplicial presheaves or sheaves, as is evidenced in the third section of this paper. The basic idea there is to apply the results of the first part of the paper to get a new description of étale K-theory and the comparison map of the Lichtenbaum–Quillen conjecture. In particular, there is an isomorphism

$$K_{i-1}^{\mathrm{et}}(S;\mathbb{Z}/l)\cong[*,\Omega^iK/l^1]_S,\quad i\geq 0,$$

for decent schemes S and primes l not dividing the residue characteristics of S. The square brackets denote morphisms in the homotopy category associated to the category of simplicial presheaves on the étale site $\text{ét}|_S$ for S, and $*$ is the terminal object on $S\,\mathrm{Pre}(\text{ét}|_S)$. K/l^1 is notation for one of the mod l K-theory presheaves on $\text{ét}|_S$. This result is Theorem 3.9 of this paper; its proof uses the main result of [28].

We therefore obtain yet another description of étale K-theory, at least in degrees above -2. The point is that, with the results of the first two sections in hand, étale K-theory may be regarded as a generalized cohomology theory for simplicial presheaves on $\text{ét}|_S$. Insofar as the mod l K-theory presheaf is defined on any of the scheme-theoretic sites which are available for S, we are entitled, via this description, to objects like 'flat' K-theory of 'Zariski' K-theory. These invariants may all be related to topos-theoretic methods. Zariski K-theory is the object of study of [5].

Another corollary of Theorem 3.9 is that the comparison map relating mod l K-theory and étale K-theory now has a very simple description. In particular, the Lichtenbaum–Quillen conjecture reduces to a 'flabbiness' assertion for the simplicial presheaf K/l^1. The reason for the concentration on simplicial presheaves will become obvious to the reader at this point. The conjecture itself asserts that the homotopy groups of the simplicial set of global sections of K/l^1 are isomorphic to the groups $[*, \Omega^i K/l^1]$, at least in high degrees. It is therefore important to use a homotopy theory which sees simplicial presheaves rather than their associated simplicial sheaves; the canonical map from a simplicial presheaf to its associated simplicial sheaf might not induce a weak equivalence in global sections.

The third section also contains technical results which lead to the Godement and Brown–Gersten descent spectral sequences for morphisms in the homotopy category on $\mathbf{SPre}(\text{ét}|_S)$. Both are essentially straightforward applications of well-known tower of fibrations techniques, modulo the technical problem that the inverse limit functor on such towers might not preserve weak equivalences in the simplicial presheaf setting. In both cases, one gets around this problem by assuming a global bound on étale cohomological dimension. This assumption is becoming standard practice [28].

The paper closes with a result that asserts that generalized cohomology groups of a simplicial scheme X over a base scheme S may be computed either in the homotopy category for the big étale site on S, or in the homotopy category associated to the étale site which is fibred over X. This result generalizes the cohomological result of [17] which led, in part, to a streamlined proof of Suslin's theorem on the K-theory of algebraically closed fields [26,27]. It also implies that Thomason's topological equivariant K-theory [30] may be interpreted as generalized simplicial presheaf cohomology of a suitable balanced product.

1. Local theory

Throughout this paper, $\mathbf{C}$ will be a fixed small Grothendieck site. $\mathbf{SPre}(\mathbf{C})$ is the category of simplicial presheaves on $\mathbf{C}$; its objects are the contravariant functors from $\mathbf{C}$ to the category $\mathbf{S}$ of simplicial sets, and its morphisms are natural transformations. Recall that the topology on $\mathbf{C}$ is specified by families $J(U)$ of subfunctors $R \subset \mathbf{C}(-, U)$ of representable functors, one for each object U of $\mathbf{C}$, and that an element R of $J(U)$ is called a covering sieve. Each such R may be identified with a subcategory of the comma category $\mathbf{C}\downarrow U$, and so each simplicial presheaf X restricts to a functor on each covering sieve. I define

$$X(U)_R = \varprojlim_{\varphi: V \to U \in R} X(V)$$

and call this the simplicial set of *R-compatible* families in $X(U)$. There is a canonical map

$$\tau_R : X(U) \to X(U)_R$$

for each U and R. A necessary and sufficient condition for X to be a simplicial sheaf is that the map τ_R is an isomorphism for each U in $\mathbf{C}$ and each covering sieve $R \subset \mathbf{C}(-, U)$. $\mathbf{SShv}(\mathbf{C})$ is the full subcategory of $\mathbf{SPre}(\mathbf{C})$ whose objects are the simplicial sheaves. Recall that the inclusion $j: \mathbf{SShv}(\mathbf{C}) \subset \mathbf{SPre}(\mathbf{C})$ has a left adjoint $X \mapsto L^2X$, called the associated sheaf functor, where the functor $L: \mathbf{SPre}(\mathbf{C}) \to \mathbf{SPre}(\mathbf{C})$ is defined by

$$L(X)(U) = \varinjlim_{R \subset \mathbf{C}(-,U) \text{ covering}} X(U)_R .$$

The colimit defining $L(X)(U)$ is filtered, and so L, and hence L^2, preserves finite limits (which are formed pointwise). The convention is to write $\tilde{X} = L^2(X)$. There is a canonical map

$$\eta_X : X \to LX,$$

and LX is a separated presheaf in the sense that η_{LX} is a pointwise monic.

A map $p: X \to Y$ of simplicial presheaves is said to be a *local fibration* if for each commutative diagram of simplicial set maps

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, phantom, "\cap"] & X(U) \arrow[d, "p(U)"] \\ \Delta^n \arrow[r, "\beta"'] & Y(U) \end{tikzcd}$$

there is a covering sieve $R \subset \mathbf{C}(-,U)$ such that for each $\varphi: V \to U$ in R there is a commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, phantom, "\cap"] & X(U) \arrow[r, "\varphi^*"] & X(V) \arrow[d, "p(V)"] \\ \Delta^n \arrow[r, "\beta"'] \arrow[urr, "\theta_\varphi"'] & Y(U) \arrow[r, "\varphi^*"'] & Y(V) \end{tikzcd}$$

In other words, $p(U)$ satisfies the lifting property of a Kan fibration, up to refinement along some covering sieve. I refer to this as a *local right lifting property*, so that $p: X \to Y$ is a local fibration if and only if p has the local right lifting property with respect to all simplicial set inclusions of the form $\Lambda^n_k \subset \Delta^n$, $n > 0$. Of course, Δ^n is the standard n-simplex generated by the n-simplex ι_n, and Λ^n_k is the subcomplex of Δ^n which is generated by all faces of ι_n except $d_k \iota_n$. A simplicial presheaf X is said to be *locally fibrant* if the map $X \to *$ is a fibration, where $*$ is the terminal object of $\mathbf{SPre}(\mathbf{C})$. Explicitly, $*(U)$ is a copy of the standard 0-simplex Δ^0. Observe that $*$ is also a simplicial sheaf.

If $q: Z \to W$ is a simplicial presheaf map which is a *pointwise Kan fibration* in the sense that each map of sections $q: Z(U) \to W(U)$, $U \in \mathbf{C}$, is a Kan fibration, then q is a local fibration; in effect, no refinements are required. On the other hand, not every local fibration is a pointwise Kan fibration. Let $\text{ét}|_S$ be the étale site for a locally Noetherian scheme S. It is easy to show that a map $p: X \to Y$ of simplicial presheaves on $\text{ét}|_S$ is a local fibration if and only if each stalk map $p_x: X_x \to Y_x$ corresponding to each geometric point x of S is a Kan fibration. This is true for all Grothendieck sites with enough 'points' or stalks. In particular, the canonical map $\eta: X \to \tilde{X}$ is a local fibration for each simplicial presheaf X on $\text{ét}|_S$. (This generalizes to arbitrary sites.) But now let F be an abelian group, and let F also denote the corresponding constant presheaf of abelian groups on $\text{ét}|_S$. Then, for each étale map $U \to S$,

$$\tilde{F}(U \to S) = \prod_{\pi_0(U)} F,$$

where $\pi_0(U)$ is the set of connected components of U. The canonical map $\eta: F \to \tilde{F}$ is given at $U \to S$ by the diagonal homomorphism

$$\Delta: F \to \prod_{\pi_0(U)} F.$$

It follows that Δ induces the canonical map

$$\eta_{BF}: BF \to B\tilde{F}$$

at $U \to S$, and so $\eta_{BF}(U \to S)$ is not a Kan fibration if U is disconnected. The simplicial presheaf BF is constructed from F by pointwise application of the usual nerve functor.

Let $\mathbf{SPre(C)}_f \subset \mathbf{SPre(C)}$ be the full subcategory of locally fibrant presheaves. The goal of this section is to show that $\mathbf{SPre(C)}_f$ satisfies the axioms [4] and [17] for a category of fibrant objects for a homotopy theory. This means that two classes of maps in $\mathbf{SPre(C)}_f$ are specified, namely fibrations and weak equivalences, which satisfy a list of axioms. This list will be written down later. The fibrations for this theory are the local fibrations, as defined above.

The weak equivalences are harder to define, since the definition is combinatorial and local. We must first arrange for a calculus of local fibrations, in the style of [14]. I say that a class $\mathscr{A}$ of simplicial set monomorphisms is *locally saturated* if it satisfies the following axioms:

(1) All isomorphisms belong to $\mathscr{A}$.
(2) $\mathscr{A}$ is closed under cobase change with respect to arbitrary maps.
(3) $\mathscr{A}$ is closed under retracts.
(4) $\mathscr{A}$ is closed under finite composition and finite direct sum.

Lemma 1.1. *The class $\mathscr{A}_p$ of simplicial set monomorphisms which has the local left lifting property with respect to a fixed simplicial presheaf map $p: X \to Y$ is locally saturated.*

Proof. $\mathcal{A}_p$ is the collection of all simplicial set inclusions $i: K \to L$ such that, for each diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "i"'] & X(U) \arrow[d, "p"] \\ L \arrow[r, "\beta"'] & Y(U) \end{tikzcd}$$

there is a covering sieve $R \subset \mathbf{C}(-, U)$ where, for each $\varphi: V \to U$ in R, there is a diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "i"'] & X(U) \arrow[r, "\varphi^*"] & X(V) \arrow[d, "p"] \\ L \arrow[r, "\beta"'] \arrow[urr, "\theta_\varphi"] & Y(U) \arrow[r, "\varphi^*"'] & Y(V) \end{tikzcd}$$

Observe that the liftings θ_φ are not required to be coherent in any way. The axioms (1), (2), and (3) trivial. To verify (4) (and to verify a lot of other things) we use a standard *refinement principle* for covering sieves. Suppose that $R \subset \mathbf{C}(-, U)$ is covering, and suppose that $S_\varphi \subset \mathbf{C}(-, V)$ is a choice of covering sieve for each $\varphi: V \to U$ in R. Let $R \circ S_*$ be the collection of all morphisms $W \to U$ of $\mathbf{C}$ having a factorization

$$\begin{tikzcd} W \arrow[rr] \arrow[dr, "\psi"'] & & U \\ & V \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where $\varphi \in R$ and $\psi \in S_\varphi$. Then the refinement principle, which is easily proved, asserts that $R \circ S_\varphi$ is covering.

Now suppose that there is a diagram

$$\begin{tikzcd} K_1 \arrow[r, "\alpha"] \arrow[d, "i_1"'] & X(U) \arrow[dd, "p"] \\ K_2 \arrow[d, "i_2"'] & \\ K_3 \arrow[r, "\beta"'] & Y(U) \end{tikzcd}$$

where i_1 and i_2 are in $\mathscr{A}_p$. Then there is a covering sieve $R \subset \mathbf{C}(-, U)$ such that, for each $\varphi: V \to U$ in R, there is a diagram

$$\begin{tikzcd}
K_1 \arrow[r] \arrow[d, "i_1"'] & X(U) \arrow[r, "\varphi^*"] & X(V) \arrow[dd, "p"] \\
K_2 \arrow[urr, "\theta_\varphi"'] \arrow[d, "i_2"'] & & \\
K_3 \arrow[r] & Y(U) \arrow[r, "\varphi^*"'] & Y(V)
\end{tikzcd}$$

There is a covering sieve $S_\varphi \subset \mathbf{C}(-, V)$ such that, for each $\varphi: W \to V$ in S_φ, there is a diagram

$$\begin{tikzcd}
K_1 \arrow[r, "\alpha"] \arrow[d, "i_1"'] & X(U) \arrow[r, "\varphi^*"] & X(V) \arrow[r, "\psi^*"] & X(W) \arrow[dd, "p"] \\
K_2 \arrow[urr, "\theta_\varphi"] \arrow[d, "i_2"'] & & & \\
K_3 \arrow[uurrr, "\theta_{\varphi,\psi}"'] \arrow[r, "\beta"'] & Y(U) \arrow[r, "\varphi^*"'] & Y(V) \arrow[r, "\psi^*"'] & Y(W)
\end{tikzcd}$$

This gives a choice of lifting $\theta_{\varphi,\psi}$ for each factorization $\gamma = \varphi \circ \psi$ of each $\gamma \in R \circ S_*$. Picking one lifting $\theta_{\varphi,\psi}$ for each γ shows that $i_2 \circ i_1$ lifts locally along $R \circ S_*$, and so $\mathscr{A}_p$ is closed under finite composition. $\mathscr{A}_p$ is closed under finite direct sums, since the covering sieves in $\mathbf{C}(-, U)$ are closed under finite intersection. $\square$

The members of the smallest locally saturated class of monomorphisms which contains the inclusions $\Lambda^n_k \subset \Delta^n$, $n > 0$, are called *strong anodyne extensions*. Standard nonsense [14, p. 61], together with Lemma 1.1, implies that all inclusions

$$(\Delta^1 \times T) \cup (\{e\} \times S) \subset \Delta^1 \times S, \quad e = 0, 1, \tag{1.2}$$

which are induced by inclusions $T \subset S$ of *finite* simplicial sets, are strong anodyne extensions. One shows, following [14] again, that if $T \subset S$ are finite, then the set of inclusions $K \subset L$ of simplicial sets such that the induced map

$$(L \times T) \cup (K \times S) \subset L \times S$$

is strong anodyne, is a locally saturated class which contains all maps of the form (1.2), and hence all inclusions $\Lambda^n_k \subset \Delta^n$, $n > 0$, giving

Corollary 1.3. *Suppose that $K \subset L$ is a strong anodyne extension and $T \subset S$ is an inclusion of finite simplicial sets. Then the induced inclusion $(L \times T) \cup (K \times S)$ is a strong anodyne extension.*

There is also the more obvious

Corollary 1.4. *Every local fibration has the local right lifting property with respect to all strong anodyne extensions.*

Now let X be a simplicial presheaf, and let K be a simplicial set. The simplicial presheaf X^K is defined at $U \in \mathbf{C}$ by

$$X^K(U) = \mathbf{hom}(K, X(U)),$$

where $\mathbf{hom}(K, X(U))$ is the simplicial function complex.

Corollary 1.5. *Suppose that $K \subset L$ is an inclusion of finite simplicial sets, and that $p: X \to Y$ is a local fibration. Then the map*

$$X^L \xrightarrow{(i^*, p_*)} X^K \times_{Y^K} Y^L$$

is a local fibration.

Proof. Use adjointness and Corollary 1.3. □

Observe that X^K is a simplicial sheaf if X is a simplicial sheaf.

