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A characterization of n-cotilting and n-tilting modules

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Abstract

We consider generalizations of the definitions of one-dimensional tilting and cotilting modules which agree with the classical notions of tilting and cotilting modules of finite homological dimension.

Keywords: Tilting modules; Cotilting modules

1. Introduction

The classical notion of tilting and cotilting modules was first considered in the case of finite-dimensional algebras by Brenner and Butler [3] and by Happel and Ringel [10] in the 80s. The tilting (cotilting) modules considered in these papers are finitely generated and of projective (injective) dimension one. In [13] Miyashita considered finitely generated tilting modules of finite projective dimension, while generalizations of tilting modules of projective dimension one over arbitrary rings have been considered by many authors: Colby and Fuller [5], Colpi and Trlifaj [6]. In [6] an infinitely generated module T is said to be tilting if $\operatorname{Gen} T = T^{\perp}$, where $\operatorname{Gen} T$ is the class of modules which are epimorphic images of direct sums of copies of T and $T^{\perp}$ is the class of modules M such that $\operatorname{Ext}^1(T, M) = 0$. This definition generalizes the classical notion of tilting modules and its natural dual generalizes the classical notion of cotilting modules. In [1] Angeleri Hügel and Coelho carry over an extensive study of infinitely generated tilting and cotilting modules of finite ho-

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mological dimension over arbitrary rings producing the important result that tilting and cotilting classes provide for special precovers and special preenvelopes (see Section 2 for definitions). In this paper we generalize to the n-dimensional case the notions introduced by Colpi and Trlifaj in the one-dimensional case and we prove that the classes of modules satisfying our new definitions coincide with the classes of tilting and cotilting modules studied by Angeleri Hügel and Coelho. Moreover, our results in the tilting case, are generalizations of results in [11].

2. Preliminaries

R will denote an associative ring with identity and R-Mod the class of left R-modules. We recall the notion of cotorsion pair introduced by Salce [14]. Given a class $\mathcal{M}$ of modules, let denote

$$^{\perp}\mathcal{M} = \{X \in R\text{-Mod} \mid \operatorname{Ext}^1_R(X, M) = 0 \text{ for all } M \in \mathcal{M}\} \quad \text{and}$$
$$\mathcal{M}^{\perp} = \{X \in R\text{-Mod} \mid \operatorname{Ext}^1_R(M, X) = 0 \text{ for all } M \in \mathcal{M}\}.$$

A pair $(\mathcal{A}, \mathcal{B})$ of classes of R-modules is called a cotorsion pair if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = A^{\perp}$. $\mathcal{A}$ is called the cotorsion-free class, while $\mathcal{B}$ is called the cotorsion class. Given a class $\mathcal{M}$ of modules, the pairs

$$\mathcal{G}_{\mathcal{M}} = \left({}^{\perp}\mathcal{M}, \left({}^{\perp}\mathcal{M}\right)^{\perp}\right) \quad \text{and} \quad \mathcal{C}_{\mathcal{M}} = \left({}^{\perp}\left(\mathcal{M}^{\perp}\right), \mathcal{M}^{\perp}\right)$$

are cotorsion pairs, called the cotorsion pairs *generated* and *cogenerated* by $\mathcal{M}$, respectively.

For every R-module M, $\operatorname{Prod} M$ $(\operatorname{Add} M)$ will denote the class of modules isomorphic to summands of direct products (direct sums) of copies of M. $\operatorname{Cogen} M$ will denote the class of the R-modules cogenerated by M, namely the class of modules which are embeddable in a product of copies of M, and $\operatorname{Gen} M$ will denote the class of the R-modules generated by M, namely the class of modules which are epimorphic images of direct sums of copies of M. It is evident that an R-module $N \in \operatorname{Cogen} M$ if and only if, for every $0 \neq x \in N$ there is a morphism $f \in \operatorname{Hom}_R(N, M)$ such that $f(x) \neq 0$ and an R-module $N \in \operatorname{Gen} M$ if and only if, for every $0 \neq x \in N$ there is a finite number of morphisms $f_i \in \operatorname{Hom}_R(M, N)$ such that $x \in \sum_i \operatorname{Im} f_i$.

We recall now the definitions of tilting and cotilting modules of dimension one introduced by Colpi and Trlifaj [6].

Definition 1. If R is any ring, an R-module U is said to be 1-*cotilting* if ${}^{\perp}U = \operatorname{Cogen} U$.

Definition *1. If R is any ring, an R-module T is said to be 1-*tilting* if $T^{\perp} = \operatorname{Gen} T$.

Thus, in the above terminology, if U is a 1-cotilting module, then ${}^{\perp}U = \operatorname{Cogen} U$ is the cotorsion-free class of the cotorsion pair generated by U. Dually, if T is a 1-tilting module, then $T^{\perp} = \operatorname{Gen} U$ is the cotorsion class of the cotorsion pair cogenerated by T. Note that for any R-module M, ${}^{\perp}M$ (respectively $M^{\perp}$) is closed under submodules (respectively epimorphic images) if and only if the injective (respectively projective) dimension $\operatorname{id} M$ (respectively $\operatorname{pd} M$) of M is less than or equal to 1; thus a 1-cotilting (respectively 1-tilting) module has injective (respectively projective) dimension at most one and this explains the terminology used in Definitions 1 and *1.

