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A note on inversion of Toeplitz matrices☆

Xiao-Guang Lv, Ting-Zhu Huang*

School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan, 610054, PR China

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Abstract

It is shown that the invertibility of a Toeplitz matrix can be determined through the solvability of two standard equations. The inverse matrix can be denoted as a sum of products of circulant matrices and upper triangular Toeplitz matrices. The stability of the inversion formula for a Toeplitz matrix is also considered.

Keywords: Toeplitz matrix; Circulant matrix; Inversion; Algorithm

1. Introduction

Let T be an n-by-n Toeplitz matrix:

$$T = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{1-n} \\ a_1 & a_0 & a_{-1} & \cdots & a_{2-n} \\ a_2 & a_1 & a_0 & \cdots & a_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix},$$

where $a_{-(n-1)}, \ldots, a_{n-1}$ are complex numbers. We use the shorthand

$$T = (a_{p-q})_{p,q=1}^{n}$$

for a Toeplitz matrix.

The inversion of a Toeplitz matrix is usually not a Toeplitz matrix. A very important step is to answer the question of how to reconstruct the inversion of a Toeplitz matrix by a low number of its columns and the entries of the original Toeplitz matrix. It was first observed by Trench [1] and rediscovered by Gohberg and Semencul [2] that T^{-1} can be reconstructed from its first and last columns provided that the first component of the first column does not vanish. Gohberg and Krupnik [3] observed that T^{-1} can be recovered from its first and second columns if the last component of the first column does not vanish.

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* Corresponding author.
E-mail address: tzhuang@uestc.edu.cn (T.-Z. Huang).

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In Heinig and Rost [4], an inversion formula was exhibited for every nonsingular Toeplitz matrix. The method requires the solution of linear systems of equations (the so-called fundamental equations), where the right-hand side of one of them is a shifted column of the Toeplitz matrix T. In [5], Ben-Artzi and Shalom proved that three columns of the inverse of a Toeplitz matrix, when properly chosen, are always enough to reconstruct the inverse. Labahn and Shalom [6], and Ng, Rost and Wen [7] presented modifications of this result. In [8], Georg Heinig discussed the problem of the reconstruction of Toeplitz matrix inverses from columns.

In this work, we give a new Toeplitz matrix inversion formula. The inverse matrix T^{-1} can be denoted as a sum of products of circulant matrices and upper triangular Toeplitz matrices. The results obtained show that this formula is numerically forward stable.

2. Toeplitz inversion formula

Lemma 1. *Let $T = (a_{p-q})_{p,q=1}^{n}$ be a $n \times n$ Toeplitz matrix; then it satisfies the formula*

$$KT - TK = fe_n^{\mathrm{T}} - e_1 f^{\mathrm{T}} J,$$

where

$$K = \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix},$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 \\ a_{n-1} - a_{-1} \\ \vdots \\ a_2 - a_{-n+2} \\ a_1 - a_{-n+1} \end{pmatrix}.$$

Theorem 1. *Let $T = (a_{p-q})_{p,q=1}^{n}$ be a Toeplitz matrix. If each of the systems of equations $Tx = f, Ty = e_1$ is solvable, $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $y = (y_1, y_2, \ldots y_n)^{\mathrm{T}}$, then*

(a) *T is invertible;*

(b) *$T^{-1} = T_1U_1 + T_2U_2$, where*

$$T_1 = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \ddots & \\ \vdots & \ddots & \ddots & y_n \\ y_n & \cdots & y_2 & y_1 \end{bmatrix}, \qquad U_1 = \begin{bmatrix} 1 & -x_n & \cdots & -x_2 \\ & 1 & \ddots & \vdots \\ & & \ddots & -x_n \\ & & & 1 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_n \\ x_n & \cdots & x_2 & x_1 \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} o & y_n & \cdots & y_2 \\ & o & \ddots & \vdots \\ & & \ddots & y_n \\ & & & o \end{bmatrix}.$$

Proof. From Lemma 1 and $Tx = f, Ty = e_1$, we have

$$\begin{aligned} KT &= TK + fe_n^{\mathrm{T}} - e_1 f^{\mathrm{T}} J \\ &= T[K + xe_n^{\mathrm{T}} - yf^{\mathrm{T}} J], \end{aligned}$$

then,

$$\begin{aligned} K^i T &= K^{i-1} T[K + x e_n^{\mathrm{T}} - y f^{\mathrm{T}} J] \\ &= \cdots \\ &= T[K + x e_n^{\mathrm{T}} - y f^{\mathrm{T}} J]^i. \end{aligned}$$

Therefore,

$$K^i e_1 = K^i T y = T[K + x e_n^{\mathrm{T}} - y f^{\mathrm{T}} J]^i y.$$

Let

$$t_i = [K + x e_n^{\mathrm{T}} - y f^{\mathrm{T}} J]^{i-1} y \quad \text{and} \quad \hat{T} = (t_1, t_2, \dots, t_n).$$

Then

$$\begin{aligned} &T t_i = T[K + x e_n^{\mathrm{T}} - y f^{\mathrm{T}} J]^{i-1} y = K^{i-1} e_1 = e_i, \\ &T \hat{T} = T(t_1, t_2, \dots, t_n) = (e_1, e_2, \dots, e_n) = I_n. \end{aligned}$$

So the matrix T is invertible, and the inverse of T is the matrix $\hat{T} = T^{-1}$.