Let $\partial\Delta^n$ be the subcomplex of Δ^n which is generated by all faces $d_i \iota_n$ of the canonical simplex ι_n. The smallest locally saturated class which contains all inclusions of the form $\partial\Delta^n \subset \Delta^n$, $n \geq 0$, also contains all inclusions $K \subset L$ of finite simplicial sets. It follows from Lemma 1.1 that all simplicial presheaf maps which have the local right lifting property with respect to all inclusions of the form $\partial\Delta^n \subset \Delta^n$, $n \geq 0$, are local fibrations. One of the better examples of such a map is given by

Lemma 1.6. *Let X be a simplicial presheaf. Then the canonical map $\eta_X: X \to \tilde{X}$ has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

Proof. This was proved in [17]. An alternative proof is given by observing that any diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, phantom, "\cap"] & X(U) \arrow[d] \\ \Delta^n \arrow[r] & LX(U) \end{tikzcd}$$

factors through a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, phantom, "\cap"] & X(U) \arrow[d, "\tau_R"] \\ \Delta^n \arrow[r] & X(U)_R \end{tikzcd}$$

for some covering sieve R. But then, for each $\varphi: V \to U$ in R, there is a commutative diagram

$$\begin{tikzcd} X(U) \arrow[r, "\varphi^*"] \arrow[d] & X(V) \arrow[d, "\tau"', "\cong"] \\ X(U)_R \arrow[r] \arrow[d] & X(V)_{\varphi^* R} \arrow[d] \\ LX(U) \arrow[r] & LX(V) \end{tikzcd}$$

where $\varphi^* R$ is the set of all morphisms $\psi: W \to V$ of $\mathbf{C}$ such that $\varphi \circ \psi$ is in R. $\varphi^* R = \mathbf{C}(-, V)$, and so the indicated map τ is an isomorphism. □

Corollary 1.7. *If $p: X \to Y$ is a local fibration, then so is $Lp: Lx \to LY$.*

Proof. Consider the diagram

$$\begin{tikzcd} X(U) \arrow[r, "\eta_X"] \arrow[d, "p"] & LX(U) \arrow[d, "L(p)"] & \Lambda^n_k \arrow[l, "\alpha"'] \arrow[d, phantom, "\cap"] \\ Y(U) \arrow[r, "\eta_Y"'] & LY(U) & \Delta^n \arrow[l, "\beta"] \end{tikzcd}$$

Lemma 1.6 and the remark preceding it imply that α lifts locally to X. Local fibrations are closed under composition, by an argument dual to that given for Lemma 1.1, and so $\eta \circ p$ is a local fibration. Thus, by refining further, one finds liftings of β to LX. □

Corollary 1.8. *If $p: X \to Y$ is a local fibration, then so is $\tilde{p}: \tilde{X} \to \tilde{Y}$. In particular, if X is a presheaf of Kan complexes, then $\tilde{X}$ is locally fibrant.*

Let K be a finite simplicial set, and let X be a locally fibrant simplicial presheaf on $\mathbf{C}$. Let

$$K \underset{g}{\overset{f}{\rightrightarrows}} X(U)$$

be a pair of simplicial set maps. f is said to be *locally homotopic* to g (write $f \simeq_{\mathrm{loc}} g$) if there is a covering sieve $R \subset \mathbf{C}(-, U)$ such that, for each $\varphi\colon V \to U$ in R, there is a diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "d^0"'] & X(U) \arrow[d, "\varphi^*"] \\ K \times \Delta^1 \arrow[r, "h_\varphi"] & X(V) \\ K \arrow[u, "d^1"] \arrow[r, "g"] & X(U) \arrow[u, "\varphi^*"'] \end{tikzcd}$$

One says that f is locally homotopic to g (rel L), where $L \subset K$, if $f|_L = g|_L$ and each homotopy h_φ is constant on L.

Lemma 1.9. *Local homotopy (rel any subcomplex) of maps $K \to X(U)$ is an equivalence relation if K is finite and X is locally fibrant.*

Proof. It suffices to show that local homotopy of vertex maps $\Delta^0 \to Y(U)$ is an equivalence relation if Y is locally fibrant. To see this, observe that each comma category $\mathbf{C}{\downarrow}U$ inherits a canonical topology from the site $\mathbf{C}$, and that if $X|_U$ is the composite functor

$$(\mathbf{C}{\downarrow}U)^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}} \xrightarrow{X} \mathbf{S}, \quad V \to U \mapsto V,$$

then $X|_U$ is locally fibrant if X is. Now suppose that there is a diagram

$$\begin{tikzcd} & K \arrow[dr, "f"] & \\ L \arrow[ur] \arrow[rr, "\sigma"] \arrow[dr] & & X(U) \\ & K \arrow[ur, "g"'] & \end{tikzcd}$$

and form the pullback diagram

$$\begin{tikzcd}
(X|_U)K/L \arrow[r] \arrow[d] & (X|_U)^K \arrow[d, "i^*"] \\
* \arrow[r, "\sigma"'] & (X|_U)^L
\end{tikzcd}$$

of simplicial presheaves on $\mathbf{C}\downarrow U$. Then f and g determine vertices of the locally fibrant simplicial presheaf $(X|_U)K/L$, and $f \simeq_{\text{loc}} g$ (rel L) if and only if the corresponding vertices are locally homotopic.

Suppose that there is a covering sieve $R \subset \mathbf{C}(-, U)$ such that, for each $\varphi: V \to U$ in R, there is a diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, "x"] \arrow[d, "d^0"'] & X(U) \arrow[d, "\varphi^*"] \\
\Delta^1 \arrow[r, "w_\varphi"] & X(V) \\
\Delta^0 \arrow[r, "y"] \arrow[u, "d^1"] & X(U) \arrow[u, "\varphi^*"']
\end{tikzcd}$$

so that $x \simeq_{\text{loc}} y$ as vertices of X. Then there is a covering sieve $S_\varphi \subset \mathbf{C}(-, V)$ such that, for each $\gamma: W \to V$ in S_φ there is a diagram

$$\begin{tikzcd}
\Lambda^2_2 \arrow[r, "{(w_\varphi, s_0\varphi^*x, -)}"] \arrow[d, phantom, "\cap"] & X(V) \arrow[d, "\psi^*"] \\
\Delta^2 \arrow[r, "v_{\gamma,\varphi}"'] & X(W)
\end{tikzcd}$$

where $(w_\varphi, s_0\varphi^*x, -)$ is the unique map on Λ^2_2 which sends $d_0\iota_2$ to $w_\varphi(\iota_1)$ and $d_1\iota_2$ to $s_0\varphi^*x$ in $X(V)$. Then there is a diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, "y"] \arrow[d, "d^0"'] & X(U) \arrow[drr, "(\varphi\circ\gamma)^*"] & & \\
\Delta^1 \arrow[r, "d^2"] & \Delta^2 \arrow[rr, "v_{\gamma,\varphi}"] & & X(W) \\
\Delta^0 \arrow[r, "x"] \arrow[u, "d^1"] & X(U) \arrow[urr, "(\varphi\circ\gamma)^*"'] & &
\end{tikzcd}$$

for each composite $\varphi \circ \gamma$ with $\varphi \in R$ and $\gamma \in S_\varphi$, and so $y \simeq_{\mathrm{loc}} x$. The transitivity is similar. Reflexivity is trivial. $\square$

Let $\mathbf{C}_t$ be a small Grothendieck site with terminal object t (like 1_U in $\mathbf{C}\downarrow U$). Let X be a locally fibrant simplicial presheaf on $\mathbf{C}_t$ and take a vertex $x \in X(t)_0$. Let x_U be the image of x in $X(U)$ under the map $X(t) \to X(U)$ which is induced by $U \to t$ in $\mathbf{C}_t$. The set of local homotopy classes of maps $(\Delta^n, \partial\Delta^n) \to (X(U), x_U)$ has elements denoted by $[(\Delta^n, \partial\Delta^n), (X(U), x_U)]_{\mathrm{loc}}$. Letting U vary gives a presheaf

$$\pi_n^p(X,x)(U) := \{[(\Delta^n, \partial\Delta^n), (X(U), x_U)]_{\mathrm{loc}}\}.$$

It is easily seen that $\pi_n^p(X,x)$ is separated for all $n \geq 1$. I define $\pi_n(X,x)$ to be the associated sheaf of $\pi_n^p(X,x)$. $\pi_n(X,x)$ may be identified with $L\pi_n^p(X,x)$. Similar considerations apply to path components; $\pi_0^p X$ is the presheaf of local homotopy classes of vertices, and $\pi_0 X$ is its associated sheaf.

A combinatorial pairing

$$m^p : \pi_n^p(X,x) \times \pi_n^p(X,x) \to \pi_n(X,x)$$

may be defined as follows. Let f and g be maps $(\Delta^n, \partial\Delta^n) \to (X(U), x_U)$ which represent local homotopy classes. There is a covering sieve $R \subset \mathbf{C}(-, U)$ such that, for each $\varphi : V \to U$ in R, there is a diagram

$$\begin{tikzcd} \Lambda_n^{n+1} \arrow[r, "{(x_U, \ldots, x_U, f, -, g)}"] \arrow[d, hook] & X(U) \arrow[d, "\varphi^*"] \\ \Delta^{n+1} \arrow[r, "w_\varphi"'] & X(V) \end{tikzcd}$$

Then $\{[d_n w_\varphi]_{\mathrm{loc}}\}_{\varphi \in R}$ is an R-compatible family, and hence defines an element $[\{[d_n w_\varphi]_{\mathrm{loc}}\}_{\varphi \in R}]$ of $\pi_n(X,x)(U)$ which is independent of the choices that have been made. It follows that

$$([f]_{\mathrm{loc}}, [g]_{\mathrm{loc}}) \mapsto [\{[d_n w_\varphi]_{\mathrm{loc}}\}_{\varphi \in R}]$$

defines the U-section component of the presheaf map m^p. Applying the associated sheaf functor to m^p gives a pairing

$$m : \pi_n(X,x) \times \pi_n(X,x) \to \pi_n(X,x).$$

The constant map $x_U : (\Delta^n, \partial\Delta^n) \to (X(U), x_U)$ determines a distinguished element $e_U \in \pi_n(X,x)(U)$ in the obvious way, and $\varphi^*(e_U) = e_V$ for each $\varphi : V \to U$.

Proposition 1.10. *$\pi_n(X,x)$, as defined above, is a sheaf of groups for $n \geq 1$ which is abelian for $n \geq 2$.*

Proof. This will only be a sketch. The idea is to take the arguments of [21, p. 9] and make them local. One gets away with this because only finitely many choices of lifts (and hence refinements) are required at each stage.

Suppose that $\{[x_\varphi]_{\mathrm{loc}}\}_{\varphi\in R}$ is an R-compatible family in $\pi_n^p(X,x)(U)$ for some covering sieve R, and that $[z]_{\mathrm{loc}}$ is an element of $\pi_n^p(X,x)(U)$. Suppose further that, for each $\varphi\colon V\to U$ in R, there is a diagram

$$\begin{tikzcd} \Lambda_n^{n+1} \arrow[r, "{(x_V,\dots,x_V,x_\varphi,-,\varphi^*z)}"] \arrow[d, phantom, "\cap"] & X(V) \\ \Delta^n \arrow[ur, "w_\varphi"'] & \end{tikzcd}$$

Then the family $\{[d_n w_\varphi]\}_{\varphi\in R}$ is R-compatible and represents the product

$$[\{[x_\varphi]_{\mathrm{loc}}\}_{\varphi\in R}]\bullet[[z]_{\mathrm{loc}}]$$

in $\pi_n(X,x)(U)$.

Now let $u, v, w\colon(\Delta^n,\partial\Delta^n)\to(X(U),x_U)$ represent elements of $\pi_n^p(X,x)(U)$. By successive refinement, there is a covering sieve $R\subset \mathbf{C}_t(-,U)$ such that, for each $\varphi\colon V\to U$ in R, there are commutative diagrams

$$\begin{tikzcd} \Lambda_n^{n+1} \arrow[r, "{(x_V,\dots,x_V,\varphi^*u,-,\varphi^*v)}"] \arrow[d, phantom, "\cap"] & X(V) \\ \Delta^{n+1} \arrow[ur, "w_{n-1}^\varphi"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Lambda_n^{n+1} \arrow[r, "{(x_V,\dots,x_V,d_n w_{n-1}^\varphi,-,\varphi^*w)}"] \arrow[d, phantom, "\cap"] & X(V) \\ \Delta^{n+1} \arrow[ur, "w_{n+1}^\varphi"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Lambda_n^{n+1} \arrow[r, "{(x_V,\dots,x_V,\varphi^*v,-,\varphi^*w)}"] \arrow[d, phantom, "\cap"] & X(V) \\ \Delta^{n+1} \arrow[ur, "w_{n+2}^\varphi"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Lambda^{n+2}_n \arrow[r, "{(x_V,\ldots,x_V,w^{\varphi}_{n-1},-,w^{\varphi}_{n+1},w^{\varphi}_{n+2})}"] \arrow[d, phantom, "\cap"] & X(V) \\ \Delta^{n+2} \arrow[ur, "u^{\varphi}"'] & \end{tikzcd}$$

But then the diagram

$$\begin{tikzcd} \partial\Lambda^{n+1}_n \arrow[r, "{(x_V,\ldots,x_V,\varphi^*u,d_nw^{\varphi}_{n+1},d_nw^{\varphi}_{n+2})}"] \arrow[d, phantom, "\cap"] & X(V) \\ \Delta^{n+1} \arrow[ur, "d_nu^{\varphi}"'] & \end{tikzcd}$$

commutes for each $\varphi: V \to U$ in R, and so

$$\begin{aligned} &[[w]_{\mathrm{loc}}] \bullet ([[v]_{\mathrm{loc}}] \bullet [[u]_{\mathrm{loc}}]) \\ &\quad = [[w]_{\mathrm{loc}}] \bullet [\{[d_n w^{\varphi}_{n-1}]_{\mathrm{loc}}\}_{\varphi\in R}] \\ &\quad = [\{[d_n w^{\varphi}_{n+1}]_{\mathrm{loc}}\}_{\varphi\in R}] \\ &\quad = [\{[d_n w^{\varphi}_{n+2}]_{\mathrm{loc}}\}_{\varphi\in R}] \bullet [[u]_{\mathrm{loc}}] \\ &\quad = ([[w]_{\mathrm{loc}}] \bullet [[v]_{\mathrm{loc}}]) \bullet [[u]_{\mathrm{loc}}] \end{aligned}$$

in $\pi_n(X,x)(u)$. It follows that the multiplication map is associative. Similar arguments give the rest of the result. □

Each of the sites $\mathbf{C}\downarrow U$ of Lemma 1.8 has a terminal object, namely the identity map $1_U: U \to U$, and so $x \in X|_U(1_U)_0$ determines a sheaf of homotopy groups $\pi_n(X|_U, x)$ if X is locally fibrant. A map $f: X \to Y$ of locally fibrant simplicial presheaves is said to be a *combinatorial weak equivalence* if each of the induced maps

$$\begin{aligned} &f_*: \pi_0(X) \to \pi_0(Y), \\ &f_*: \pi_n(X|_U, x) \to \pi_n(Y|_U, fx), \quad U \in \mathbf{C}, \quad x \in X(U)_0 \end{aligned}$$

are isomorphisms of sheaves. Recall that, if $\mathbf{C}$ is a site such that the sheaf category $\mathrm{Shv}(\mathbf{C})$ has enough points, or stalks, then X is locally fibrant if and only if each of the stalks X_y is a Kan complex. In this case, a map $f: X \to Y$ is a combinatorial weak equivalence if and only if each of the induced stalk maps $f_y: X_y \to Y_y$ is a weak equivalence of Kan complexes in the simplicial set category $\mathbf{S}$.

Proposition 1.11. *Suppose given a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

of morphisms of locally fibrant simplicial presheaves. If any two of f, g, or h are combinatorial weak equivalences, then so is the third.

Proof. Suppose that g and h are combinatorial weak equivalences. Then $g_*: \pi_0(X) \to \pi_0(Y)$ is an isomorphism, so that, for each $y \in Y(U)_0$, $U \in \mathbf{C}$, there is a covering sieve R such that, for each $\varphi: V \to U$ in R, there is a diagram

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "(\varphi^* y, g(x_\varphi))"] \arrow[d, phantom, "\cap"] & Y(V) \\ \Delta^1 \arrow[ur, "w_\varphi"'] & \end{tikzcd}$$

where $x_\varphi \in X(V)_0$. Each such w_φ induces an isomorphism of sheaves

$$(w_\varphi)_*: \pi_n(Y|_V, \varphi^* y) \cong \pi_n(Y|_V, g(x_\varphi)).$$

More generally, if Y' is a locally fibrant simplicial presheaf on $\mathbf{C}_t$ and there is a diagram

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "(x, x')"] \arrow[d, phantom, "\cap"] & Y'(t) \\ \Delta^1 \arrow[ur, "w"'] & \end{tikzcd}$$

then there is an induced isomorphism

$$w_*: \pi_n(Y', x) \cong \pi_n(Y', x')$$

which is natural in the obvious sense. In effect, if U is an object of $\mathbf{C}_t$ and $\alpha: (\Delta^n, \partial\Delta^n) \to (Y'(U), x_U)$ represents an element of $\pi_n^p(Y', x)(U)$, then there is a covering sieve $S \subset \mathbf{C}_t(-, U)$ such that, for each $\varphi: V \to U$ in S there is a commutative diagram

$$\begin{tikzcd} (\Delta^n \times \Delta^0) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "(\alpha, w_U)"] \arrow[d, phantom, "\cap"] & Y'(U) \arrow[r, "\varphi^*"] & Y'(V) \\ \Delta^n \times \Delta^1 \arrow[urr, "w_\varphi"'] & & \end{tikzcd}$$

where w_U is the composite

$$\partial\Delta^n \times \Delta^1 \xrightarrow{\mathrm{pr}} \Delta^1 \xrightarrow{w} Y'(t) \longrightarrow Y'(U).$$

Then $\{[w_\varphi \circ d^1]_{\mathrm{loc}}\}_{\varphi\in R}$ is an S-compatible family in $\pi_n^p(Y',x)(U)$, and so determines an element $w_*[\alpha]_{\mathrm{loc}}$ of $\pi_n(Y',x')(U)$; $w_*[\alpha]_{\mathrm{loc}}$ is independent of the representative of $[\alpha]_{\mathrm{loc}}$. The induced presheaf map

$$w_*\colon \pi_n^p(Y',x) \to \pi_n(Y',x')$$

is monic and locally epimorphic. Finally, observe that, for each $\varphi\colon V\to U$ in the original covering sieve R, the site isomorphism $(\mathbf{C}{\downarrow}U){\downarrow}(\varphi\colon V\to U) \cong \mathbf{C}{\downarrow}V$ induces an isomorphism

$$\pi_n(Y|_U, y)(\varphi\colon V\to U) \cong \pi_n(Y|_V, \varphi^*y)(1_V)$$

which is natural in Y.