As proved in [2,6,7], the above definitions are respectively equivalent to the following.

Definition 2. An R-module U is 1-cotilting if the following three conditions hold:

(1) $\operatorname{id} U \leqslant 1$;
(2) $\operatorname{Ext}^1_R(U^{\lambda}, U) = 0$ for every cardinal λ;
(3) there exists an exact sequence

$$0 \to U_1 \to U_0 \to E \to 0,$$

where E is an injective cogenerator of R-Mod and $U_0, U_1 \in \operatorname{Prod} U$.

Definition *2. An R-module T is 1-tilting if the following three conditions hold:

(1) $\operatorname{pd} T \leqslant 1$;
(2) $\operatorname{Ext}^1_R(T, T^{(\lambda)}) = 0$ for every cardinal λ;
(3) there exists an exact sequence

$$0 \to R \to T_0 \to T_1 \to 0,$$

where $T_0, T_1 \in \operatorname{Add} T$.

In the one-dimensional case the following alternative definitions are available.

Definition 3. An R-module U is 1-cotilting if and only if U satisfies conditions (1), (2) of Definition 2 and

(3′) for any R-module M, $\operatorname{Hom}_R(M, U) = 0$ and $\operatorname{Ext}^1_R(M, U) = 0$ imply $M = 0$.

Definition *3. An R-module T is 1-tilting if and only if T satisfies conditions (1), (2) of Definition *2 and

(3′) for any R-module M, $\operatorname{Hom}_R(T, M) = 0$ and $\operatorname{Ext}^1_R(T, M) = 0$ imply $M = 0$.

We recall the notions of *precover*, *special precover*, and *cover* introduced by Enochs and Xu in [9,15]. If $\mathcal{X}$ is any class of modules and $X \in \mathcal{X}$, a homomorphism $\phi \in \operatorname{Hom}_R(X, M)$ is called an *$\mathcal{X}$-precover* of the R-module M, if for every homomorphism $\phi' \in \operatorname{Hom}_R(X', M)$ with $X' \in \mathcal{X}$ there exists a homorphism $f : X' \to X$ such that $\phi' = \phi f$.

An $\mathcal{X}$-precover, $\phi \in \operatorname{Hom}_R(X, M)$ is called an *$\mathcal{X}$-cover* of M if for every endomorphism f of X such that $\phi = \phi f$, f is an automorphism of X. An $\mathcal{X}$-precover ϕ of M is said to be *special* if ϕ is surjective and $\operatorname{Ker} \phi \in \mathcal{X}^{\perp}$.

The notions of *$\mathcal{X}$-preenvelope*, *special $\mathcal{X}$-preenvelope*, and *$\mathcal{X}$-envelope* are defined dually.

A class $\mathcal{X}$ is said to be a *precovering* (*preenveloping*, *covering*, *enveloping*) if every R-module admits an $\mathcal{X}$-precover ($\mathcal{X}$-preenvelope, $\mathcal{X}$-cover, $\mathcal{X}$-envelope).

The following two results, dual to each other, will be used throughout.

Lemma 2.1 [4, Proposition 1.8]. *Let N, M be R-modules. Assume that $N \in \operatorname{Cogen} M$ and $M^{\lambda} \in {}^{\perp}M$, for every cardinal λ. Then there exists an exact sequence*

$$0 \to N \to M^{I} \to L \to 0, \quad \textit{where } L \in {}^{\perp}M.$$

Proof. It is enough to let $I = \operatorname{Hom}_R(N, M)$. □

Lemma 2.2 [6, Lemma 1.2]. *Let N, M be R-modules. Assume that $N \in \operatorname{Gen} M$ and $M^{(\lambda)} \in M^{\perp}$, for every cardinal λ. Then there exists an exact sequence*

$$0 \to L \to M^{(I)} \to N \to 0, \quad \textit{where } L \in M^{\perp}.$$

Proof. It is enough to let $I = \operatorname{Hom}_R(M, N)$. □

3. n-Cotilting and n-tilting modules

We recall the generalization of the notion of tilting and cotilting modules to modules of finite homological dimension introduced by Angeleri Hügel and Coelho in [1] and investigated also by Krause and Solberg in [12].

Definition 4. An R-module U is n-cotilting if and only if the following three conditions hold:

(C1) $\operatorname{id} U \leqslant n$;
(C2) $\operatorname{Ext}^i_R(U^{\lambda}, U) = 0$ for each $i > 0$ and for every cardinal λ;
(C3) there exists a long exact sequence

$$0 \to U_r \to \cdots \to U_1 \to U_0 \to E \to 0,$$

where E is an injective cogenerator of R-Mod, $U_i \in \operatorname{Prod} U$, for every $0 \leqslant i \leqslant r$.

U is said to be partial n-cotilting if it satisfies conditions (C1) and (C2).

Definition *4. An R-module T is n-tilting if and only if the following three conditions hold:

(T1) $\operatorname{pd} T \leqslant n$;
(T2) $\operatorname{Ext}^i_R(T, T^{(\lambda)}) = 0$ for each $i > 0$ and for every cardinal λ;

(T3) there exists a long exact sequence

$$0 \to R \to T_0 \to T_1 \to \cdots \to T_r \to 0,$$

where $T_i \in \operatorname{Add} T$, for every $0 \leqslant i \leqslant r$.