For (b): First of all, it is easy to see that

$$\begin{aligned} &t_1 = y, \qquad t_i = [K + x e_n^{\mathrm{T}} - y f^{\mathrm{T}} J] t_{i-1} \quad (i = 1, 2, \dots, n), \\ &t_i = T^{-1} e_i, \qquad J e_i = e_{n-i+1}, \\ &J T J = T^{\mathrm{T}}, \qquad J J = I, \quad J^{\mathrm{T}} = J. \end{aligned}$$

Then, for $i > 1$

$$\begin{aligned} t_i &= K t_{i-1} + x e_n^{\mathrm{T}} t_{i-1} - y f^{\mathrm{T}} J t_{i-1} \\ &= K t_{i-1} + x e_n^{\mathrm{T}} T^{-1} e_{i-1} - y f^{\mathrm{T}} J T^{-1} e_{i-1} \\ &= K t_{i-1} + x e_n^{\mathrm{T}} J J T^{-1} J e_{n-i+2} - y f^{\mathrm{T}} T^{-T} J e_{i-1} \\ &= K t_{i-1} + x e_1^{\mathrm{T}} T^{-T} e_{n-i+2} - y f^{\mathrm{T}} T^{-T} e_{n-i+2} \\ &= K t_{i-1} + x y^{\mathrm{T}} e_{n-i+2} - y x^{\mathrm{T}} e_{n-i+2} \\ &= K t_{i-1} + y_{n-i+2} x - x_{n-i+2} y. \end{aligned}$$

So we have

$$\begin{aligned} &t_1 = y, \qquad t_2 = K y + y_n x - x_n y, \\ &\dots, \\ &t_n = K^{n-1} y + K^{n-2} x y_n - K^{n-2} y x_n + \cdots + x y_2 - y x_2 \\ &T^{-1} = (t_1, t_2, \dots t_n) \\ &\quad = (y, K y, \dots, K^{n-1} y) \begin{bmatrix} 1 & -x_n & \cdots & -x_2 \\ & 1 & \ddots & \vdots \\ & & \ddots & -x_n \\ & & & 1 \end{bmatrix} + (x, K x, \dots, K^{n-1} x) \begin{bmatrix} o & y_n & \cdots & y_2 \\ & o & \ddots & \vdots \\ & & \ddots & y_n \\ & & & o \end{bmatrix} \\ &\quad = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \ddots & \\ \vdots & \ddots & \ddots & y_n \\ y_n & \cdots & y_2 & y_1 \end{bmatrix} \begin{bmatrix} 1 & -x_n & \cdots & -x_2 \\ & 1 & \ddots & \vdots \\ & & \ddots & -x_n \\ & & & 1 \end{bmatrix} + \begin{bmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_n \\ x_n & \cdots & x_2 & x_1 \end{bmatrix} \begin{bmatrix} o & y_n & \cdots & y_2 \\ & o & \ddots & \vdots \\ & & \ddots & y_n \\ & & & o \end{bmatrix}. \quad \square \end{aligned}$$

Remark. In Theorem 1, let the Toeplitz matrix $T = (a_{p-q})_{p,q=1}^{n}$ be a circulant Toeplitz matrix. That is to say, the elements of the matrix $T = (a_{p-q})_{p,q=1}^{n}$ satisfy $a_i = a_{i-n}$ for all $i = 1, \ldots, n-1$. It is easy to see that $f = 0$. Thus, $x = T^{-1} f = 0$. From (b) of Theorem 1, we get

$$T^{-1} = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \ddots & \\ \vdots & \ddots & \ddots & y_n \\ y_n & \cdots & y_2 & y_1 \end{bmatrix}.$$

We conclude: The inverses of the circulant Toeplitz matrices are circulant Toeplitz matrices.

3. Stability analysis

In this section, we will show that the Toeplitz inversion formula presented in Section 2 is evaluation forward stable. An algorithm is called forward stable if for all well conditioned problems, the computed solution $\tilde{x}$ is close to the true solution x in the sense that the relative error $\|x - \tilde{x}\|_2 / \|x\|_2$ is small. In the matrix computation, round-off errors occur. Let $A, B \in C^{n,n}$ and $\alpha \in C$. If we neglect the $O(\varepsilon^2)$ terms, then for any floating-point arithmetic with machine precision ε, then (cf. [9])

$$\text{fl}(\alpha A) = \alpha A + E, \qquad \|E\|_F \le \varepsilon\, |\alpha|\, \|A\|_F \le \varepsilon \sqrt{n}\, |\alpha|\, \|A\|_2\,,$$
$$\text{fl}(A + B) = A + B + E, \qquad \|E\|_F \le \varepsilon\, \|A + B\|_F \le \varepsilon \sqrt{n}\, \|A + B\|_2\,,$$
$$\text{fl}(AB) = AB + E, \qquad \|E\|_F \le \varepsilon n\, \|A\|_F\, \|B\|_F\,.$$

According to the floating-point arithmetic, we have the following bound.