Putting all of the above together gives commutative diagrams

$$\begin{tikzcd}
\pi_n(Y|_U, y)(1_U) \arrow[r, "\varphi^*"] \arrow[d, "f_*"'] & \pi_n(Y|_U, y)(\varphi\colon V\to U) \arrow[d, "f_*"] \\
\pi_n(Z|_U, fy)(1_U) \arrow[r, "\varphi^*"] & \pi_n(Z|_U, fy)(\varphi\colon V\to U)
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_n(Y|_U, y)(\varphi\colon V\to U) \arrow[r, phantom, "\cong"] \arrow[d, "f_*"'] & \pi_n(Y|_V, \varphi^*y)(1_V) \arrow[d, "f_*"] \\
\pi_n(Z|_U, fy)(\varphi\colon V\to U) \arrow[r, phantom, "\cong"] & \pi_n(Z|_V, f\varphi^*y)(1_V)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \pi_n(X|_V, x_\varphi)(1_V) \arrow[dl, "g_*"'] \arrow[ddl, "h_*"] \\
\pi_n(Y|_V, \varphi^*y)(1_V) \arrow[r, "(w_\varphi)_*"] \arrow[d, "f_*"'] & \pi_n(Y|_V, gx_\varphi)(1_V) \arrow[d, "f_*"] & \\
\pi_n(Z|_V, f\varphi^*y)(1_V) \arrow[r, "(fw_\varphi)_*"] & \pi_n(Z|_V, fgx_\varphi)(1_V) &
\end{tikzcd}$$

for each $\varphi\colon V\to U$ in the covering sieve R. Thus, all of the maps f_* are isomorphisms, and so f is a combinatorial weak equivalence if g and h are. The other cases are trivial. $\square$

Local fibrations between locally fibrant simplicial presheaves are characterized by having the local right lifting property with respect to all simplicial set inclusions of the form $\Lambda_k^n \subset \Delta^n$, $n > 0$. The following result, which is the key to this theory, implies that a map of locally fibrant simplicial presheaves is a local fibration and a combinatorial weak equivalence if and only if it has the local right lifting property with respect to *all* inclusions of finite simplicial sets; such maps will be called *trivial local fibrations.*

Theorem 1.12. *A map $p: X \to Y$ between locally fibrant simplicial presheaves is a local fibration and a combinatorial weak equivalence if and only if it has the local right lifting property with respect to all inclusions of the form $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

Proof. A map $(\Delta^n, \partial\Delta^n) \to (X(U), x)$ represents the trivial element of $\pi_n(X|_U, x)(1_U)$ if and only if there is a covering sieve $R \subset \mathbf{C}(-, U)$ such that, for each $\varphi: V \to U$ in R, there is a diagram

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r, "{(\varphi^*\alpha, x_V, \dots, x_V)}"] \arrow[d, phantom, "\cap"] & X(V) \\ \Delta^{n+1} \arrow[ur, "w_\varphi"'] & \end{tikzcd}$$

It follows that a map $p: X \to Y$ which has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$, is a combinatorial weak equivalence. Such a map is clearly a local fibration, by the observation preceding the proof of Lemma 1.6.

For the converse, say that a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, phantom, "\cap"] & X(U) \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"'] & Y(U) \end{tikzcd}$$

has a *local lifting* if there is a covering sieve $R \subset \mathbf{C}(-, U)$ such that, for each $\varphi: V \to U$ in R, there is a commutative diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\varphi^* \circ \alpha"] \arrow[d, phantom, "\cap"] & X(V) \arrow[d, "p"] \\ \Delta^n \arrow[r, "\varphi^* \circ \beta"'] \arrow[ur] & Y(V) \end{tikzcd}$$

The idea is to show that, if $p: X \to Y$ is a combinatorial weak equivalence and a local fibration, then every diagram of the form D has a local lifting.

First of all, if D is locally homotopic to diagrams D_φ having local liftings, then D has a local lifting. In effect, there is a covering sieve $R \subset \mathbf{C}(-, U)$ such that, for each $\varphi: V \to U$ in R, there is a commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[dr, "d^0"'] \arrow[drr, "\varphi^*\circ\alpha"] \arrow[dd, hook] & & \\
& \partial\Delta^n\times\Delta^1 \arrow[r, "h_\varphi"] \arrow[dd, hook] & X(V) \arrow[dd, "p"] \\
\Delta^n \arrow[dr, "d^0"'] \arrow[drr, "\varphi^*\circ\beta"] & & \\
& \Delta^n\times\Delta^1 \arrow[r, "h'_\varphi"'] & Y(V)
\end{tikzcd}$$

Furthermore, one is assuming that, for each φ in R, there is a covering sieve $R_\varphi \subset \mathbf{C}(-, V)$ such that, for each $\psi: W \to V$ in R_φ, there is a diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\psi^*\circ h_\varphi\circ d^1"] \arrow[d, hook] & X(W) \arrow[d, "p"] \\
\Delta^n \arrow[r, "\psi^*\circ h'_\varphi\circ d^1"'] \arrow[ur, "\theta_{\psi,\varphi}"] & Y(W)
\end{tikzcd}$$

Then there is a covering sieve $R_{\psi,\varphi} \subset \mathbf{C}(-, W)$ such that, for each $\gamma: W' \to W$ in $R_{\psi,\varphi}$, there is a diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "d^0"] \arrow[d, hook] & (\partial\Delta^n\times\Delta^1)\cup(\Delta^n\times 0) \arrow[r, "{(\gamma^*\psi^*h_\varphi,\, \gamma^*\theta_{\psi,\varphi})}"] \arrow[d, hook] & X(W') \arrow[d, "p"] \\
\Delta^n \arrow[r, "d^0"'] & \Delta^n\times\Delta^1 \arrow[r, "\gamma^*\psi^*h'_\varphi"'] \arrow[ur, "w_{\gamma,\psi,\varphi}"] & Y(W')
\end{tikzcd}$$

Composing the refinements gives the claim.

Now consider the diagram

$$D \qquad \begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook] & X(U) \arrow[d, "p"] \\
\Delta^n \arrow[r, "\beta"'] & Y(U)
\end{tikzcd} \qquad n \geq 1.$$

D is locally homotopic to diagrams of the form

$$D_\varphi \qquad \begin{array}{ccc} \partial\Delta^n & \xrightarrow{(\alpha_\varphi, x_V, \dots, x_V)} & X(V) \\ \cap & & \downarrow p \\ \Delta^n & \xrightarrow[\beta_\varphi]{} & Y(V) \end{array} \qquad \varphi: V \to U,$$

where x is the image of the vertex 0 of $\partial\Delta^n$. In effect, the subcomplex Λ_0^n of $\partial\Delta^n$ contracts onto the vertex 0, and the homotopy extends locally to a homotopy of diagrams. But then α_φ represents the trivial element of $\pi_{n-1}(X|_V, x_V)(1_V) \cong \pi_{n-1}(Y|_V, px_V)(1_V)$ and so each D_φ is locally homotopic to diagrams of the form

$$D_{\psi,\varphi} \qquad \begin{array}{ccc} \partial\Delta^n & \xrightarrow{x_W} & X(W) \\ \cap & & \downarrow p \\ \Delta^n & \xrightarrow[\beta_{\psi,\varphi}]{} & Y(W) \end{array} \qquad \psi: W \to V.$$

Finally, $p_*: \pi_n(X|_W, x_W) \to \pi_n(Y|_W, p(x_W))$ is a sheaf epi, and so each $D_{\psi,\varphi}$ has a local lifting. Thus D has a local lifting. The sheaf isomorphism $p_*: \pi_0 X \xrightarrow{\cong} \pi_0 Y$ gives the required local liftings for every vertex of Y. $\square$

Recall [4], [17] that a *category of fibrant objects* (for a homotopy theory) is a category $\mathscr{C}$ with pullbacks and a terminal object $*$, equipped with two classes of maps, called fibrations and weak equivalences, such that the following axioms are satisfied:

(A) Given maps $f: X \to Y$ and $g: Y \to Z$ in $\mathscr{C}$, if any two of f, g, or $g \circ f$ are weak equivalences, then so is the third.

(B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.

(C) Fibrations and trivial fibrations (i.e., maps which are fibrations and weak equivalences) are closed under pullback.

(D) For any object X of $\mathscr{C}$, there is a commutative diagram

$$\begin{tikzcd} & X^I \arrow[d, "{(d_0, d_1)}"] \\ X \arrow[ur, "s"] \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

where Δ is the diagonal map, s is a weak equivalence, and (d_0, d_1) is a fibration.

(E) For each object X of $\mathscr{C}$, the map $X \to *$ is a fibration.

The point of what we have done so far in this section has been to prove:

Theorem 1.13. *The category* SPre(C)$_f$ *of locally fibrant simplicial presheaves on an arbitrary Grothendieck site* **C**, *together with the classes of combinatorial weak equivalences and local fibrations, satisfies the axioms for a category of fibrant objects for a homotopy theory.*

Proof. Axiom (A) is Proposition 1.11. The non-trivial part of Axiom (B) was observed in the proof of Corollary 1.7. Local fibrations and trivial local fibrations are defined by local lifting properties, by Theorem 1.12, and are therefore closed under base change, giving Axiom (C). There is a commutative diagram

$$\begin{tikzcd} & X^{\Delta^1} \arrow[d, "i^*"] \arrow[ddr, "{(d_0,d_1)}"] & \\ X\cong X^{\Delta^0} \arrow[ur, "s"] \arrow[r] \arrow[drr, "\Delta"'] & X^{\partial\Delta^1} \arrow[dr, "\cong"] & \\ & & X\times X \end{tikzcd}$$

for each locally fibrant simplicial presheaf X, which is induced by the diagram of finite simplicial sets

$$\begin{tikzcd} \Delta^1 \arrow[dr] & \\ & \Delta^0 \\ \partial\Delta^1 \arrow[uu, hook, "i"] \arrow[ur] & \end{tikzcd}$$

i^* is a local fibration by Corollary 1.5. On the other hand, $d_0: X^{\Delta^1} \to X$ may be identified with the map $(d^0)^*: X^{\Delta^1} \to X^{\Delta^0}$. This map $(d^0)^*$ has the local left lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$, by adjointness, and so d_0 is a trivial local fibration by Theorem 1.12. But then s is a weak equivalence by Proposition 1.11, and so Axiom (D) is verified. Axiom (E) is an assumption. $\square$

The object X^I of Axiom (D) is called a *path object* for X. It is important to note that the path object construction of the proof of Theorem 1.13 is functorial and classifies natural simplicial homotopy, by adjointness. More precisely, there is a diagram

$$\begin{tikzcd} & Y^{\Delta^1} \arrow[d, "{(d_0,d_1)}"] \\ X \arrow[ur, "h"] \arrow[r, "{(f,g)}"'] & Y\times Y \end{tikzcd}$$

if and only if there is a diagram

$$\begin{tikzcd} X \arrow[d, "d^0"'] \arrow[dr, "f"] & \\ X\times\Delta^1 \arrow[r, "h_*"] & Y \\ X \arrow[u, "d^1"] \arrow[ur, "g"'] & \end{tikzcd}$$

where $X\times\Delta^1$ is the presheaf which is defined by $X\times\Delta^1(U)=X(U)\times\Delta^1$. Let $\pi(X,Y)$ be the set $\hom(X,Y)$ in $\mathbf{SPre}(\mathbf{C})_{\mathrm{f}}$, collapsed by the smallest equivalence relation which is generated by the simplicial homotopy relation. The naturality of Y^{Δ^1} implies that $\pi(X,Y)$ is the set of morphisms from X to Y of a category $\pi\mathbf{SPre}(\mathbf{C})_{\mathrm{f}}$, whose objects are those of the original category $\mathbf{SPre}(\mathbf{C})_{\mathrm{f}}$. This category approximates the associated homotopy category $\mathrm{Ho}(\mathbf{SPre}(\mathbf{C})_{\mathrm{f}})$ in the sense that there is an isomorphism

$$[X,Y]\cong \varinjlim_{[\pi]:Z\to X\in \mathrm{Triv}\downarrow X} \pi(Z,Y),$$

where $\mathrm{Triv}\downarrow X$ is the full (filtered) subcategory of the comma category $\pi\mathbf{SPre}(\mathbf{C})_{\mathrm{f}}\downarrow X$ whose objects consist of maps which are represented by trivial fibrations, and $[X,Y]$ denotes morphisms from X to Y in $\mathrm{Ho}(\mathbf{SPre}(\mathbf{C})_{\mathrm{f}})$. The naturality of γ^{Δ^1} implies that the homotopy category may be approximated by $\pi\mathbf{SPre}(\mathbf{C})_{\mathrm{f}}$ via a calculus of fractions (see also [4, p. 425]). The corresponding point for simplicial sheaves is central to the cup product constructions of [17].

Theorem 1.13 implies the analogous result [17] for simplicial sheaves.

Corollary 1.14. *The category* $\mathbf{SShv}(\mathbf{C})_{\mathrm{f}}$ *of locally fibrant simplicial sheaves on an arbitrary Grothendieck site* $\mathbf{C}$*, together with the classes of local fibrations and combinatorial weak equivalences as defined above, satisfies the axioms for a category of fibrant objects for a homotopy theory. Moreover,* $\mathrm{Ho}(\mathbf{SShv}(\mathbf{C})_{\mathrm{f}})$ *is equivalent to* $\mathrm{Ho}(\mathbf{SPre}(\mathbf{C})_{\mathrm{f}})$.

Proof. All finite limits in $\mathbf{SShv}(\mathbf{C})$ are formed as they are in $\mathbf{SPre}(\mathbf{C})$, and so X^K is a simplicial sheaf if X is. This implies the path object Axiom (D) for simplicial sheaves. The rest of the axioms are trivial.