T is said to be partial n-tilting if it satisfies conditions (T1) and (T2).

It is easy to show (see Proposition 3.5), that if U is an n-cotilting module, then in the long exact sequence in (C3), r can be chosen to be less than or equal to n. Thus, in the case $n = 1$, the above definition agrees with the one introduced in Section 2. Analogously, the same remark holds for 1-tilting modules.

For any class $\mathcal{M}$ of R-modules we will consider the following classes:

$$\begin{aligned} {}^{\perp_\infty}\mathcal{M} &= \{X \in R\text{-Mod} \mid \operatorname{Ext}^i_R(X, M) = 0,\ \forall M \in \mathcal{M},\ \forall i \geqslant 1\}, \\ {}^{\perp_j}\mathcal{M} &= \{X \in R\text{-Mod} \mid \operatorname{Ext}^i_R(X, M) = 0,\ \forall M \in \mathcal{M}\}, \\ {}^{\perp_{\geqslant j}}\mathcal{M} &= \{X \in R\text{-Mod} \mid \operatorname{Ext}^i_R(X, M) = 0,\ \forall M \in \mathcal{M},\ \forall i \geqslant j\}. \end{aligned}$$

$\mathcal{M}^{\perp_\infty}$, $\mathcal{M}^{\perp_j}$, and $\mathcal{M}^{\perp_{\geqslant j}}$ are defined dually. If $\mathcal{M} = \{M\}$, we will use the notations ${}^{\perp_\infty}M$, ${}^{\perp_j}M$, ${}^{\perp_{\geqslant j}}M$ and $M^{\perp_\infty}$, $M^{\perp_j}$, $M^{\perp_{\geqslant j}}$.

Useful generalizations of Lemmas 2.1 and 2.2 are given by the following result which is a slight generalization of [1, Lemma 2.4].

Lemma 3.1 [1, Lemma 2.4]. *Let N, M be R-modules.*

(i) *Assume that $N \in \operatorname{Cogen} M$, $N \in {}^{\perp_\infty}M$, and $M^\lambda \in {}^{\perp_\infty}M$, for every cardinal λ. Then there exists an exact sequence*

$$0 \to N \to M^I \to L \to 0, \quad \textit{where } L \in {}^{\perp_\infty}M.$$

(ii) *Assume that $N \in \operatorname{Gen} M$, $N \in M^{\perp_\infty}$, and $M^{(\lambda)} \in M^{\perp_\infty}$, for every cardinal λ. Then there exists an exact sequence*

$$0 \to L \to M^{(I)} \to N \to 0, \quad \textit{where } L \in M^{\perp_\infty}.$$

An application of the preceding lemma yields the following result.

Lemma 3.2. *Let U be an n-cotilting module. An R-module M belongs to ${}^{\perp_\infty}U$ if and only if there exists an infinite exact sequence of the form*

$$0 \to M \to U^{\alpha_1} \to U^{\alpha_2} \to \cdots \to U^{\alpha_n} \to \cdots,$$

for some cardinals α_i. In particular, ${}^{\perp_\infty}U$ is closed under direct products.

Dually, let T be an n-tilting module. An R-module M belongs to $T^{\perp_\infty}$ if and only if there exists an infinite exact sequence of the form

$$\cdots \to T^{(\alpha_n)} \to \cdots \to T^{(\alpha_2)} \to T^{(\alpha_1)} \to M \to 0,$$

for some cardinals α_i. In particular, $T^{\perp_\infty}$ is closed under direct sums.

Proof. The statement concerning n-cotilting modules has been noted in [12, Proposition 5.4]. The dual statement for an n-tilting module T follows easily by the fact that $T^{\perp_\infty} \subseteq \operatorname{Gen} T$ (see [1, Lemma 2.3]), by Lemma 3.1(ii), and by dimension shifting. □

If U is a 1-cotilting module, then ${}^{\perp_\infty}U = {}^{\perp}U = \operatorname{Cogen} U$. If U is an n-cotilting module, then it is no longer true that ${}^{\perp_\infty}U = \operatorname{Cogen} U$, but as proved in [1, Lemma 2.3], ${}^{\perp_\infty}U \subseteq \operatorname{Cogen} U$. Dually, if T is a 1-tilting module, then $T^{\perp_\infty} = T^{\perp} = \operatorname{Gen} T$, and if T is an n-tilting module, then $T^{\perp_\infty} \subseteq \operatorname{Gen} T$. In Proposition 3.6 we will see that suitable notions of $\operatorname{Cogen}_n U$ and of $\operatorname{Gen}_n T$ will yield the equalities ${}^{\perp_\infty}U = \operatorname{Cogen}_n U$ and $T^{\perp_\infty} = \operatorname{Gen}_n T$, for n-cotilting modules U and n-tilting modules T.

For any n-cotilting module U, let $\mathcal{X} = {}^{\perp_\infty}U$ and $\mathcal{X}_j = {}^{\perp_{\geqslant j}}U$. Similarly, for any n-tilting module T, let $\mathcal{X} = T^{\perp_\infty}$, $\mathcal{X}_j = T^{\perp_{\geqslant j}}$.