Theorem 2. *Let $T = (a_{p-q})_{p,q=1}^{n}$ be a nonsingular Toeplitz matrix and be well conditioned; then the formula in Theorem 1 is forward stable.*

Proof. Assume that we have computed the solutions $\tilde{x}$, $\tilde{y}$ in Theorem 1 which are perturbed by the normwise relative errors bounded by $\tilde{\varepsilon}$,

$$\|\tilde{x}\|_2 \le \|x\|_2\,(1 + \tilde{\varepsilon}), \qquad \|\tilde{y}\|_2 \le \|y\|_2\,(1 + \tilde{\varepsilon}).$$

Therefore, we have

$$\|T_1\|_F = \sqrt{n}\, \|y\|_2\,, \qquad \|T_2\|_F = \sqrt{n}\, \|x\|_2\,,$$
$$\|U_1\|_F \le \sqrt{n}\sqrt{1 + \|x\|_2^2}, \qquad \|U_2\|_F \le \sqrt{n}\, \|y\|_2\,.$$

Using the perturbed solutions $\tilde{x}$, $\tilde{y}$, the inversion formula in Theorem 1 can be expressed as

$$\begin{aligned} \tilde{T}^{-1} &= fl\left(\tilde{T}_1 \tilde{U}_1 + \tilde{T}_2 \tilde{U}_2\right) \\ &= fl\left((T_1 + \Delta T_1)(U_1 + \Delta U_1) + (T_2 + \Delta T_2)(U_2 + \Delta U_2)\right) \\ &= T^{-1} + \Delta T_1 U_1 + T_1 \Delta U_1 + \Delta T_2 U_2 + T_2 \Delta U_2 + E + F. \end{aligned}$$

Here, E is the matrix containing the error which results from computing the matrix products, and F contains the error from subtracting the matrices. For the error matrices ΔT_1, ΔU_1, ΔT_2 and ΔU_2, we have

$$\|\Delta T_1\|_F \le \tilde{\varepsilon}\, \|T_1\|_F = \tilde{\varepsilon} \sqrt{n}\, \|y\|_2\,,$$
$$\|\Delta T_2\|_F \le \tilde{\varepsilon}\, \|T_2\|_F = \tilde{\varepsilon} \sqrt{n}\, \|x\|_2\,,$$
$$\|\Delta U_1\|_F \le \tilde{\varepsilon}\, \|U_1\|_F \le \tilde{\varepsilon} \sqrt{n}\sqrt{1 + \|x\|_2^2},$$
$$\|\Delta U_2\|_F \le \tilde{\varepsilon}\, \|U_2\|_F \le \tilde{\varepsilon} \sqrt{n}\, \|y\|_2\,.$$

It follows that

$$\begin{aligned}\|E\|_2 &\leq \|E\|_F \\ &\leq \varepsilon n(\|T_1\|_F \|U_1\|_F + \|T_2\|_F \|U_2\|_F) \\ &\leq \varepsilon n^2 \|y\|_2 \left(\sqrt{1+\|x\|_2^2} + \|x\|_2\right) \\ &\leq \varepsilon n^2 \|y\|_2 (1 + 2\|x\|_2), \qquad \|F\|_2 \leq \varepsilon\sqrt{n} \left\|T^{-1}\right\|_2.\end{aligned}$$

Adding all these error bounds, we have

$$\left\|\tilde{T}^{-1} - T^{-1}\right\|_2 \leq n(2\tilde{\varepsilon} + n\varepsilon) \|y\|_2 (1 + 2\|x\|_2) + \varepsilon\sqrt{n} \left\|T^{-1}\right\|_2.$$

Note that $Ty = e_1$ and $Tx = f$; then

$$\|y\|_2 \leq \left\|T^{-1}\right\|_2 \quad \text{and} \quad \|x\|_2 \leq \left\|T^{-1}\right\|_2 \|f\|_2.$$

Thus, the relative error is

$$\frac{\left\|\tilde{T}^{-1} - T^{-1}\right\|_2}{\left\|T^{-1}\right\|_2} \leq n(2\tilde{\varepsilon} + n\varepsilon)\left(1 + 2\left\|T^{-1}\right\|_2 \|f\|_2\right) + \varepsilon\sqrt{n}.$$

As T is well conditioned, thus, $\left\|T^{-1}\right\|_2$ is finite. Obviously, $\|f\|_2$ is finite. Therefore, the formula presented in Theorem 1 is forward stable. □

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