A map $p: X\to Y$ of simplicial presheaves is a trivial local fibration if and only if the maps

$$X^{\Delta^n} \xrightarrow{(i^*,p_*)} X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$$

are local epimorphisms in degree 0; this follows from Theorem 1.12. By adjointness and Corollaries 1.3 and 1.4, this is equivalent to (i^*, p_*) being a local epimorphism in all degrees. On the other hand, the canonical map

$$\eta_*: X^K \to \tilde{X}^K$$

induces an isomorphism of the sheaf associated to X^K with $\tilde{X}^K$ if K is finite, since the associated sheaf functor commutes with finite limits. It follows that the map

$$\tilde{X}^{\Delta^n} \xrightarrow{(\tilde{i}^*, \tilde{p}_*)} \tilde{X}^{\partial\Delta^n} \times_{\tilde{Y}^{\partial\Delta^n}} \tilde{Y}^{\Delta^n}$$

is a degree-wise local epi if (i^*, p_*) is, and so $\tilde{p}$ is a trivial local fibration if p is. We have already seen in Corollary 1.7 that the associated sheaf functor preserves local fibrations. Every map $g: X \to Y$ of $\mathbf{SPre}(\mathbf{C})_{\mathrm{f}}$ has a factorization $g = q \circ i$, where q is a local fibration and i is right inverse to a trivial local fibration; this is the factorization lemma of [4]. Therefore, if g is a combinatorial weak equivalence, then so are $\tilde{q}$ and $\tilde{i}$ and hence $\tilde{g}$. Thus, the associated sheaf functor preserves combinatorial weak equivalences, and so there are induced functors

$$\mathrm{Ho}(\mathbf{SPre}(\mathbf{C})_{\mathrm{f}}) \underset{\tilde{}_*}{\overset{j_*}{\rightleftarrows}} \mathrm{Ho}(\mathbf{SShv}(\mathbf{C})_{\mathrm{f}}).$$

Lemma 1.6 implies that this is an equivalence of categories. □

Any of the classical constructions of simplicial homotopy theory which involve only finitely many solutions of the Kan extension condition carry over to the locally fibrant simplicial presheaf setting. The long exact sequence of a fibration is an example; one constructs the boundary homomorphism locally by analogy with the construction of [20], giving

Lemma 1.15. *Suppose that $\mathbf{C}_t$ is a Grothendieck site with terminal object t. Suppose that $p: X \to Y$ is a local fibration of locally fibrant simplicial presheaves on $\mathbf{C}_t$ and that $x \in X(t)_0$ is a global choice of base point for X. Let F_x be defined by the Cartesian square*

$$\begin{tikzcd} F_x \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\ * \arrow[r, "px"'] & Y \end{tikzcd}$$

Then there is a sequence of pointed sheaves

$$\cdots \xrightarrow{\partial} \pi_1(F_x, x) \xrightarrow{i_*} \pi_1(X, x) \xrightarrow{p_*} \pi_1(Y, px)$$
$$\xrightarrow{\partial} \pi_0 F_x \xrightarrow{i_*} \pi_0 X \xrightarrow{p_*} \pi_0 Y$$

which is exact as a sequence of sheaves, and consists of group homomorphisms in the usual range.

For $\mathbf{C}_t$, X and x as in Lemma 1.15, observe that there are pullback squares

$$\begin{tikzcd} PX \arrow[r, "\mathrm{pr}"] \arrow[d] & X^{\Delta^1} \arrow[r, "d_1"] \arrow[d, "d_0"] & X \\ * \arrow[r, "x"] & X & \end{tikzcd}$$

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & PX \arrow[d, "d_1 \mathrm{pr}"] \\ * \arrow[r, "x"] & X \end{tikzcd}$$

which define the path space PX and the loop space ΩX respectively, relative to the choice of x. Then PX is trivially locally fibrant, and so there are isomorphisms of sheaves

(1.16)
$$\begin{aligned} &\pi_i(X,x) \cong \pi_{i-1}(\Omega X,x), \quad i \geq 2, \\ &\pi_1(X,x) \cong \pi_0(\Omega X) \end{aligned}$$

by Theorem 1.12 and Lemma 1.15. One can show that $\pi_i(X,x)$ is abelian for $i \geq 2$ by using this fact, for then $\pi_{i-1}(\Omega X,x)$ has two group multiplications which have a common identity and satisfy an interchange law.

Now let $\mathbf{C}$ be arbitrary and suppose that Y is a locally fibrant simplicial presheaf on $\mathbf{C}$. Kan's Ex^∞ functor [19] may be used to construct a presheaf of Kan complexes $\mathrm{Ex}^\infty Y$ and a canonical map $v: Y \to \mathrm{Ex}^\infty Y$.

Proposition 1.17. *The map $v: Y \to \mathrm{Ex}^\infty Y$ is a combinatorial weak equivalence if Y is a locally fibrant simplicial presheaf on an arbitrary Grothendieck site $\mathbf{C}$.*

Proof. Recall [19] that $v: Y \to \mathrm{Ex}^\infty Y$ is a filtered colimit of maps of the form $Y \to \mathrm{Ex}\, Y$, where Ex is right adjoint to the subdivision functor and $Y \to \mathrm{Ex}\, Y$ is induced pointwise by the last vertex maps $\mathrm{sd}\,\Delta^n \to \Delta^n$. The idea is to show that $\mathrm{Ex}\, Y$ is locally fibrant and that the map $Y \to \mathrm{Ex}\, Y$ is a combinatorial weak equivalence. The result then follows from the fact that, if

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots$$

is a filtered system in $\mathbf{SPre}(\mathbf{C})_f$ such that each f_i is a combinatorial weak equivalence, then $\varinjlim X_i$ is locally fibrant, and the canonical map

$$X_0 \to \varinjlim X_i$$

is a combinatorial weak equivalence.

The inclusion $\mathrm{sd}\,\Lambda_k^n \subset \mathrm{sd}\,\Delta^n$ is a strong anodyne extension; this is implicit in Kan's Lemma 3.4 [19], but a less combinatorial proof may be given which takes advantage of the fact that $\mathrm{sd}\,\Delta^n$ is a cone on $\mathrm{sd}\,\partial\Delta^n$. It follows that Ex preserves local fibre sequences, and that $\mathrm{Ex}\,Y$ is locally fibrant in particular.

It is trivial to show that, if X is locally fibrant, then $X \to \mathrm{Ex}\,X$ induces an isomorphism in π_0. Suppose that $x \in X(U)_0$ for an object U of $\mathbf{C}$. Ex commutes with restriction to $\mathbf{C}\downarrow U$, and so we may suppose that $\mathbf{C} = \mathbf{C}_t$, X and x are as in the statement of Lemma 1.15. Then there is a commutative diagram of sheaf homomorphisms

$$\begin{tikzcd}
\pi_1(X,x) \arrow[r] \arrow[d, "\cong"'] & \pi_1(\mathrm{Ex}\,X,x) \arrow[d, "\partial"] \\
\pi_0(\Omega X) \arrow[r, "\cong"] & \pi_0(\mathrm{Ex}\,\Omega X)
\end{tikzcd}$$

But $\mathrm{Ex}\,PX \to *$ is a trivial local fibration, by adjointness and Theorem 1.12, and so ∂ is an isomorphism. Iterating this procedure shows that all of the induced maps

$$\pi_i(X,x) \to \pi_i(\mathrm{Ex}\,X,x), \quad i \geq 1,$$

are isomorphisms. □

Observe that Proposition 1.17 is a triviality if $\mathrm{Shv}(\mathbf{C})$ has enough points, for then Ex^∞ commutes with all stalk constructions, and Kan's theorem that $X \to \mathrm{Ex}^\infty X$ is a weak equivalence for all Kan complexes X may be just quoted. This theorem had to be reproved in the context above. It will become important when various homotopy categories are compared in the next section. The sheaves of homotopy groups which are associated to presheaves of Kan complexes are also very easy to describe.

Proposition 1.18. *Let X be a presheaf of Kan complexes on a site $\mathbf{C}_t$ with terminal object t, and take $x \in X(t)_0$. Let $\pi_n^{\mathrm{simp}}(X,x)$ be the presheaf of simplicial homotopy groups of X, based at x. Then the sheaf associated to $\pi_n^{\mathrm{simp}}(X,x)$ is canonically isomorphic to $\pi_n(X,x)$.*

Proof. Consider the map

$$\eta : \pi_n^{\mathrm{simp}}(X,x) \to L\pi_n^{\mathrm{simp}}(X,x).$$

The simplices $\alpha, \beta : (\Delta^n, \partial\Delta^n) \to (X(U), x_U)$ represent the same element of

$L\pi_n^{\text{simp}}(X,x)(U)$ if and only if they are locally homotopic (rel $\partial\Delta^n$). Thus, there is a factorization of presheaf maps

$$\begin{tikzcd} \pi_n^{\text{simp}}(X,x) \arrow[rr, "\eta"] \arrow[dr] & & L\pi_n^{\text{simp}}(X,x) \\ & \pi_n^p(X,x) \arrow[ur, tail] & \end{tikzcd}$$

All of the maps in this diagram become isomorphisms when the associated sheaf functor is applied. $\square$

2. Global theory

Let $\mathbf{C}$ be an arbitrary small Grothendieck site. The global homotopy theory for the full category $\mathbf{SPre}(\mathbf{C})$ of simplicial presheaves is essentially a theory of cofibrations. These are easy to define; a *cofibration* is a map of simplicial presheaves which is a pointwise monomorphism. Associated to any simplicial presheaf X on $\mathbf{C}$ and $x\in X(U)_0$ is a sheaf $\pi_n^{\text{top}}(X|_U,x)$ on $\mathbf{C}\downarrow U$. $\pi_n^{\text{top}}(X|_U,x)$ is the sheaf associated to the presheaf which is defined by

$$\varphi: V\to U \mapsto \pi_n(|X(V)|,x_V),$$

where $|X(V)|$ is the realization of the simplicial set $X(V)$, and $\pi_n(|X(V)|,x_V)$ is the usual nth homotopy group of the space $|X(V)|$, based at $x_V=\varphi^*(x)$. $\pi_n^{\text{top}}(X|_U,x)$ is a sheaf of groups which is abelian if $n\geq 2$. The sheaf $\pi_0^{\text{top}}(X)$ of topological path components is defined similarly. A map $f:X\to Y$ of simplicial presheaves is said to be a *topological weak equivalence* if it induces isomorphisms of *sheaves*

$$f_*: \pi_n^{\text{top}}(X|_U,x) \xrightarrow{\cong} \pi_n^{\text{top}}(X|_U,fx), \quad U\in\mathbf{C},\ x\in X(U)_0,$$

$$f_*: \pi_0^{\text{top}}(X) \xrightarrow{\cong} \pi_0^{\text{top}}(Y).$$

There is a canonical isomorphism

$$\pi_n^{\text{top}}(X|_U,x)\cong \pi_n(S|X|\,|_U,x),$$

where S is the singular functor, in view of Proposition 1.18 and the usual adjointness tricks. It follows that $f:X\to Y$ is a topological weak equivalence if and only if the associated map $S|f|:S|X|\to S|Y|$ is a combinatorial weak equivalence. Thus, Proposition 1.11 implies

Lemma 2.1. *Given simplicial presheaf maps $f:X\to Y$ and $g:Y\to Z$, if any two of f, g, or $g\circ f$ are topological weak equivalences, then so is the third.*

A *trivial cofibration* $i:A\to B$ of simplicial presheaves is a map which is both a cofibration and a topological weak equivalence. We are now working towards a closed model structure on SPre(**C**) which is based on cofibrations and topological weak equivalences. We begin by proving

Proposition 2.2. *Trivial cofibrations are closed under pushout.*

Proof. Pointwise weak equivalences are topological weak equivalences, so it is enough to consider any pushout diagram of the form

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & C \arrow[d, "i'"] \\ B \arrow[r] & D \end{tikzcd}$$

with j a *cofibration,* and show that i' is a trivial cofibration if i is. i' is a trivial cofibration if and only if for every diagram of topological spaces of the form

$$E \qquad \begin{tikzcd} |\partial\Delta^n| \arrow[r, "\alpha"] \arrow[d, phantom, "\cap"] & |C(U)| \arrow[d, "|i'|"] \\ |\Delta^n| \arrow[r, "\beta"'] & |D(U)| \end{tikzcd}$$

there is a covering sieve $R \subset \mathbf{C}(-, U)$ such that, for each $\varphi: V \to U$ in R, there are diagrams

$$\begin{tikzcd} |\partial\Delta^n| \arrow[dr, "d^0"] \arrow[drr, "|\varphi^*|\alpha"] \arrow[ddd] & & \\ & |\partial\Delta^n| \times |\Delta^1| \arrow[r, "c_{|\varphi^*|\alpha}"'] \arrow[d] & |C(V)| \arrow[d, "|i'|"] \\ & |\Delta^n| \times |\Delta^1| \arrow[r, "h_\varphi"] & |D(V)| \\ |\Delta^n| \arrow[ur, "d^0"] \arrow[urr, "|\varphi^*|\beta"'] & & \end{tikzcd}$$

$$\begin{tikzcd} |\partial\Delta^n| \arrow[r, "|\varphi^*|\alpha"] \arrow[d] & |C(V)| \arrow[d, "|i'|"] \\ |\Delta^n| \arrow[r, "h_\varphi d^1"'] \arrow[ur, "\theta_\varphi"] & |D(V)| \end{tikzcd}$$

where $c_{|\varphi^*|\alpha}$ is the constant homotopy on $|\varphi^*|\alpha$. This is shown by applying Theorem 1.12, together with a local simplex choice argument, to the local fibration associated to the map $S|i'|: S|C| \to S|D|$ of presheaves of Kan complexes.

Now take a diagram of the form E. Then there is a homotopy

$$\begin{tikzcd}
{|\partial\Delta^n|} \arrow[dddd] \arrow[dr, "d^0"] \arrow[drr, "\alpha"] & & \\
& {|\partial\Delta^n| \times |\Delta^1|} \arrow[r, "c_\alpha"'] \arrow[dd] & {|C(U)|} \arrow[dd, "|i'|"] \\
& & \\
& {|\Delta^n| \times |\Delta^1|} \arrow[r, "h_D"] & {|D(U)|} \\
{|\Delta^n|} \arrow[ur, "d^0"] \arrow[urr, "\beta"'] & &
\end{tikzcd}$$

and a subdivision $|L| \cong |\Delta^n|$ (in the classical sense [25]) such that, in the diagram

$$\begin{tikzcd}
{|K| \cong |\partial\Delta^n|} \arrow[r, "\alpha"] \arrow[d, phantom, "\cap \quad \cap"] & {|C(U)|} \arrow[d, "|i'|"] \\
{|L| \cong |\Delta^n|} \arrow[r, "h_D d^1"'] & {|D(U)|}
\end{tikzcd}$$

the image of the realization $|\sigma|$ of each simplex σ of L is contained either in $|C(U)|$ or $|B(U)|$, where $|K|$ is the induced subdivision of $|\partial\Delta^n|$. It follows that the homotopy lifting property for E may be replaced by the corresponding problem for diagrams of the form

$$E' \qquad \begin{tikzcd}
{|K|} \arrow[r, "\alpha'"] \arrow[d] & {|C(U)|} \arrow[d, "|i'|"] \\
{|L|} \arrow[r, "\beta'"'] & {|D(U)|}
\end{tikzcd}$$

such that β' maps each $|\sigma|$ into either $|C(U)|$ or $|B(U)|$.

There is a sequence of subcomplexes

$$K = K_0 \subset K_1 \subset \cdots \subset K_n = L$$

of L, where K_{i+1} is obtained from K_i by adjoining a simplex. Suppose that, for the induced diagram

$$E_i \qquad \begin{tikzcd} {|K|} \arrow[r, "\alpha'"] \arrow[d] & {|C(U)|} \arrow[d, "|i'|"] \\ {|K_i|} \arrow[r, "\beta_i'"'] & {|D(U)|} \end{tikzcd}$$

there is a covering sieve $R \subset \mathbf{C}(-, U)$ such that, for each $\varphi: V \to U$ in R, there are diagrams

$$\begin{tikzcd} {|K|} \arrow[dr, "d^0"] \arrow[drr, "|\varphi^*|\,\alpha'"] \arrow[ddd] & & \\ & {|K| \times |\Delta^1|} \arrow[r] \arrow[d] & {|C(V)|} \arrow[d, "|i'|"] \\ & {|K_i| \times |\Delta^1|} \arrow[r, "h_\varphi"] & {|D(V)|} \\ {|K_i|} \arrow[ur, "d^0"] \arrow[urr, "|\varphi^*|\,\beta_i'"'] & & \end{tikzcd}$$

$$\begin{tikzcd} {|K|} \arrow[r, "|\varphi^*|\,\alpha'"] \arrow[d] & {|C(V)|} \arrow[d, "|i'|"] \\ {|K_i|} \arrow[r, "h_\varphi d^1"'] \arrow[ur, "\theta_\varphi"] & {|D(V)|} \end{tikzcd}$$

such that

(1) if $\beta_i'|\sigma| \subset |B(U)|$, then $h_\varphi(|\sigma| \times |\Delta^1|) \subset |B(V)|$,

(2) if $\beta_i'|\sigma| \subset |C(U)|$, then h_φ is constant on $|\sigma|$.