Remark 1. Note that, if $\mathcal{X} = {}^{\perp_\infty}M$, for some module M, then $\mathcal{X}^{\perp_\infty} = \mathcal{X}^{\perp}$ and similarly, if $\mathcal{X} = M^{\perp_\infty}$, then ${}^{\perp_\infty}\mathcal{X} = {}^{\perp}\mathcal{X}$ (see [1, Lemma 1.2]).

In [1, Theorem 3.1, Proposition 3.3] it is proved that $\mathcal{X} = {}^{\perp_\infty}U$ (respectively $\mathcal{X} = T^{\perp_\infty}$) is precovering (respectively preenveloping) and, moreover, that for every R-module M there exists a special $\mathcal{X}$-precover $X \xrightarrow{\phi} M$ of M such that $\operatorname{Ker}\phi \in \mathcal{X}^{\perp}$ (respectively a special $\mathcal{X}$-preenvelope $M \xrightarrow{\phi} X$ of M such that $\operatorname{Coker}\phi \in {}^{\perp}\mathcal{X}$).

Another important result proved in [1, Lemmas 2.3, 2.4] states that if U is an n-cotilting module, then $\mathcal{X} \cap \mathcal{X}^{\perp} = \operatorname{Prod} U$ and if T is an n-tilting module, then $\mathcal{X} \cap {}^{\perp}\mathcal{X} = \operatorname{Add} T$. We will use this result throughout the paper.

For any R-module M of injective dimension at most n, we choose an injective resolution

$$0 \to M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \to \cdots \xrightarrow{f_n} I_n \to 0,$$

where for every $j \geqslant 0$, I_j is injective and we let $C_j = \operatorname{Ker} f_{j+1}$ for every $j \geqslant 0$.

Lemma 3.3. *In the above notations, we have ${}^{\perp_{i+k}}M = {}^{\perp_i}C_k$ and ${}^{\perp_{\geqslant i+k}}M = {}^{\perp_{\geqslant i}}C_k$, for every $i \geqslant 1$, $k \geqslant 0$.*

Proof. It follows immediately by considering the long exact sequences induced by applying the functor $\operatorname{Hom}_R(-, M)$ to the short exact sequences $0 \to C_r \to I_r \to C_{r+1} \to 0$, for each $r \geqslant 0$. □

We now turn to the classes ${}^{\perp_{\geqslant j}}U$ and $T^{\perp_{\geqslant j}}$ defined above.

Lemma 3.4. *Assume that U is an n-cotilting R-module and let $\mathcal{X}_j = {}^{\perp_{\geqslant j}}U$. For every $j \geqslant 2$, $\mathcal{X}_j$ consists of the R-modules M such that there exists an exact sequence of the form*

$$0 \to U_{j-1} \to X_{j-2} \to \cdots \to X_1 \to X_0 \to M \to 0, \tag{1}$$

where $U_{j-1} \in \operatorname{Prod} U$ and $X_i \in {}^{\perp_\infty}U$, for every $0 \leqslant i \leqslant j-2$. In particular, $\mathcal{X}_j$ is closed under products, for every $j \geqslant 1$.

Dually, assume that T is an n-tilting R-module and let $\mathcal{X}_j = T^{\perp_{\geqslant j}}$. For every $j \geqslant 2$, $\mathcal{X}_j$ consists of the R-modules M such that there exists an exact sequence of the form

$$0 \to M \to X_0 \to X_1 \to \cdots \to X_{j-2} \to T_{j-1} \to 0, \tag{1}$$

where $T_{j-1} \in \operatorname{Add} T$ and $X_i \in T^{\perp_\infty}$, for every $0 \leqslant i \leqslant j-2$. In particular, $\mathcal{X}_j$ is closed under direct sums, for every $j \geqslant 1$.

Proof. By a dimension shifting argument it is immediate to check that the sequence (1) yields $\operatorname{Ext}_R^{i+j-1}(M,U) \cong \operatorname{Ext}_R^i(U_{j-1},U) = 0$ for every $i \geqslant 1$, hence $M \in \mathcal{X}_j$. To prove the converse we proceed by induction on j. Let $j = 2$ and let $M \in \mathcal{X}_2$. Consider a special precover $0 \to Y \to X \to M \to 0$ of M where $X \in \mathcal{X}$, $Y \in \mathcal{X}^\perp$. Clearly, $\operatorname{Ext}_R^i(Y,U) \cong \operatorname{Ext}_R^{i+1}(M,U)$, for every $i \geqslant 1$. Thus, $Y \in {}^{\perp_\infty}U = \mathcal{X}$, hence $Y \in \mathcal{X} \cap \mathcal{X}^\perp$ which coincides with $\operatorname{Prod} U$ by [1, Lemmas 2.3, 2.4]. So $0 \to Y \to X \to M \to 0$ is a sequence of type (1) for M. Assuming the statement true for any $2 \leqslant k \leqslant j$, we prove it for $j+1$. Let $M \in \mathcal{X}_{j+1}$ and let $0 \to Y \to X' \to M \to 0$ be a special $\mathcal{X}$-precover of M. Since $X' \in \mathcal{X}$ and $M \in {}^{\perp_{\geqslant j+1}}U$, it is evident that $Y \in {}^{\perp_{\geqslant j}}U = \mathcal{X}_j$. Thus, by induction, there exists a sequence

$$0 \to U_{j-1} \to X_{j-2} \to \cdots \to X_0 \to Y \to 0, \tag{2}$$

with $U_{j-i} \in \operatorname{Prod} U$, $X_i \in {}^{\perp_\infty}U$. From (2) we obtain the sequence

$$0 \to U'_j \to X'_{j-1} \to \cdots \to X'_1 \to X'_0 \to M \to 0,$$

where $U'_j = U_{j-1}$, $X'_{i+1} = X_i$, for $1 \leqslant i \leqslant j-2$, $X'_0 = X'$ which satisfies the wanted conditions.

To prove the second statement note that, by [12, Lemma 3.2], $\mathcal{X} = \mathcal{X}_1$ is closed under products. Let now $\{M_\alpha\}_{\alpha\in\Lambda}$ be a family of modules belonging to $\mathcal{X}_j$, for $j \geqslant 2$. By the first part of the proof, for each α, there exist sequences

$$0 \to U_{j-1,\alpha} \to X_{j-2,\alpha} \to \cdots \to X_{0,\alpha} \to M_\alpha \to 0;$$

hence we obtain the sequence

$$0 \to \prod_\alpha U_{j-1,\alpha} \to \prod_\alpha X_{j-2,\alpha} \to \cdots \to \prod_\alpha X_{0,\alpha} \to \prod_\alpha M_\alpha \to 0.$$

Since $\mathcal{X}$ is closed under products, the sequence (2) shows that $\prod_\alpha M_\alpha \in \mathcal{X}_j$.
The dual statement is easily seen to be true. □

Using the preceding lemma we can now prove the following result.

Proposition 3.5. *Let U be an n-cotilting R-module. Let E be an injective cogenerator of R-*Mod *for which condition* (C3) *is satisfied, i.e., E fits in the exact sequence*

$$0 \to U_r \to \cdots \to U_1 \to U_0 \to E \to 0,$$

with $U_i \in \operatorname{Prod} U$, for every $0 \leqslant i \leqslant r$. Then $r \leqslant \operatorname{id} U$ can be chosen and the minimal length r of any such sequence is exactly $\operatorname{id} U$.

Dually, let T be an n-tilting R-module. Consider the exact sequence given by condition (T3)

$$0 \to R \to T_0 \to T_1 \to \cdots \to T_r \to 0,$$

where $T_i \in \operatorname{Add} T$, for every $0 \leqslant i \leqslant r$. Then $r \leqslant \operatorname{pd} T$ can be chosen and the minimal length r of any such sequence is exactly $\operatorname{pd} T$.

Proof. The fact that r can be chosen so that $r \leqslant \operatorname{id} U$ is well-known (see [13]), but for convenience we recall its proof. Consider the sequence

$$0 \to U_r \xrightarrow{f_r} \cdots \to U_1 \xrightarrow{f_1} U_0 \xrightarrow{f_0} E \to 0,$$

satisfying condition (C3) and assume $r > \operatorname{id} U$. Let $K_{i+1} = \operatorname{Ker} f_i$, hence $K_r = U_r \in \mathcal{X}^\perp$, where $\mathcal{X} = {}^{\perp_\infty} U$. Since $\mathcal{X}^\perp$ is closed under cokernels of monomorphisms, we get $K_i \in \mathcal{X}^\perp$, for every $1 \leqslant i \leqslant r$; thus, in particular, if $m = \operatorname{id} U$, $K_m \in \mathcal{X}^\perp$. By dimension shifting we have

$$\operatorname{Ext}^i_R(K_m, U) \cong \operatorname{Ext}^{i+m-1}_R(K_1, U) \cong \operatorname{Ext}^{i+m}_R(E, U),$$

for every $i \geqslant 1$; hence $K_m \in \mathcal{X}$. Since, $\mathcal{X} \cap \mathcal{X}^\perp = \operatorname{Prod} U$ (see [1]), we conclude that $K_m \in \operatorname{Prod} U$ and thus $r = m$ can be chosen in the above sequence.

We show now that r cannot be strictly smaller than $\operatorname{id} U$. Assume $\operatorname{id} U = m$ and $r < m$. By dimension shifting we obtain, as above,

$$0 = \operatorname{Ext}^i_R(K_r, U) \cong \operatorname{Ext}^{i+r}_R(E, U),$$

for every $i \geqslant 1$; hence $E \in \mathcal{X}_{r+1} = {}^{\perp_{\geqslant r+1}} U$. By Lemma 3.4, $\mathcal{X}_{r+1}$ is closed under products, hence $E^\gamma \in \mathcal{X}_{r+1}$, for every cardinal γ. Let N be an arbitrary R-module; since E is an injective cogenerator, there exists an exact sequence

$$0 \to N \to E^{\gamma_0} \to N_1 \to 0,$$

which gives rise to the exact sequence

$$0 = \mathrm{Ext}_R^{r+1}\big(E^{\gamma_0}, U\big) \to \mathrm{Ext}_R^{r+1}(N, U) \to \mathrm{Ext}_R^{r+2}(N_1, U).$$

Repeating the same argument we can embed N_1 in a product E^{γ_1} with cokernel N_2 obtaining the exact sequence

$$0 = \mathrm{Ext}_R^{r+2}\big(E^{\gamma_1}, U\big) \to \mathrm{Ext}_R^{r+2}(N_1, U) \to \mathrm{Ext}_R^{r+3}(N_2, U).$$

After $0 < k = m - r$ steps we obtain

$$0 = \mathrm{Ext}_R^{r+k}\big(E^{\gamma_{k-1}}, U\big) \to \mathrm{Ext}_R^{r+k}(N_{k-1}, U) \to \mathrm{Ext}_R^{r+k+1}(N_k, U).$$

Since $\mathrm{id}\,U = m$, $\mathrm{Ext}_R^{r+k+1}(N_k, U) = 0$, hence going back k steps we conclude that $\mathrm{Ext}_R^{r+1}(N, U) = 0$. Since N was arbitrary, we get the contradiction $\mathrm{id}\,U \leqslant r$.