These conditions are compatible, since $|A(U)| = |B(U)| \cap |C(U)|$. Suppose that K_{i+1} is obtained from K_i by the pushout

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, phantom, "\cap"] & K_i \arrow[d] \\ \Delta^n \arrow[r, "\sigma"'] & K_{i+1} \end{tikzcd}$$

If $\beta_i'|\sigma| \subset |C(U)|$, then h_φ may be extended by a homotopy $h_\varphi': |K_{i+1}| \times |\Delta^1| \to |D(V)|$ which is constant on $|\sigma|$. If $\beta_i'|\sigma|$ is contained in $|B(U)|$ but not in $|C(U)|$, then h_φ may be extended to a homotopy $g_\varphi: |K_{i+1}| \times |\Delta^1| \to |D(V)|$ such that $g_\varphi(|\sigma| \times |\Delta^1|) \subset |B(V)|$. Thus, since $i: A \to B$ is a trivial cofibration, there is a covering sieve $S_\varphi \subset \mathbf{C}(-, V)$ such that, for each $\psi: W \to V$ in S_φ, there are commutative diagrams

$$
\begin{tikzcd}
{|K_i|} \arrow[dddd] \arrow[dr, "d^0"] \arrow[drr, "|\psi^*|\,\theta_\varphi"] & & \\
& {|K_i| \times |\Delta^1|} \arrow[r, "c_{|\psi^*|\,\theta_\varphi}"'] \arrow[d] & {|C(W)|} \arrow[d, "|i'|"] \\
& {|K_{i+1}| \times |\Delta^1|} \arrow[r, "h_\psi"] & {|D(W)|} \\
& & \\
{|K_{i+1}|} \arrow[uur, "d^0"] \arrow[uurr, "|\psi^*|\,g_\varphi d^1"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
{|K_i|} \arrow[r, "|\psi^*|\,\theta_\varphi"] \arrow[d] & {|C(W)|} \arrow[d, "|i'|"] \\
{|K_{i+1}|} \arrow[r, "h_\psi d^1"'] \arrow[ur, "\theta_{\varphi,\psi}"'] & {|D(W)|}
\end{tikzcd}
$$

Composing the homotopies h_ψ and $|\psi^*|\, g_\varphi$ along the covering sieve $R \circ S_\varphi$ solves the local lifting problem for the inclusion $K \to K_{i+1}$. $\square$

Lemma 2.1 and Proposition 2.2 already imply that the category SPre(**C**), together with the classes of cofibrations and topological weak equivalences as defined above, satisfies a list of axioms which are dual to the axioms (A)–(E) of the last section, making SPre(**C**) a category of cofibrant objects for a homotopy theory. But more is true. Say that a map $p: X \to Y$ is a *global fibration* if p has the right lifting property with respect to all trivial cofibrations. We shall prove

Theorem 2.3. SPre(**C**), *with the classes of cofibrations, topological weak equivalences and global fibrations as defined above, satisfies the axioms for a closed model category.*

Recall that a *closed model category* is a category $\mathscr{M}$, together with three classes of maps, called cofibrations, fibrations and weak equivalences, such that the following axioms hold:

CM1. $\mathscr{M}$ is closed under finite direct and inverse limits.

CM2. Given $f: X \to Y$ and $g: Y \to Z$ in $\mathscr{M}$, if any two of f, g or $g \circ f$ are weak equivalences, then so is the third.

CM3. If f is a retract of g in the category of arrows of $\mathscr{M}$, and g is a cofibration, fibration or weak equivalence, then so is f.

CM4. Given any solid arrow diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of $\mathcal{M}$, where i is a cofibration and p is a fibration, then the dotted arrow exists making the diagram commute if either i or p is a weak equivalence.

CM5. Any map f of $\mathcal{M}$ may be factored as

(1) $f = p \circ i$, where p is a fibration and i is a cofibration and a weak equivalence,

(2) $f = q \circ j$, where q is a fibration and a weak equivalence and j is a cofibration.

CM1 and CM3 are trivial for **SPre(C)**. The part of CM4 that is not the definition of global fibration is proved with a trick of Joyal [18]. In effect, given a diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r, "\beta"'] & Y \end{tikzcd}$$

where p is a trivial global fibration (i.e., p is a global fibration and a topological weak equivalence, as usual) and i is a cofibration, form the diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[ddd, "i"'] & X \arrow[rr, "1_X"] \arrow[dd, "i'"'] \arrow[dr, "ji'"] & & X \arrow[dd, "p"] \\ & & Z \arrow[ur, dashed] \arrow[dr, "q"] & \\ & V \cup_U X \arrow[ur, "j"] \arrow[rr, "\theta"] & & Y \\ V \arrow[ur] \arrow[urrr, "\beta"'] & & & \end{tikzcd}$$

where θ is the canonical map, q is a trivial fibration, and j is a cofibration. Then i' is a cofibration, so ji' is a trivial cofibration. Thus, the dotted arrow exists, making the diagram commute, and CM4 is proved (modulo CM5). The proof of Theorem 2.3 is therefore reduced to proving the factorization axiom CM5.

The site **C** is 'small', so that there is a cardinal number α such that α is larger than the cardinality of the set of subsets **PMor**(**C**) of the set of morphisms Mor(**C**) of **C**. A simplicial presheaf is said to be *α-bounded* if the cardinality of each $X_n(U)$, $U \in \mathbf{C}$, $n \geq 0$, is smaller than α. Observe that if X is α-bounded, then so is the associated sheaf $\tilde{X}$. The key point in the proof of CM5(1) and Theorem 2.3 is

Lemma 2.4. *A map $p: X \to Y$ is a global fibration if and only if it has the right lifting property with respect to all trivial cofibrations $i: U \to V$ such that V is α-bounded.*

Proof. First of all, let $j: A \to C$ be a trivial cofibration, and suppose that B is an α-bounded subobject of C. I claim that there is an α-bounded subobject B_ω of C such that $B \subset B_\omega \subset C$ and such that $B_\omega \cap A \to B_\omega$ is a trivial cofibration. In effect, given $\gamma \in \pi_i(B(U), B \cap A(U), x)$, there is a covering sieve $R \subset \mathbf{C}(-, U)$ such that $\varphi^* j_* \gamma$ is trivial in $\pi_i(C(V), A(V), x_V)$ for each $\varphi: V \to U$ in R. (The relative homotopy groups are topological; the realization notation $|-|$ has been dropped for notational convenience. In addition, i can be 0. For example, $\pi_0(B(U), B \cap A(U), x)$ is defined to be the quotient $\pi_0(B(U))/\pi_0(B \cap A(U))$.) C is a filtered colimit of its α-bounded subobjects and R is α-bounded, so that there is a subobject B_γ of C which contains B, such that the image γ' of γ in $\pi_i(B_\gamma(U), B_\gamma \cap A(U), x)$ vanishes in $\pi_i(B_\gamma(V), B_\gamma \cap A(V), x_V)$ for each $\varphi: V \to U$ in R. Let $B_1 = \bigcup B_\gamma$, where the union is taken over all $\gamma \in \pi_i(B(U), B \cap A(U), x)$, $U \in \mathbf{C}$, $x \in B \cap A(U)_0$, $i \geq 0$. Then B_1 is α-bounded. Iterate the procedure to produce α-bounded objects

$$B \subset B_1 \subset B_2 \subset \cdots,$$

and let

$$B_\omega = \bigcup_{i \geq 1} B_i.$$

Then B_ω is an α-bounded subobject of C, and any element of $\pi_i(B_\omega(U), B_\omega \cap A(U), y)$ vanishes along some covering sieve, so $B_\omega \cap A \to B_\omega$ is a trivial cofibration.

Now suppose that $p: X \to Y$ has the right lifting property with respect to all α-bounded trivial cofibrations, and consider the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ A \arrow[r] & Y \end{tikzcd}$$

where i is a trivial cofibration. Consider the set of partial lifts

$$\begin{tikzcd}
C \arrow[rr] \arrow[dr, "i'"] \arrow[ddr] & & X \arrow[dd, "p"] \\
& B \arrow[ur, "\theta"'] \arrow[d] & \\
& A \arrow[r] & Y
\end{tikzcd}$$

such that i' is a trivial cofibration and $B \neq C$. This set is inductively ordered. To see that it is non-empty, observe that A is a filtered colimit of its α-bounded sub-complexes, and so there is a pushout diagram

$$\begin{tikzcd}
B' \cap C \arrow[r] \arrow[d, "i''"'] & C \arrow[d, "i'"] \\
B' \arrow[r] & B
\end{tikzcd}$$

where B' is an α-bounded subobject of A which is not in C, i'' is a trivial cofibration by the above, and i' is a trivial cofibration by Proposition 2.2. But then the same argument implies that the maximal partial lifts have the form

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
A \arrow[r] & Y
\end{tikzcd}$$

These exist, by Zorn's Lemma. □

Lemma 2.5. *Every simplicial presheaf map $f: X \to Y$ may be factored*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& Z \arrow[ur, "p"'] &
\end{tikzcd}$$

where i is a trivial cofibration and p is a global fibration.

Proof. This proof is a transfinite small object argument. Choose a cardinal $\beta > 2^{\alpha}$, and define a functor $F: \beta \to \mathbf{SPre(C)}\downarrow Y$ on the partially ordered set β by setting

$$F(0) = f: X \to Y, \qquad X = X(0),$$

$$X(\zeta) = \varinjlim_{\gamma < \zeta} X(\gamma) \quad \text{for limit ordinals } \zeta,$$

and by requiring that the map $X(\gamma) \to X(\gamma+1)$ be defined by the pushout diagram

$$\begin{tikzcd} \coprod_D U_D \arrow[r, "\coprod_D i_D"] \arrow[d, "(\alpha_D)"'] & \coprod_D V_D \arrow[d] \\ X(\gamma) \arrow[r] & X(\gamma+1) \end{tikzcd}$$

such that the index set consists of all diagrams of the form

$$\begin{tikzcd} U_D \arrow[r] \arrow[d, "i_D"'] & X(\gamma) \arrow[d, "F(\gamma)"] \\ V_D \arrow[r] & Y \end{tikzcd}$$

where the i_D are chosen from a list of representatives of isomorphism classes of α-bounded trivial cofibrations. Let

$$X(\beta) = \varinjlim_{\gamma<\beta} X(\gamma),$$

and consider the induced factorization

$$\begin{tikzcd} X \arrow[rr, "i(\beta)"] \arrow[dr, "f"'] & & X(\beta) \arrow[dl, "F(\beta)"] \\ & Y & \end{tikzcd}$$

of f. Then $i(\beta)$ is a trivial cofibration, since it is a filtered colimit of such. Also, for any diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X(\beta) \arrow[d, "F(\beta)"] \\ V \arrow[r] & Y \end{tikzcd}$$

such that V is α-bounded and i is a trivial cofibration, the map $U \to X(\beta)$ must factor through some $X(\gamma) \to X(\beta)$, $\gamma < \beta$, for otherwise U has too many subobjects. The result follows. □

The proof of CM5(2) is relatively easy by comparison. First of all, observe that, for each object U of $\mathbf{C}$, the U-sections functor $X \mapsto X(U)$ has a left adjoint $?_U: \mathbf{S} \to \mathbf{SPre(C)}$ which sends the simplicial set Y to the simplicial presheaf Y_U, which is defined by

$$Y_U(V) = \coprod_{\varphi: V \to U} Y.$$

A map $q: Z \to X$ has the right lifting property with respect to all cofibrations if and only if it has the right lifting property with respect to all inclusions $S \subset \Delta_U^n$ of subobjects of the Δ_U^n, $U \in \mathbf{C}$, $n \geq 0$. One uses a Zorn's Lemma argument on an inductively ordered set of partial lifts to see this, as in the proof of Lemma 2.4. A transfinite small object argument, as in Lemma 2.5, shows that every map $f: Y \to X$ has a factorization

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, "f"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

where j is a cofibration and q has the right lifting property with respect to all cofibrations. In other words, q is an injective resolution of f in $\mathbf{SPre(C)}\downarrow X$ (see also [15]; this is really just the same argument). But then q is a weak equivalence as well as a global fibration. In effect, q has the right lifting property with respect to all inclusions $\partial\Delta_U^n \subset \Delta_U^n$, $U \in \mathbf{C}$, $n \geq 0$, so that each map of sections $q: Z(U) \to X(U)$ is a trivial fibration of simplicial sets. Thus, q is a pointwise, hence topological, weak equivalence, and CM5(2) is proved.

The proof of Theorem 2.3 is also complete. Its argument is roughly parallel to that given by Joyal for the corresponding result about simplicial sheaves. More explicitly, a cofibration (resp. topological weak equivalence) of simplicial sheaves is just a cofibration (resp. topological weak equivalence) in the simplicial presheaf category. A global fibration $p: X \to Y$ of simplicial sheaves is a map which has the right lifting property with respect to all trivial cofibrations of simplicial sheaves. This is equivalent to saying that p is a global fibration of simplicial presheaves by the following:

Lemma 2.6. *Suppose that Z is a simplicial presheaf on $\mathbf{C}$. Then the canonical map $Z \to \tilde{Z}$ is a topological weak equivalence.*

Proof. It suffices to show that the map $\eta: Z \to LZ$ is a topological weak equivalence. But

$$LZ(U) = \varinjlim_{R} Z(U)_R,$$

and every element of $\pi_n(|Z(U)_R|)$ lifts locally along R. On the other hand, if $\alpha \in \pi_n(|Z(U)|, x)$ vanishes in $\pi_n(|LZ(U)|, \eta x)$, then α vanishes in $\pi_n(|Z(U)_r|, x_R)$

for some covering sieve R, and so α vanishes locally in $\pi_n^{\mathrm{top}}(|Z(U)|, x)$ along R. It follows that each induced map of sheaves

$$\eta_* : \pi_n^{\mathrm{top}}(Z|_U, x) \to \pi_n^{\mathrm{top}}(LZ|_U, \eta x)$$

is an isomorphism. The argument for π_0^{top} is similar. □

Corollary 2.7 (Joyal). *With the definitions given above, the category* SShv(**C**) *of simplicial sheaves on an arbitrary Grothendieck site* **C** *is a closed model category.*

Proof. CM1, CM2 and CM3 are trivial again. CM4 is a consequence of the corresponding axiom for SPre(**C**) and Lemma 2.6. To prove CM5(1), construct a factorization

$$\begin{tikzcd} X \arrow[rr, "i_\beta"] \arrow[dr, "f"'] & & \tilde{X}_\beta \arrow[dl, "p_\beta"] \\ & Y & \end{tikzcd}$$

of a map f by defining a functor $P : \beta \to \mathrm{SShv}(\mathbf{C}) \downarrow Y$. The maps $P(\gamma) : \tilde{X}(\gamma) \to Y$ are defined such that

$$\tilde{X}(\zeta) = \varinjlim_{\gamma < \zeta} \tilde{X}(\gamma)$$

in the simplicial sheaf category if ζ is a limit ordinal, and such that the diagram

$$\begin{tikzcd} \tilde{X}(\gamma) \arrow[rr] \arrow[dr, "i_{\gamma+1}"] \arrow[ddr, "P(\gamma)"'] & & \tilde{X}(\gamma+1) \arrow[ddl, "P(\gamma+1)"] \\ & X(\gamma+1) \arrow[ur, "\eta"] \arrow[d, "p'(\gamma+1)"] & \\ & Y & \end{tikzcd}$$

commutes. Here, $i_{\gamma+1}$ is a trivial cofibration and $p'(\gamma+1)$ is a global fibration in SPre(**C**), and η is the canonical map from $X(\gamma+1)$ to its associated sheaf $\tilde{X}(\gamma+1)$. Taking

$$\tilde{X}_\beta = \varinjlim_{\gamma < \beta} \tilde{X}(\gamma)$$

gives the desired factorization.

The cofibration i'_γ in each diagram

$$\begin{tikzcd} \tilde{X}(\gamma) \arrow[rr, "i'_\gamma"] \arrow[dr, "P(\gamma)"'] & & \tilde{X}(\gamma+1) \arrow[dl, "P(\gamma+1)"] \\ & Y & \end{tikzcd}$$

is trivial by Lemma 2.6, and so i_β is a trivial cofibration. On the other hand, P_β has the right lifting property with respect to all α-bounded trivial cofibrations, by the choice of the cardinal β, and CM5(1) is proved. The proof of CM5(2) is similar; it uses the fact that a map $p: X \to Y$ of simplicial sheaves is a trivial global fibration if and only if it has the right lifting property with respect to all subobjects of all members of the generating family for $\mathbf{SPre(C)}\downarrow Y$. This is a detail in the proof of CM5(2) for $\mathbf{SPre(C)}$. □

The results introduced so far lead to specific constructions of homotopy categories for each of the categories $\mathbf{SPre(C)}_f$, $\mathbf{SShv(C)}_f$, $\mathbf{SPre(C)}$ and $\mathbf{SShv(C)}$. The constructions themselves will be used later. In the meantime, it is easy to see that the associated homotopy categories are all equivalent.