The dual result is proved by dual arguments. □

Definition 5. For every R-module U denote by $\mathrm{Cogen}_n U$ the class consisting of the R-modules M for which there exists an exact sequence of the form

$$0 \to M \to U^{\alpha_1} \to U^{\alpha_2} \to \cdots \to U^{\alpha_n}$$

for some cardinals α_i; and by $\mathrm{Cogen}_\infty U$ the class of R-modules M for which there exists an infinite exact sequence of the form

$$0 \to M \to U^{\alpha_1} \to U^{\alpha_2} \to \cdots \to U^{\alpha_n} \to \cdots$$

for some cardinals α_i.

Dually, for every R-module T denote by $\mathrm{Gen}_n T$ the class consisting of the R-modules M for which there exists an exact sequence of the form

$$T^{(\alpha_n)} \to \cdots \to T^{(\alpha_2)} \to T^{(\alpha_1)} \to M \to 0$$

for some cardinals α_i; and by $\mathrm{Gen}_\infty T$ the class of R-modules M for which there exists an infinite exact sequence of the form

$$\cdots \to T^{(\alpha_n)} \to \cdots \to T^{(\alpha_2)} \to T^{(\alpha_1)} \to M \to 0$$

for some cardinals α_i.

First, we note the following.

Proposition 3.6. *Let U be an n-cotilting R-module. Then* ${}^{\perp_\infty}U = \mathrm{Cogen}_n U$. *Moreover*, $\mathrm{Cogen}_n U = \mathrm{Cogen}_{n+k} U = \mathrm{Cogen}_\infty U$, *for every $k \geqslant 0$.*

If T is an n-tilting R-module, then $T^{\perp_\infty} = \mathrm{Gen}_n T$. *Moreover*, $\mathrm{Gen}_n T = \mathrm{Gen}_{n+k} T = \mathrm{Gen}_\infty T$, *for every $k \geqslant 0$.*

Proof. By Lemma 3.2, every module M in $\mathcal{X} = {}^{\perp_\infty}U$ fits in an infinite exact sequence of the form

$$0 \to M \to U^{\alpha_1} \to U^{\alpha_2} \to \cdots \to U^{\alpha_i} \to \cdots$$

for some cardinals α_i, thus there exists also a sequence of the same type and of length n. So $\mathcal{X} \subseteq \mathrm{Cogen}_n U$. For the other implication, let $M \in \mathrm{Cogen}_n U$ and consider an exact sequence

$$0 \to M \xrightarrow{f_1} U^{\gamma_1} \xrightarrow{f_2} U^{\gamma_2} \to \cdots \xrightarrow{f_n} U^{\gamma_n}.$$

Let $L_i = \mathrm{Coker}\, f_i$; by dimension shifting, $\mathrm{Ext}^i_R(M, U) \cong \mathrm{Ext}^{i+n}(L_n, U)$, for every $i \geqslant 1$. Hence, $M \in \mathcal{X}$, since $\mathrm{id}\, U \leqslant n$. Thus, $\mathrm{Cogen}_n U = \mathcal{X}$. To prove the second statement, note that, clearly, $\mathrm{Cogen}_\infty U \subseteq \mathrm{Cogen}_{n+k} U \subseteq \mathrm{Cogen}_n U$. Conversely, if $M \in \mathrm{Cogen}_n U$, then $M \in \mathcal{X}$, hence as noted at the beginning of the proof, $M \in \mathrm{Cogen}_\infty U$. The statement about n-tilting modules is proved dually. □

Our next goal is to prove the converse of Proposition 3.6. The final result (see Theorem 3.11) will be proved in several steps. First, we need two lemmas.

Lemma 3.7. *Let M be an R-module and let $0 \to A \to B \xrightarrow{\pi} C \to 0$ be an exact sequence. If $A, C \in \mathrm{Cogen}\, M$ and $C \in {}^{\perp}M$, then $B \in \mathrm{Cogen}\, M$.*

Dually, if $A, C \in \mathrm{Gen}\, M$ and $A \in M^{\perp}$, then $B \in \mathrm{Gen}\, M$.

Proof. Let $0 \neq x \in B$; if $x \in A$, then there exists $f \in \mathrm{Hom}_R(A, M)$ such that $f(x) \neq 0$. Since $\mathrm{Ext}^1_R(C, M) = 0$, f is extendible to a map $f' : B \to M$, hence $f'(x) \neq 0$. If $x \notin A$, then $\pi(x) \neq 0$. Since $C \in \mathrm{Cogen}\, M$, there is a map $g : C \to M$ such that $g(\pi(x)) \neq 0$. Thus, $g' = g \circ \pi \in \mathrm{Hom}_R(B, M)$ and $g'(x) \neq 0$. The dual statement is proved accordingly. □

A stronger version of the preceding lemma is given by the following.

Lemma 3.8. *Let M be an R-module such that $M^\lambda \in {}^{\perp_\infty}M$ for every cardinal λ and ${}^{\perp_\infty}M \subseteq \mathrm{Cogen}\, M$. Let $0 \to A \to B \to C \to 0$ be an exact sequence. If $A \in \mathrm{Cogen}_m M$ and $C \in {}^{\perp_\infty}M$, then $B \in \mathrm{Cogen}_m M$, for every $m \geqslant 1$.*

Dually, let M be an R-module such that $M^{(\lambda)} \in M^{\perp_\infty}$ for every cardinal λ and $M^{\perp_\infty} \subseteq \mathrm{Gen}\, M$. Let $0 \to A \to B \to C \to 0$ be an exact sequence. If $C \in \mathrm{Gen}_m M$ and $A \in M^{\perp_\infty}$, then $B \in \mathrm{Gen}_m M$, for every $m \geqslant 1$.

Proof. The proof is by induction on m. The case $m = 1$ follows by Lemma 3.7. Assuming the result true for any $1 \leqslant j \leqslant m$, we prove it for $m + 1$. Consider an exact sequence

$$0 \to A \to B \xrightarrow{\pi} C \to 0,$$

where $A \in \operatorname{Cogen}_{m+1} M$ and $C \in {}^{\perp_\infty} M$. Choose a sequence

$$0 \to A \xrightarrow{\mu} M^J \to A_1 \to 0$$

with $A_1 \in \operatorname{Cogen}_m M$. Since, $\operatorname{Ext}^1_R(C, M^J) \cong \prod_J \operatorname{Ext}^1_R(C, M) = 0$, we have an epimorphism

$$\operatorname{Hom}_R\bigl(B, M^J\bigr) \to \operatorname{Hom}_R\bigl(A, M^J\bigr) \to 0,$$

thus μ is extendible to a map $\rho : B \to M^J$. Our hypotheses now allow to apply Lemma 3.1(i); hence there exists an exact sequence

$$0 \to C \xrightarrow{\nu} M^I \to C_1 \to 0,$$

where $C_1 \in {}^{\perp_\infty} M$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & M^J & \longrightarrow & M^J \oplus M^I & \longrightarrow & M^I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & N & \longrightarrow & C_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array},$$

where ϕ is defined by $\phi(b) = \rho(b) + (\nu \circ \pi)(b)$, for every $b \in B$, and the third row is obtained by letting $N = \operatorname{Coker} \phi$. In the third row we have $A_1 \in \operatorname{Cogen}_m M$, $C_1 \in {}^{\perp_\infty} M$, hence, by inductive hypothesis, $N \in \operatorname{Cogen}_m M$. Thus, the second column yields $B \in \operatorname{Cogen}_{m+1} M$.

The dual statement is proved accordingly. □

We can now prove the following result.

Lemma 3.9. *Let M be an R-module such that ${}^{\perp_\infty} M = \operatorname{Cogen}_n M$. Then M is a partial n-cotilting R-module.*

Dually, let M be an R-module such that $M^{\perp_\infty} = \operatorname{Gen}_n M$. Then M is a partial n-tilting R-module.

Proof. We have to show that M satisfies conditions (C1) and (C2) of Definition 4. Clearly, M satisfies (C2), since $M^\lambda \in \mathrm{Cogen}_n M$ for every cardinal λ. We show now that $\mathrm{id}\, M \leqslant n$. Let N be an arbitrary R-module and consider a projective resolution of N:

$$P_j \xrightarrow{f_j} P_{j-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} N \to 0;$$

let $K_{m+1} = \mathrm{Ker}\, f_m$ for every $m \leqslant j$. For every projective module P_i in the above sequence there is a cardinal α_i and an exact sequence

$$0 \to P_i \to M^{\alpha_i} \to C_i \to 0 \qquad (\mathrm{E}_i)$$

with $C_i \in {}^{\perp_\infty}M$. In fact, $P_i \in {}^{\perp_\infty}M$, hence, by hypothesis, $P_i \in \mathrm{Cogen}_n M \subseteq \mathrm{Cogen}\, M$. Applying Lemma 3.1(i), we obtain $C_i \in {}^{\perp_\infty}M$. We now claim:

(A) $K_m \in \mathrm{Cogen}_m M$, for every m.

We prove the claim by induction on m. If $m = 1$, then $K_1 \subseteq P_0 \in \mathrm{Cogen}\, M$. Assume the claim true for every $1 \leqslant j \leqslant m$. The sequences

$$0 \to K_{m+1} \to P_m \to K_m \to 0$$

and (E_i) yield the exact sequence

$$0 \to \frac{P_m}{K_{m+1}} \to \frac{M^{\alpha_m}}{K_{m+1}} \to C_m \to 0.$$

Since $P_m/K_{m+1} \cong K_m$ and $C_m \in {}^{\perp_\infty}M$, the inductive step and Lemma 3.8 allow to conclude that $M^{\alpha_m}/K_{m+1} \in \mathrm{Cogen}_m M$. It follows that $K_{m+1} \in \mathrm{Cogen}_{m+1} M$ and claim (A) is proved.