Proposition 2.8. *The inclusions in the diagram*

$$\begin{array}{ccc} \mathbf{SShv(C)}_f & \subset & \mathbf{SPre(C)}_f \\ \cap & & \cap \\ \mathbf{SShv(C)} & \subset & \mathbf{SPre(C)} \end{array}$$

induce equivalences of categories

$$\begin{array}{ccc} \mathrm{Ho}(\mathbf{SShv(C)}_f) & \longrightarrow & \mathrm{Ho}(\mathbf{SPre(C)}_f) \\ \downarrow & & \downarrow \\ \mathrm{Ho}(\mathbf{SShv(C)}) & \longrightarrow & \mathrm{Ho}(\mathbf{SPre(C)}) \end{array}$$

Proof. A map $f: X \to Y$ of presheaves of Kan complexes is a topological weak equivalence if and only if it is a combinatorial weak equivalence; this follows from Proposition 1.18 and the remarks preceding Lemma 2.1. Thus, Proposition 1.17 implies that a map of locally fibrant simplicial presheaves is a topological weak equivalence if and only if it is a combinatorial weak equivalence, and so the vertical functors of homotopy categories displayed above are defined.

The functor $\mathrm{Ho}(\mathbf{SShv(C)}_f) \to \mathrm{Ho}(\mathbf{SPre(C)}_f)$ was shown to be an equivalence of categories in Corollary 1.14. The canonical map $\eta: X \to \tilde{X}$ is a topological weak equivalence by Lemma 2.6. Thus, the associated sheaf functor preserves topological weak equivalences and hence induces a functor on the homotopy category level which is inverse to the functor $\mathrm{Ho}(\mathbf{SShv(C)}) \to \mathrm{Ho}(\mathbf{SPre(C)})$, up to natural isomorphism. Finally, the Ex^∞ functor induces an inverse up to natural isomorphism for the functor $\mathrm{Ho}(\mathbf{SPre(C)}_f) \to \mathrm{Ho}(\mathbf{SPre(C)})$, since the natural map $X \to \mathrm{Ex}^\infty X$ is a topological weak equivalence. □

I shall stop making the distinction between topological and combinatorial weak equivalences henceforth; they will simply be called *weak equivalences*.

Let X be a simplicial presheaf, and let $\mathbb{Z}X$ be the simplicial abelian presheaf which is obtained by applying the free abelian group functor pointwise. It follows from Proposition 1.18 that there are natural isomorphisms

$$L^2H_n(\mathbb{Z}X)\cong\pi_n(\mathbb{Z}X,0),\quad n>0,$$
$$L^2H_0(\mathbb{Z}X)\cong\pi_0(\mathbb{Z}X),$$

where $L^2H_n(\mathbb{Z}X)$ is the abelian sheaf associated to the homology presheaf $H_n(\mathbb{Z}X)$ of the associated presheaf of Moore chain complexes. An argument given in [17], together with Theorem 1.12, implies that the functor $X\mapsto\mathbb{Z}X$ takes trivial local fibrations to weak equivalences of chain complexes, and hence takes weak equivalences of locally fibrant objects to weak equivalences of chain complexes. This observation may be expanded in the following way:

Proposition 2.9. *Suppose that $f:X\to Y$ is an arbitrary weak equivalence of simplicial presheaves on* **C**. *Then the induced maps $f_*:L^2H_*(\mathbb{Z}X)\to L^2H_*(\mathbb{Z}Y)$ are isomorphisms of abelian sheaves.*

Proof. In the diagram

$$\begin{tikzcd} X \arrow[r, "v_X"] \arrow[d, "f"'] & \mathrm{Ex}^\infty X \arrow[d, "\mathrm{Ex}^\infty f"] \\ Y \arrow[r, "v_Y"'] & \mathrm{Ex}^\infty Y \end{tikzcd}$$

the maps v_X and v_Y are pointwise weak equivalences, and hence induce isomorphisms of homology presheaves. On the other hand, $\mathrm{Ex}^\infty f$ is a weak equivalence of locally fibrant simplicial presheaves, and therefore induces isomorphisms of homology sheaves. □

The proof of Proposition 2.9 is a triviality, but the relation of weak equivalences to homology in the general context where the ambient topos might not have enough points was a problem at one time (see [16, p. 26]; Illusie's conjecture is proved in [32]). This result is the mechanism by which sheaf cohomology is related to morphisms in the homotopy category [17]. For example, if F is a sheaf of abelian groups on $\text{ét}|_S$, then there is an isomorphism

$$[*,K(F,n)]\cong H^n_{\text{et}}(S;F),$$

where $H^n_{\text{et}}(S;F)$ is the nth étale cohomology group of the scheme S with coefficients in F, and $[*,K(F,n)]$ denotes morphisms from the terminal simplicial presheaf $*$ to the Eilenberg–MacLane presheaf $K(F,n)$ in the homotopy category

$\mathrm{Ho}(\mathbf{SPre}(\text{ét}\,|_S))$. This is one of the motivating examples for the description of non-abelian 'sheaf' cohomology which is given in the next section.

3. Flabby simplicial presheaves

The results of the previous two sections imply that there is a commutative diagram of functors

$$\begin{tikzcd}
 & \pi\mathbf{SPre}(\mathbf{C})_f \arrow[r] \arrow[dd, "i_*"] & \mathrm{Ho}(\mathbf{SPre}(\mathbf{C})_f) \arrow[dd, "i_*"] \\
\mathbf{SPre}(\mathbf{C})_f \arrow[ur] \arrow[urr] \arrow[dd, "i"'] & & \\
 & \pi\mathbf{SPre}(\mathbf{C}) \arrow[r, "\kappa"] & \mathrm{Ho}(\mathbf{SPre}(\mathbf{C})) \\
\mathbf{SPre}(\mathbf{C}) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

where $\pi\mathbf{SPre}(\mathbf{C})$ is the category of simplicial homotopy classes of maps in $\mathbf{SPre}(\mathbf{C})$. It contains $\pi\mathbf{SPre}(\mathbf{C})_f$ as a full subcategory. Moreover, the functor

$$i_*: \mathrm{Ho}(\mathbf{SPre}(\mathbf{C})_f) \to \mathrm{Ho}(\mathbf{SPre}(\mathbf{C}))$$

is an equivalence of categories.

It will be convenient to think of $\pi\mathbf{SPre}(\mathbf{C})$ as an approximation of $\mathrm{Ho}(\mathbf{SPre}(\mathbf{C}))$, in much the same way as $\pi\mathbf{SPre}(\mathbf{C})_f$ approximates $\mathrm{Ho}(\mathbf{SPre}(\mathbf{C})_f)$. One is led, in particular, to consider the map on morphism sets

$$\kappa_*: \pi(X, Y) \to [X, Y]$$

which is induced by the functor κ. Every object of $\mathbf{SPre}(\mathbf{C})$ is cofibrant by construction, so that standard closed model category results [23] imply that κ_* is a bijection if Y is globally fibrant.

I want to extend this idea a little bit. Suppose that Y is a presheaf of Kan complexes on $\mathbf{C}$ with a global choice of base point y. I say that Y is *flabby for X* if the maps

$$\kappa_*: \pi(X, \Omega^i Y) \to [X, \Omega^i Y], \quad i \geq 0,$$

are isomorphisms. If $\mathbf{C}$ has terminal object t, then $\pi(*, \Omega^i Y)$ is isomorphic to $\pi_i(|Y(t)|, y)$, so that, in this case, Y is flabby for $*$ if and only if κ_* induces isomorphisms

$$\pi_i(|Y(t)|, y) \cong [*, \Omega^i Y], \quad i \geq 0.$$

This notion of flabbiness extends standard cohomological ideas [22]. There are 'enough' flabby simplicial presheaves; this essentially follows from the fact that $\mathbf{SPre}(\mathbf{C})$ is a closed simplicial model category.

For simplicial presheaves U and Z, define a simplicial set $\mathbf{hom}(U, Z)$ by requiring that the n-simplices be simplicial presheaf maps of the form $U \times \Delta^n \to Z$, where Δ^n is identified with a constant simplicial presheaf. Suppose that $j: K \to L$ is an inclusion of simplicial sets and that $i: U \to V$ is a cofibration of simplicial presheaves. Then the induced map

$$(V \times K) \cup_{U \times L} (U \times L) \to V \times L$$

is a cofibration of $\mathbf{SPre(C)}$ which is trivial if either i or j is trivial, giving

Lemma 3.1. *Suppose that $p: Z \to W$ is a global fibration and that $i: U \to V$ is a cofibration of simplicial presheaves. Then the induced map*

$$\mathbf{hom}(V, Z) \xrightarrow{(i^*, p_*)} \mathbf{hom}(U, Z) \times_{\mathbf{hom}(U, W)} \mathbf{hom}(V, W)$$

is a fibration of simplicial sets, which is trivial if either i or p is trivial.

Corollary 3.2. *If Y is globally fibrant, then Y is flabby for all X.*

Proof. Any global fibration has the right lifting property with respect to all maps of the form $(\Lambda_k^n)_U \subset (\Delta^n)_U$, $n > 0$, $U \in \mathbf{C}$. In particular, Y is a presheaf of Kan complexes, so the statement of the corollary makes sense. But then the map $Y^{\Delta^n} \to Y^{\partial \Delta^n}$ is a fibration by Lemma 3.1, so the fibre $\Omega^n Y$ is globally fibrant as well. $\Box$

It also follows from Lemma 3.1 that $\mathbf{hom}(X, Y)$ is a Kan complex if Y is globally fibrant. If y is a global choice of base point for Y as above, then there is an isomorphism

$$\pi_n(\mathbf{hom}(X, Y), y) \cong \pi(X, \Omega^n Y),$$

so that the homotopy groups of $\mathbf{hom}(X, Y)$ may be identified, in this case, with sets of morphisms $[X, \Omega^n Y]$ in the homotopy category. A cheap consequence is that $[Z, \Omega^n W]$ is a group for $n \geq 1$, which is abelian for $n \geq 2$ where it makes sense, namely if W is globally fibrant or if Z and W are both locally fibrant. Lemma 3.1 may also be used to show that every local fibration sequence

$$\begin{array}{ccc} F & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & Y \end{array}$$

with Y locally fibrant gives rise to a long exact sequence

$$\cdots \to [Z, \Omega X] \to [Z, \Omega Y] \to [Z, F] \to [Z, X] \to [Z, Y].$$

In effect, $Y \to *$ and then p may be replaced up to weak equivalence by global fibrations. This works because every global fibration is a pointwise and hence local fibration. Finally, it should be observed that the model which is used in the proof of Corollary 3.2 for $\Omega^n Y$ is defined by a pullback diagram

$$\begin{tikzcd} \Omega^n Y \arrow[r] \arrow[d] & Y^{\Delta^n} \arrow[d] \\ * \arrow[r] & Y^{\partial\Delta^n} \end{tikzcd}$$

instead of the iterated construction $\Omega \cdots \Omega Y$. This creates no difficulties, since $\Omega \cdots \Omega Y$ is a globally fibrant object which is weakly equivalent to $\Omega^n Y$. The point is that any function space model for $\Omega^n Y$ will do.

Henceforth, suppose that S is a scheme which has a Zariski open cover $S = \bigcup U_i$ by Noetherian schemes U_i. Suppose further that U_i has a number N_i which bounds the étale cohomological dimension for l-torsion sheaves on all étale patches $V \to U_i$, where l is a prime which does not divide any of the residue characteristics of S. This will hold, for example, if the l-torsion Galois cohomology of all residue fields of U_i has a global bound on cohomological dimension [28,1]. This list of assumptions, while highly technical, is usually met in practice. It seems to be exactly the device that is needed to deal with towers of fibrations in $\mathbf{SPre}(\text{ét}|_S)$. The fact that the topos of sheaves on $\text{ét}|_S$ has enough points is not quite enough.

Let X be a presheaf of Kan complexes on $\text{ét}|_S$ such that all of the sheaves $\pi_n(X|_U, x)$, $U \to S$ étale, are annihilated by l in some range $n > K$, and such that X is locally connected in the sense that $\pi_0(X) = *$. The examples that I shall concentrate on are the simplicial presheaves $K/l^1, K/l^2, \ldots$ which appear in the mod l K-theory presheaf of spectra $K/l = \{K/l^0, K/l^1, \ldots\}$ on $\text{ét}|_S$.

The basic philosophy behind the construction of K/l is, first of all, to produce a presheaf of spectra $K = \{K^0, K^1, \ldots\}$ such that $K^1(U)$ is weakly equivalent to $BQ\mathbf{P}(U)$ for each $U \to S$ in $\text{ét}|_S$, where $\mathbf{P}(U)$ is the category of vector bundles on the scheme U, and Q is Quillen's Q-construction [24]. This can be done by applying the Γ-space techniques of [21] to the pseudo-category of automorphisms of $\mathbf{P}$ on $\text{ét}|_S$ (see also [31]). The weak equivalence at U is natural in U up to coherent homotopy. One also finds, in [21], a method for showing that K has the structure of a presheaf of ring spectra. The mod l Moore spectrum Y^l has a ring spectrum structure for $l > 3$, and so the presheaf of spectra $K \wedge Y^l$ has a presheaf of ring spectra structure in that range (see the appendix of [28]). Y^l is self-dual, so Spanier–Whitehead duality implies that there is a pointwise stable homotopy equivalence $K \wedge Y^l \to \mathbf{hom}_*(Y^l, K)$, where the nth object $\mathbf{hom}_*(Y^l, K)^n$ of the presheaf of function spectra may be identified with the simplicial presheaf $\mathbf{hom}_*(Y_2^l, K^{n+2})$, and Y_2^l is the cofibre of multiplication by l on the circle. I define $K/l^n =$

$\mathbf{hom}_*(Y_2^l, K^{n+2})$. This model for K/l is a presheaf of connective Ω-spectra. In particular, its presheaves of stable homotopy groups may be identified with presheaves of homotopy groups of its constituent simplicial presheaves.

Any simplicial presheaf Y on $\text{ét}|_S$ has associated to it a presheaf of cosimplicial spaces

$$p_* p^* Y \rightrightarrows p_* p^* p_* p^* Y \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots$$

This is the Godement resolution for Y [22,28]. It is the standard cosimplicial object which is assigned to the adjoint pair of functors

$$\mathbf{SPre}(\text{ét}|_S) \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathbf{S}^{\text{Geom}(S)},$$

where the objects of $\mathbf{S}^{\text{Geom}(S)}$ consist of families of simplicial sets $\{Z_x\}$, indexed by the geometric points $x: \text{Sp}(\Omega_x) \to S$ of S. p^* is all of the stalk functors, collected together, so that $p^* Y = \{Y_x\}$, where

$$Y_x = \varinjlim Y(U).$$

$$\begin{tikzcd} & U \arrow[d, "\text{ét}"] \\ \text{Sp}(\Omega_x) \arrow[ur] \arrow[r, "x"'] & S \end{tikzcd}$$

$p_*\{Z_x\}$ may be defined by

$$p_*\{Z_x\}(U \to S) = \prod_{\text{Sp}(\Omega_x) \xrightarrow{x} S} \prod Z_x.$$

$$\begin{tikzcd} & U \arrow[d, "\text{ét}"] \\ \text{Sp}(\Omega_x) \arrow[ur] \arrow[r, "x"'] & S \end{tikzcd}$$

It is easily checked that p^* is left adjoint to p_*.

The simplicial presheaf $G(Y)$ is defined to be the homotopy limit

$$\underset{\overleftarrow{n}}{\text{holim}}((p_* p^*)^n Y)$$

of the above cosimplicial diagram in the sense of [3]. To simplify the notation, let

$$G^n Y = (p_* p^*)^n Y,$$

and denote the corresponding cosimplicial object on $\mathbf{SPre}(\text{ét}|_S)$ by $G^* Y$. Then, in the notation of [3],

$$G(Y) = \text{Tot} \prod{}^* G^* Y,$$

where $\prod^* G^* Y$ is the cosimplicial object with

$$\prod{}^n G^*Y = \prod_{\substack{i_0\to i_1\to\cdots\to i_n\\ \in B\Delta_n}} G^{i_0}(Y)$$

such that Δ is the category of finite ordinal numbers. There is a canonical map $Y\to G(Y)$ which is induced by the augmentation of G^*Y.

Proposition 3.3. *Under the conditions on the simplicial presheaf X and on the underlying scheme S given above, $G(X)$ is globally fibrant and the map $X\to G(X)$ is a weak equivalence.*

Proof. We show that $G(X)$ is globally fibrant by proving that the functor G takes local fibrations to global fibrations. The functor $(p_*p^*)^n$ takes local fibrations to global fibrations, by adjointness and the fact that global fibrations are local fibrations. In general, if $f: X\to Y$ is a natural transformation of functors $I\to \mathbf{SPre(C)}$ defined on a small category I such that each map $f(i): X(i)\to Y(i)$, $i\in I$, is a global fibration, then the induced maps

$$\prod{}^{n+1}X \xrightarrow{(f,s)} \prod{}^{n+1}Y \underset{M^n\prod^*Y}{\times} M^n\prod{}^*X, \quad n\geq 0,$$

are global fibrations, where $M^n\prod^*X$ is the subobject of

$$\prod_{i=0}^{n} \prod{}^n X$$

consisting, in each degree, of $(n+1)$-tuples $(x_0,\dots,x_n)$ such that $s^ix_j = s^{j-1}x_i$ if $i<j$. The map $s: \prod^{n+1}X\to M^n\prod^*X$ is defined by $s(x)=(s^0x,\dots,s^nx)$, as in [3]. The fact that (f,s) is a global fibration is proved by analogy with the corresponding result for diagrams of simplicial sets. One shows that, for the diagram

$$\begin{tikzcd}
U \arrow[r, "(\alpha_\beta)"] \arrow[d, "i"'] & \prod^{n+1}X \arrow[r, "s"] \arrow[d, "f"] & M^n\prod^*X \arrow[d, "f"] \\
V \arrow[r, "(\gamma_\beta)"'] \arrow[ur, dashed, "\theta"] \arrow[urr] & \prod^{n+1}Y \arrow[r, "s"'] & M^n\prod^*Y
\end{tikzcd}$$

with i a trivial cofibration, the dotted arrow exists making the diagram commute. The existence of θ amounts to the existence of θ_β in each diagram

$$\begin{tikzcd}
U \arrow[r, "\alpha_\beta"] \arrow[d, "i"'] & X(i_0) \arrow[d, "f"] \\
V \arrow[r, "\gamma_\beta"'] \arrow[ur, dashed, "\theta_\beta"] & Y(i_0)
\end{tikzcd}$$

corresponding to a *non-degenerate* simplex $\beta : i_0 \to i_1 \to \cdots \to i_n$ of BI. If $g : Z \to W$ is a map of cosimplicial objects in SPre(**C**) such that each induced map

$$Z^{n+1} \xrightarrow{(g,s)} W^{n+1} \underset{M^n W}{\times} M^n Z, \quad n \geq 0,$$

is a global fibration, then the map of total objects $\mathrm{Tot}(g) : \mathrm{Tot}(Z) \to \mathrm{Tot}(W)$ is a global fibration. In effect, one shows that the dotted arrow exists in each diagram

$$\begin{tikzcd} A \times \Delta \arrow[r, "\alpha"] \arrow[d, "i \times 1"'] & Z \arrow[d, "g"] \\ B \times \Delta \arrow[r, "\beta"'] \arrow[ur, dashed, "\theta"] & W \end{tikzcd}$$

where $i : A \to B$ is a trivial cofibration of simplicial sets and Δ is the cosimplicial object $n \mapsto \Delta^n$ of [3], by inductively constructing maps

$$\theta^i : B \times \Delta^i \to Z^i, \quad 0 \leq i \leq s,$$

of truncated cosimplicial objects. θ^{s+1} is the dotted arrow in the diagram

$$\begin{tikzcd} (B \times \partial\Delta^{s+1}) \underset{A \times \partial\Delta^{s+1}}{\bigcup} (A \times \Delta^{s+1}) \arrow[r] \arrow[d, "j"'] & Z^{s+1} \arrow[d, "(g,s)"] \\ B \times \Delta^{s+1} \arrow[r] \arrow[ur, dashed, "\theta^{s+1}"] & W^{s+1} \underset{M^s W}{\times} M^s Z \end{tikzcd}$$

where the horizontal arrows are induced by θ^i, $0 \leq i \leq s$. This proves the initial claim, since $G(X) = \mathrm{Tot} \prod^* G^* X$.

To show that $X \to G(X)$ is a weak equivalence, it is enough to suppose that there is a global bound on l-torsion étale cohomological dimension for all schemes $U \to S$ étale over S, since the problem is local for the Zariski topology on S. The tower of pointwise (in fact, global) fibrations

$$\cdots \to \mathrm{Tot}_2 \textstyle\prod^* G^*(X) \to \mathrm{Tot}_1 \prod^* G^*(X) \to \mathrm{Tot}_0 \prod^* G^*(X) \to *$$

gives rise to a presheaf of Bousfield–Kan spectral sequences on $\text{ét}|_S$, with

$$E_2^{s,t} \cong H^s(\pi_t^{\text{simp}}(G^*X)) \cong \pi_{t-s} F_s^{(1)}, \quad t \geq s \geq 0,$$

since $\pi_t^{\text{simp}}(G^*X) \cong G^*(\pi_t(X))$ is the Godement resolution of $\pi_t(X)$ (see Proposition 1.18). F_s is the fibre over a global choice of base point in X of the map $\mathrm{Tot}_s \prod^* G^* X \to \mathrm{Tot}_{s-1} \prod^* G^* X$. The choice of base point may be suppressed, by the connectedness assumption on X. There is, in particular, a commutative diagram

$$\begin{tikzcd} & \pi_t^{\mathrm{simp}}(GX) = \pi_t^{\mathrm{simp}}(\mathrm{Tot} \prod {}^*G^*X) \arrow[d] \\ \pi_t^{\mathrm{simp}}(X) \arrow[ur] \arrow[r, "\cong"'] & E_2^{0,t} \cong \pi_t F_0^{(1)} = \pi_t^{\mathrm{simp}}(\mathrm{Tot}_0 \prod {}^*G^*X)^{(1)} \end{tikzcd}$$

in the notation of [3]. But $\pi_t(X)$ is l-torsion for t sufficiently large, and l-torsion cohomological dimension is globally bounded, so there is a number N such that $E_2^{s,t}(U)=0$ for $s \geq N$ and all étale patches $U \to S$ of S. Then the derived long exact sequences of presheaves

$$\cdots \to \pi_i^{\mathrm{simp}} F_n^{(1)} \to \pi_i^{\mathrm{simp}} \mathrm{Tot}_n \prod {}^*G^*(X)^{(1)}$$
$$\to \pi_i^{\mathrm{simp}} \mathrm{Tot}_{n-1} \prod {}^*G^*(X)^{(1)} \xrightarrow{\partial} \pi_{i-1}^{\mathrm{simp}} F_{n-1}^{(1)} \to \cdots$$

and the $\varprojlim^1$ short exact sequences imply that there are *pointwise* isomorphisms of presheaves

$$\pi_i^{\mathrm{simp}} G(X) \cong \pi_i^{\mathrm{simp}} \mathrm{Tot}_{N-1} \prod {}^*G^*(X)^{(1)}, \quad i \geq 1,$$

and an inclusion

$$\pi_0^{\mathrm{simp}} G(X) \subset \pi_0^{\mathrm{simp}} \mathrm{Tot}_{N-1} \prod {}^*G^*(X)^{(1)}.$$

But stalkwise

$$(\pi_{t-s}^{\mathrm{simp}} F_s^{(1)})_x \cong H_{\mathrm{et}}^s(?\,;\, \pi_t(X|_?))_x,$$

which groups vanish if $s>0$. Moreover, X is locally connected, and so the maps

$$\pi_i^{\mathrm{simp}} \mathrm{Tot}_{N-1} \prod {}^*G^*(X)^{(1)} \to \pi_i^{\mathrm{simp}} \mathrm{Tot}_0 \prod {}^*G^*(X)^{(1)} \cong \pi_i(X)$$

induce isomorphisms of sheaves for $i \geq 0$. □

It is likely that there is a closed model structure, at least, on the category of cosimplicial objects in **SPre(C)**, for arbitrary sites **C**. The usefulness of such a thing would appear to be constrained by the observation that it is not clear that the inverse limit functor for towers of fibrations preserves (stalkwise) weak equivalences. The problem is a common one; inverse limits cannot be commuted with filtered colimits in general. This is the issue that is skirted above by using the global cohomological dimension assumption. It comes up also in the study of Postnikov towers.

Suppose again that the simplicial presheaf X on $\text{ét}|_S$ and the underlying scheme S satisfy the assumptions preceding Proposition 3.3. X has associated to it a presheaf of Postnikov towers

$$\cdots \to P_2X \to P_1X \to P_0X \to *$$

which is defined pointwise according to the recipe which appears in [20]. Choose a global base point x for X. Then the fibre F_n over x of the fibration $P_nX \to P_{n-1}X$ is a presheaf of Kan complexes, with

$$\pi_i^{\mathrm{simp}}(F_n) \cong \begin{cases} \pi_n^{\mathrm{simp}}(X), & i=n, \\ 0, & i \neq n. \end{cases}$$

By 'standard techniques', F_n is weakly equivalent to the Eilenberg–MacLane presheaf $K(\pi_n^{\mathrm{simp}}(X), n)$ (this is trivial if $n=0$; if $n=1$, use the canonical map $X \to BG(X)$ to the nerve of the fundamental groupoid [14, p. 76]; in higher degrees, use the Hurewicz map [28, 5.52]). One would like, at least, to be able to recover $[*, \Omega^i X]$ from the cohomology of $*$ with coefficients in the sheaves $\pi_n(X)$ by using the Postnikov tower for X in the usual way, and it can almost be done.

Recall that the fibre sequences

$$F_n \to P_nX \xrightarrow{p} P_{n-1}X$$

give rise to long exact sequences

$$\cdots \to [*, \Omega^{i+1}P_{n-1}X] \to [*, \Omega^i F_n] \to [*, \Omega^i P_nX]$$
$$\to [*, \Omega^i P_{n-1}X] \to \cdots \to [*, P_nX] \to [*, P_{n-1}X].$$

Recall also that the most efficient way to construct the long exact sequence is to replace the original fibre sequence up to weak equivalence by a fibre sequence

$$GF_n \to GP_nX \xrightarrow{Gp} GP_{n-1}X,$$

of globally fibrant simplicial presheaves such that Gp is a global fibration. Then the long exact sequence above is the ordinary long exact sequence which is associated to the fibre sequence

$$\mathbf{hom}(*, GF_n) \to \mathbf{hom}(*, GP_nX) \to \mathbf{hom}(*, GP_{n-1}X)$$

of simplicial sets. This tells us what to do in general, namely to replace the original tower up to weak equivalence by a tower

$$\cdots \to GP_2X \to GP_1X \to GP_0X \to *$$

of global fibrations of globally fibrant objects. Then one is entitled to a Bousfield–Kan spectral sequence for the tower of Kan fibrations

$$\cdots \to \mathbf{hom}(*, GP_2X) \to \mathbf{hom}(*, GP_1X) \to \mathbf{hom}(*, GP_0X) \to *.$$

The inverse limit of the tower is $\mathbf{hom}(*, \varprojlim GP_iX)$. The non-trivial part is therefore to demonstrate

Lemma 3.4. *Under the above assumptions on X and S, the map*

$$X \to \varprojlim_i GP_i X$$

is a weak equivalence.

Proof. Consider the fibre sequence

$$GF_n \to GP_n \to GP_{n-1}.$$

The restriction $GF_n|_U$ of GF_n to ét$|_U$ is globally fibrant for each $\varphi : U \to V$ in ét$|_S$. To see this, observe that the restriction functor has a left adjoint

$$\varphi_! : \mathrm{SPre}(\text{ét}\,|_U) \to \mathrm{SPre}(\text{ét}\,|_S)$$

(extension by 0) which is defined by

$$\varphi_! Y(\gamma : V \to S) = \coprod_{\begin{tikzcd}[row sep=small, column sep=small] V \arrow[rr, "\psi"] \arrow[dr, "\gamma"'] & & U \arrow[dl] \\ & S & \end{tikzcd}} Y(\psi)$$

It is apparent from the definition and Theorem 1.12 that $\varphi_!$ preserves pointwise weak equivalences. It follows that Ex^∞ may be used to show that $\varphi_!$ preserves all weak equivalences of simplicial presheaves. But then $\varphi_!$ preserves trivial cofibrations, and so the restriction functor preserves global fibrations, by adjointness.

It follows that there are isomorphisms

$$\pi_i(GF_n(U), x) \cong [*, \Omega^i GF_n|_U]_U$$

for $i \geq 0$, and so there are presheaf isomorphisms

$$\pi_i(GF_n(U), x) \cong \begin{cases} H_{\mathrm{et}}^{n-i}(U; \pi_n(X, x)|_U), & 0 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

The hypotheses imply that there are positive integers N and M such that $H_{\mathrm{et}}^k(U; \pi_n(X, x)|_U) = 0$ for $k \geq N$, $n \geq M$ and all $U \to S$ in ét$|_S$. Thus, for each $i \geq 0$, there is a sufficiently large $n > i$ such that the map

$$\pi_i^{\mathrm{simp}}(\varprojlim GP_n(X)) \to \pi_i^{\mathrm{simp}}(GP_n(X))$$

is an isomorphism of presheaves. But the map of presheaves

$$\pi_i^{\mathrm{simp}}(X) \to \pi_i^{\mathrm{simp}} GP_n(X)$$

induces a sheaf isomorphism by construction, and so the map

$$\pi_i^{\mathrm{simp}}(X) \to \pi_i^{\mathrm{simp}}(\varprojlim GP_n(X))$$

induces a sheaf isomorphism. $\square$

The argument for Lemma 3.4 is essentially due to Brown and Gersten [5]. Here, as in op. cit., it is the key step in the proof of

Corollary 3.5. *Suppose that the simplicial presheaf X and the underlying scheme S are as above. Then the Postnikov tower of X induces a Bousfield-Kan spectral sequence, with*

$$E_1^{s,t} \cong H_{\text{et}}^{2s-t}(S; \pi_s(X)) \Rightarrow [*, \Omega^{t-s}X], \quad t \geq s \geq 0.$$

This is the Brown-Gersten spectral sequence for the étale topology on S. Recall that, by quite a different method, we constructed a presheaf of spectral sequences for the Godement resolution $G(X)$ of X. Since $G(X)$ is globally fibrant and $X \to G(X)$ is a weak equivalence, taking global sections of this presheaf of spectral sequences gives

Corollary 3.6. *With the assumptions on X and S above, the Godement resolution $G(X)$ of X determines a spectral sequence, with*

$$E_2^{s,t} \cong H_{\text{et}}^{s}(S; \pi_t(X)) \Rightarrow [*, \Omega^{t-s}X], \quad t \geq s \geq 0.$$

The spectral sequences of the corollaries above are cohomological descent spectral sequences for the invariants $[*, \Omega^i X]$, $i \geq 0$, determined by the simplicial presheaf X. In particular, if X is the mod l K-theory presheaf K/l^1, then a straightforward application of Corollary 3.6 gives a spectral sequence, with

$$E_2^{s,t} \cong H_{\text{et}}^{s}(S; \tilde{K}_{t-1}(-; \mathbb{Z}/l)) \Rightarrow [*, \Omega^{t-s}K/l^1],$$

since $\pi_i K/l^1(U) \cong K_{i-1}(U; \mathbb{Z}/l)$ for $i \geq 0$.

Suppose now, that the following list of conditions holds for the scheme S:

(3.7) S is separated, Noetherian and regular. $1/l \in \mathcal{O}_S$, and $\sqrt{(-1)} \in \mathcal{O}_S$ if $l=2$. S has finite Krull dimension, and a uniform bound on l-torsion étale cohomological dimension of all residue fields. Each residue field of S has a Tate-Tsen filtration.