In particular, $K_n \in \mathrm{Cogen}_n M$, hence $K_n \in {}^{\perp_\infty}M$. Applying a dimension shifting argument to the projective resolution of N considered at the beginning of the proof, we obtain $\mathrm{Ext}_R^{n+1}(N, M) \cong \mathrm{Ext}_R^1(K_n, M) = 0$; hence $\mathrm{id}\, M \leqslant n$, since N was arbitrary.

The dual statement is proved similarly, starting with an injective resolution. □

Proposition 3.10. *Let M be an R-module such that ${}^{\perp_\infty}M = \mathrm{Cogen}_n M$. Then M is an n-cotilting R-module.*

Dually, let M be an R-module such that $M^{\perp_\infty} = \mathrm{Gen}_n M$. Then M is an n-tilting R-module.

Proof. In view of Lemma 3.9, M is a partial n-cotilting R-module and, moreover, ${}^{\perp_\infty}M \subseteq \mathrm{Cogen}\, M$. Thus, as remarked in the last four lines of [1], the proof of [1, Proposition 3.3] carries over giving the conclusion that M is n-cotilting.

The dual statement is proved analogously, but applying [1, Theorem 4.4]. □

We can now state our main result which follows immediately by Propositions 3.6 and 3.10.

Theorem 3.11. *Let U be an R-module. Then U is n-cotilting if and only if* $\mathrm{Cogen}_n U = {}^{\perp_\infty}U$. *Dually, an R-module T is n-tilting if and only if* $\mathrm{Gen}_n T = T^{\perp_\infty}$.

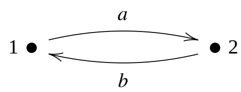
In Section 2 we recalled that there are three equivalent definitions of 1-tilting and 1-cotilting modules. We ask now whether Definitions 3 and *3 have a correspondent formulation for n-tilting and n-cotilting modules.

To this aim, consider the conditions:

(C3′) $\mathrm{Hom}_R(N, U) = 0$ and $\mathrm{Ext}^i_R(N, U) = 0$, for every $i \geqslant 1$ imply $N = 0$.
(T3′) $\mathrm{Hom}_R(T, N) = 0$ and $\mathrm{Ext}^i_R(T, N) = 0$, for every $i \geqslant 1$ imply $N = 0$.

It is immediate to check that if U is an n-cotilting module, then U satisfies (C1), (C2), (C3′); analogously, if T is an n-tilting module, then T satisfies (T1), (T2), (T3′) but the converse is not true as it is shown by the following example due to G. D'Este.

Example 1 (*G. D'Este* [8]). Let R denote the K-algebra given by the quiver



with relation $ab = 0$. Let $U = {}^{2}_{1}$ be the indecomposable projective corresponding to the vertex 2. Then $\mathrm{id}\, U = 2$ and U satisfies (C2). U satisfies (C3′), since the simple module 2 is the unique indecomposable such that $\mathrm{Hom}_R(2, U) = 0$ and $\mathrm{Ext}^2_R(2, U) \cong \mathrm{Ext}^1_R(1, U) \neq 0$. But U is not 2-cotilting, since it is not faithful, hence it does not cogenerate R. Similarly, let, $T = {}^{1}_{2}$ be the indecomposable injective corresponding to the vertex 2. Then, $\mathrm{pd}\, T = 2$ and T satisfies (T1), (T2). T satisfies (T3′), since the simple module 2 is the unique indecomposable such that $\mathrm{Hom}_R(T, 2) = 0$ and $\mathrm{Ext}^2_R(T, 2) \cong \mathrm{Ext}^1_R(T, 1) \neq 0$. But T is not 2-tilting, since it does not generate the injective envelope of R.

The next result shows that conditions (C3) and (T3) can be replaced by ${}^{\perp_\infty}U \subseteq \mathrm{Cogen}\, U$ and $T^{\perp_\infty} \subseteq \mathrm{Gen}\, T$, respectively.

Lemma 3.12. *Let U be a partial n-cotilting R-module. Then*:

(1) $\mathrm{Cogen}_n U \subseteq {}^{\perp_\infty}U$.
(2) *U is n-cotilting if and only if* ${}^{\perp_\infty}U \subseteq \mathrm{Cogen}\, U$.

Dually, let T be a partial n-tilting R-module. Then:

(*1) $\mathrm{Gen}_n T \subseteq T^{\perp_\infty}$.
(*2) *T is n-tilting if and only if* $T^{\perp_\infty} \subseteq \mathrm{Gen}\, U$.

Proof. 1. Let $M \in \mathrm{Cogen}_n U$ and consider an exact sequence

$$0 \to M \xrightarrow{f_0} U^{\alpha_1} \xrightarrow{f_1} U^{\alpha_2} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} U^{\alpha_n}.$$

Let $M_{i+1} = \mathrm{Coker}\, f_i$. By dimension shifting, $\mathrm{Ext}^i_R(M, U) \cong \mathrm{Ext}^{n+i}_R(M_n, U)$, for every $i \geqslant 1$; hence $M \in {}^{\perp_\infty}U$. Statement (2) is proved in [1, p. 249].

Dually, (*1) is proved by a dimension shifting argument and (*2) is condition (ii) of Theorem 4.4 in [1]. □

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