Let ζ be a primitive lth root of unity. Recall [7,28], that there is an element $\omega \in K_2(\mathbb{Z}(l^{-1}); \mathbb{Z}/l)$ which base changes to β^{l-1} in $K_{2(l-1)}(\mathbb{Z}(l^{-1}, \zeta); \mathbb{Z}/l)$, where $\beta \in K_2(\mathbb{Z}(l^{-1}, \zeta); \mathbb{Z}/l)$ is the Bott element. β is defined in such a way that it restricts to $\zeta \in \mathrm{Tor}(\mathbb{Z}/l, K_1(\mathbb{Z}(l^{-1}, \zeta))$. Multiplication by ω determines maps of simplicial presheaves

$$K/l^m \xrightarrow{\omega} \Omega^n K/l^m \xrightarrow{\Omega^n \omega} \Omega^{2n} K/l^m \xrightarrow{\Omega^{2n}\omega} \cdots \tag{3.8}$$

where $n = 2(l-1)$. The filtered colimit of this system in the category $\mathrm{SPre}(\text{ét}\,|_S)$ is the mth object $K/l(1/\beta)^m$ of the mod l Bott periodic K-theory presheaf of spectra $K/l(1/\beta)$. The notation reflects the fact, locally, inverting ω coincides with inverting β. In particular, by the Gabber-Gillet-Thomason rigidity theorem [13,11], the maps in (3.8) are weak equivalences of simplicial presheaves when $m=1$. Thomason's descent theorem (the main result of [28]) asserts that the map of presheaves of spectra induced by the maps

$$K/l(1/\beta)^m \to G(K/l(1/\beta)^m)$$

is a *pointwise* stable homotopy equivalence in the sense that it induces stable homotopy equivalences

$$K/l(1/\beta)^m(U) \to G(K/l(1/\beta)^m)(U)$$

for all $U \to S$ in $\text{ét}|_S$. A periodicity argument shows that this is equivalent to the statement that the map

$$K/l(1/\beta)^1 \to G(K/l(1/\beta)^1)$$

is a pointwise weak equivalence of simplicial presheaves.

Consider the diagram

$$\begin{tikzcd}
K/l^1 \arrow[r] \arrow[d] & \Omega^n K/l^1 \arrow[r] \arrow[d] & \Omega^{2n} K/l^1 \arrow[r] \arrow[d] & \cdots \\
G(K/l^1) \arrow[r] & G(\Omega^n K/l^1) \arrow[r] & G(\Omega^{2n} K/l^1) \arrow[r] & \cdots
\end{tikzcd}$$

The bottom horizontal maps are weak equivalences of globally fibrant simplicial presheaves, and are therefore pointwise weak equivalences. There are several ways to see this. One could, for example, use an adjointness argument to show that each section functor takes trivial global fibrations to trivial fibrations of simplicial sets, and hence preserves weak equivalences of globally fibrant objects. The argument in Proposition 1.17 now implies that there is a commutative diagram of weak equivalences

$$\begin{tikzcd}
K/l^1 \arrow[r] \arrow[dd] & \varinjlim \Omega^{kn} K/l^1 = K/l(1/\beta)^1 \arrow[dd, "\varepsilon_*"] \arrow[dr] \arrow[dr, shift left=2, "\varepsilon"] & \\
& & G(\varinjlim \Omega^{kn} K/l^1) = G(K/l(1/\beta)^1) \arrow[dd, "G(\varepsilon_*)"] \\
G(K/l^1) \arrow[r] \arrow[drr, "\tau"'] & \varinjlim G(\Omega^{kn} K/l^1) \arrow[dr] & \\
& & G(\varinjlim G(\Omega^{kn} K/l^1))
\end{tikzcd}$$

But the maps $G(\varepsilon_*)$ and τ are weak equivalences of globally fibrant objects, hence pointwise weak equivalences, and ε is a pointwise weak equivalence by Thomason's descent theorem. In conclusion, we have

Theorem 3.9. *Suppose that the scheme S satisfies the conditions of* (3.7). *Then there are isomorphisms*

$$[*, \Omega^i K/l^1] \cong K/l_{i-1}(S)(1/\beta), \quad i \geq 0.$$

The groups $K/l_*(S)(1/\beta)$ are the stable homotopy groups of the global sections of the presheaf of spectra $K/l(1/\beta)$. It follows [29] that the groups $[*, \Omega^i K/l^1]$ coincide with the étale K-groups $K_{i-1}^{\mathrm{et}}(S; \mathbb{Z}/l)$ if $i \geq 0$.

The existence of the induced isomorphisms

$$[*, \Omega^{i+3} K/l^1] \cong K_{i+2}^{\mathrm{et}}(S; \mathbb{Z}/l), \quad i \geq 0,$$

can be proven much more directly if S is defined over an algebraically closed field k, by using Friedlander's original definition of étale K-theory [8] and the rigidity theorem. In that case, there is a weak equivalence of $\Omega^3 K/l^1$ with the constant simplicial sheaf $\Gamma^* \mathbf{hom}_*(Y_2^l, \Omega S|BQ\mathbf{P}(k)|)$ on the simplicial set $\mathbf{hom}_*(Y_2^l, \Omega S|BQ\mathbf{P}(k)|)$. This simplicial set is weakly equivalent to $\mathbf{hom}_*(Y_2^l, BU)$, by rigidity again. On the other hand, $K_{i+2}^{\mathrm{et}}(S; \mathbb{Z}/l)$ is defined to be the filtered colimit

$$\varinjlim\, [\pi_0 V, \Omega^i \mathbf{hom}_*(Y_2^l, BU)],$$

of homotopy classes of maps of simplicial sets, indexed over the *representable* objects $[\pi] : V \to *$ in the category $\mathrm{Triv}\!\downarrow\! *$ which corresponds to $\mathrm{SShv}(\text{ét}|_S)_f$. A hypercovering V of the scheme S is, after all, nothing but a representable trivial local fibration $V \to *$ in $\mathrm{SShv}(\text{ét}|_S)_f$. π_0 is, in this case, the Verdier functor, so 'Verdier duality' implies that the filtered colimit above is isomorphic to

$$\varinjlim_{\substack{V \to * \text{ representable} \\ \in \mathrm{Triv}\downarrow *}} \pi(V, \Gamma^* \Omega^i \mathbf{hom}_*(Y_2^l, BU)).$$

The representable objects in $\mathrm{Triv}\!\downarrow\! *$ are cofinal, by an Artin–Mazur style argument [2, Ch. 8], and so $K_{i+2}^{\mathrm{et}}(S; \mathbb{Z}/l)$ coincides with the morphism group

$$[*, \Gamma^* \Omega^i \mathbf{hom}_*(Y_2^l, BU)]$$

in $\mathrm{Ho}(\mathrm{SShv}(\text{ét}|_S)_f)$. This morphism group coincides with the morphism group

$$[*, \Gamma^* \Omega^i \mathbf{hom}_*(Y_2^l, BU)]$$

in $\mathrm{Ho}(\mathrm{SPre}(\text{ét}|_S))$, and thus with $[*, \Omega^{i+3} K/l^1]$, as claimed. One expects that a variant of this argument can be produced to handle the more general definition of $K_*^{\mathrm{et}}(X; \mathbb{Z}/l)$ of a simplicial scheme X which is given in [6]. The key will be to think about local coefficients appropriately.

Another thing to observe is that the statement of Theorem 3.9 is best possible in some sense. One defines the generalized cohomology group $H^i(*; K/l)$ of the terminal object $*$ in $\mathrm{SPre}(\text{ét}|_S)$ with coefficients in the presheaf of spectra K/l by

$$H^i(*; K/l) = \varinjlim_n\, [*, \Omega^{n+i} K/l^n]$$

for $i \in \mathbb{Z}$. Then $H^*(*; K/l)$ has a graded ring structure, since K/l is a presheaf of ring spectra. Moreover, $H^i(*; K/l)$ coincides with $K/l_i(S)(1/\beta)$ for $i \geq -1$ by Theorem 3.9. But $H^i(*; K/l) = 0$ if $i < -N$, where N is a global bound on l-torsion cohomological dimension for ét$|_S$; this is proved by looking at the descent spectral sequence of Corollary 3.6 for $[*, \Omega^i K/l^n]$, with $n > N$. In fact, if S is a smooth complete curve over the algebraically closed field k, then there is an isomorphism

$$H^{-2}(*; K/l) \cong \mathbb{Z}/l,$$

whereas

$$H^0(*; K/l) \cong K/l_0(S)(1/\beta) \cong \mathbb{Z}/l \oplus \mathbb{Z}l.$$

Thus, one does not expact to find Bott periodicity in $H^i(*; K/l)$ for $i < -1$ in general.

I claim, nevertheless, that $H^*(*; K/l)$ is a good description of mod l étale K-theory for the scheme S. The definition has been made with respect to the étale topology on S, so that one could properly rename the invariant $H^*_{\mathrm{et}}(S; K/l)$. The presheaves K/l^i are, moreover, defined for all of the Grothendieck topologies that one usually associates with S, so that one is entitled to groups of the form $H^*_{\mathrm{Zar}}(S; K/l)$ (which we 'know' about [5]) for the Zariski topology on S, and to groups $H^*_{\mathrm{fl}}(S; K/l)$ for the flat topology. The construction may also be promoted to the big étale site $(\mathrm{Sch}|_S)_{\mathrm{et}}$ for S, and to the fibred étale site which is associated to a simplicial scheme X.

Suppose that X is a simplicial S-scheme which is locally of finite type over S, meaning that X is a simplicial object of the big site $(\mathrm{Sch}|_S)_{\mathrm{et}}$. Then X represents a simplicial sheaf on $(\mathrm{Sch}|_S)_{\mathrm{et}}$. I define groups $H^i_S(X; K/l)$ by

$$H^i_S(X; K/l) = \varinjlim [X, \Omega^{n+i} K/l^n]_S, \quad i \in \mathbb{Z},$$

by analogy with the definition of $H^i(*; K/l)$ which is given above. In this case, $[X, \Omega^{n+i} K/l^n]_S$ means morphisms in $\mathrm{Ho}(\mathrm{SPre}((\mathrm{Sch}|_S)_{\mathrm{et}}))$. One could, of course, define corresponding invariants $H^*_S(Y; K/l)$ for any simplicial presheaf Y on $(\mathrm{Sch}|_S)_{\mathrm{et}}$. On the other hand, the restriction $K/l|_X$ of the mod l K-theory presheaf on $(\mathrm{Sch}|_S)_{\mathrm{et}}$ to the fibred étale ét$|_X$ is weakly equivalent to the mod l K-theory presheaf on ét$|_X$. This construction apparently gives another invariant $H^*_{\mathrm{et}}(X; K/l)$, defined by

$$H^i_{\mathrm{et}}(X; K/l) = H^i_X(*; K/l) = \varinjlim [*, \Omega^{n+i} K/l^n]_X.$$

The following result implies that $H^i_{\mathrm{et}}(X; K/l)$ is isomorphic to $H^i_S(X; K/l)$:

Theorem 3.10. *Suppose that S is a locally Noetherian scheme, and that X is a simplicial S-scheme which is locally of finite type. Suppose that Y is a simplicial presheaf on* $(\mathrm{Sch}|_S)_{\mathrm{et}}$. *Then there is an isomorphism*

$$[X, Y]_S \cong [*, Y|_X]_X.$$

Proof. Recall that $Y|_X$ is defined by

$$Y|_X(U \to X_n) = Y(U \to X_n \to S),$$

where $X_n \to S$ is the S-structure map for the scheme X_n of n-simplices of X. This functor is exact, so it preserves trivial local fibrations and hence weak equivalences. We may therefore assume that Y is a globally fibrant object of $\mathbf{SPre}((\mathrm{Sch}|_S)_{\mathrm{et}})$. In that case, $[X, Y]_S$ is isomorphic to $\pi_0\mathbf{hom}(X, Y)$. $\mathbf{hom}(X, Y)$ is the total space of the cosimplicial space

$$\hom_S(X_n, Y_m), \quad n, m \geq 0,$$

which is constructed by using homomorphisms of $\mathbf{SPre}((\mathrm{Sch}|_S)_{\mathrm{et}})$. This cosimplicial space is fibrant in the sense of [3]. In effect, the matching space $M^p\hom_S(X_*, Y_*)$ is the simplicial set $\hom_S(DX_{p+1}, Y_*)$, where DX_{p+1} is the subpresheaf of degeneracies in X_{p+1}. Thus, if $p: Y \to Z$ is a global fibration, then showing that the map

$$\hom_S(X_{p+1}, Y_*) \xrightarrow{(p_*, s)} \hom_S(X_{p+1}, Z_*) \underset{M^p\hom_S(X_*, Z_*)}{\times} M^p\hom_S(X_*, Y_*)$$

is a fibration of simplicial sets amounts to observing that the presheaf inclusions

$$(\Delta^n \times DX_{p+1}) \underset{(\Lambda^n_k \times DX_{p+1})}{\cup} (\Lambda^n_k \times X_{p+1}) \subset (\Delta^n \times X_{p+1})$$

are pointwise anodyne extensions.

Now let 1_X be the simplicial presheaf on $\mathrm{\acute{e}t}|_X$ which is represented by X [17]. More explicitly, the presheaf 1_{X_n} of n-simplices of 1_X is represented by the étale map $1_{X_n}: X_n \to X_n$. There is an isomorphism of cosimplicial spaces

$$\hom_S(X_n, Y_m) \cong \hom_X(1_{X_n}, Y_m|_X),$$

where $\hom_X(1_{X_n}, Y_m|_X)$ is homomorphisms in $\mathbf{SPre}(\mathrm{\acute{e}t}|_X)$. In particular, the cosimplicial space on the right is fibrant; its total space is $\mathbf{hom}(1_X, Y|_X)$. The theorem is proved if we can show that the canonical map

$$\pi_0\mathbf{hom}(1_X, Y|_X) \cong \pi(1_X, Y|_X) \to [1_X, Y|_X]$$

is an isomorphism, since the map $1_X \to *$ is a weak equivalence. In other words, we want to show that $Y|_X$ is flabby for 1_X under the assumption that Y is globally fibrant.

Choose a trivial cofibration $i: Y|_X \to Z$, where Z is globally fibrant on $\mathrm{\acute{e}t}|_X$. The presheaf restriction functor $W \mapsto W|_{X_n}$ along the inclusion $\mathrm{\acute{e}t}|_{X_n} \subset \mathrm{\acute{e}t}|_X$ is exact and has a left adjoint which preserves inclusions, pointwise weak equivalences and trivial local fibrations (see [17, §3] and the proof of Lemma 3.4). This is also true for restriction along the functor $\mathrm{\acute{e}t}|_{X_n} \to (\mathrm{Sch}|_S)_{\mathrm{et}}$ which is defined by

$$U \to X_n \mapsto U \to X_n \to S$$

(see [10, p. 16]). It follows that both restriction functors preserve global fibrations, so that, for each n, the induced map

$$i|_{X_n}: Y|_{X_n} = Y|_X|_{X_n} \to Z|_{X_n}$$

is a weak equivalence of globally fibrant objects, and is therefore a pointwise weak equivalence. But then the map

$$i_*: \hom(1_{X_n}, Y_m|_X) \to \hom(1_{X_n}, Z_m)$$

of fibrant cosimplicial spaces is a weak equivalence, and thus induces an isomorphism

$$\pi_0 \mathbf{hom}(1_X, Y|_X) \cong \pi_0 \mathbf{hom}(1_X, Z).$$

$\pi_0 \mathbf{hom}(1_X, Z)$ may be identified with $[1_X, Y|_X]_X$, and so the theorem is proved. □

Theorem 3.10 generalizes Theorem 3.2 of [17]. It also implies that there is an isomorphism

$$(3.11) \qquad [X, \Omega^n K/l^1]_S \cong \pi_n \underset{\underset{r}{\longleftarrow}}{\operatorname{holim}}\, GK^1/l(X_r)$$

if S and the simplicial S-scheme X are as in the statement of the theorem and GK/l^1 is a globally fibrant model for K/l^1 in SPre(ét$|_X$). It follows that the equivariant topological K-groups (really G-groups) of [30] may be computed in this way. If X is an S-scheme with an action by an algebraic group H over S, subject to the usual constraints [30], then there are isomorphisms

$$(3.12) \qquad [EH \underset{H}{\times} X, \Omega^n K/l^1]_S \cong G/l^{\mathrm{top}}_{n-1}(H, X), \quad n \geq 0,$$

since the groups $G/l^{\mathrm{top}}_*(H, X)$ are defined via the right-hand side of (3.11). Observe that the left-hand side of (3.12) is, once again, a very general object.

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