A note on inversion of Toeplitz matrices

Xiao-Guang Lv, Ting-Zhu Huang*

School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan, 610054, PR China

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Abstract

It is shown that the invertibility of a Toeplitz matrix can be determined through the solvability of two standard equations. The inverse matrix can be denoted as a sum of products of circulant matrices and upper triangular Toeplitz matrices. The stability of the inversion formula for a Toeplitz matrix is also considered.

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1. Introduction

Let $T$ be an $n$-by-$n$ Toeplitz matrix:

$$
T = \begin{bmatrix}
    a_0 & a_{-1} & a_{-2} & \cdots & a_{1-n} \\
    a_1 & a_0 & a_{-1} & \cdots & a_{2-n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0
\end{bmatrix},
$$

where $a_{-(n-1)}, \ldots, a_{n-1}$ are complex numbers. We use the shorthand

$$
T = (a_{p-q})_{p,q=1}^n
$$

for a Toeplitz matrix.

The inversion of a Toeplitz matrix is usually not a Toeplitz matrix. A very important step is to answer the question of how to reconstruct the inversion of a Toeplitz matrix by a low number of its columns and the entries of the original Toeplitz matrix. It was first observed by Trench [1] and rediscovered by Gohberg and Semencul [2] that $T^{-1}$ can be reconstructed from its first and last columns provided that the first component of the first column does not vanish. Gohberg and Krupnik [3] observed that $T^{-1}$ can be recovered from its first and second columns if the last component of the first column does not vanish.

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* Corresponding author.

E-mail address: tzhuang@uestc.edu.cn (T.-Z. Huang).
In Heinig and Rost [4], an inversion formula was exhibited for every nonsingular Toeplitz matrix. The method requires the solution of linear systems of equations (the so-called fundamental equations), where the right-hand side of one of them is a shifted column of the Toeplitz matrix $T$. In [5], Ben-Artzi and Shalom proved that three columns of the inverse of a Toeplitz matrix, when properly chosen, are always enough to reconstruct the inverse. Labahn and Shalom [6], and Ng, Rost and Wen [7] presented modifications of this result. In [8], Georg Heinig discussed the problem of the reconstruction of Toeplitz matrix inverses from columns.

In this work, we give a new Toeplitz matrix inversion formula. The inverse matrix $T^{-1}$ can be denoted as a sum of products of circulant matrices and upper triangular Toeplitz matrices. The results obtained show that this formula is numerically forward stable.

2. Toeplitz inversion formula

**Lemma 1.** Let $T = (a_{p-q})_{p,q=1}^n$ be a $n \times n$ Toeplitz matrix; then it satisfies the formula

$$KT - TK = f e_n^T - e_1 f^T J,$$

where

$$K = \begin{bmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & & & & 1 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots \\ & & & & 1 \end{bmatrix},$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad f = \begin{pmatrix} a_{n-1} - a_{-1} \\ \vdots \\ a_2 - a_{-n+2} \end{pmatrix}.$$

**Theorem 1.** Let $T = (a_{p-q})_{p,q=1}^n$ be a Toeplitz matrix. If each of the systems of equations $Tx = f, Ty = e_1$ is solvable, $x = (x_1, x_2, \ldots, x_n)^T, y = (y_1, y_2, \ldots, y_n)^T$, then

(a) $T$ is invertible;
(b) $T^{-1} = T_1 U_1 + T_2 U_2$, where

$T_1 = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ y_n & \cdots & y_2 & y_1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ x_n & \cdots & x_2 & x_1 \end{bmatrix}$

and

$U_1 = \begin{bmatrix} 1 & -x_n & \cdots & -x_2 \\ & \ddots & \ddots & \ddots \\ & & 1 & -x_n \\ & & & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & y_n & \cdots & y_2 \\ & \ddots & \ddots & \ddots \\ & & \ddots & y_1 \\ & & & 0 \end{bmatrix}.$

**Proof.** From Lemma 1 and $Tx = f, Ty = e_1$, we have

$$KT = TK + f e_n^T - e_1 f^T J = T [K + x e_n^T - y f^T J],$$
then,
\[
K^i T = K^{i-1} T [K + xe_n^T - yf^T J]
\]
\[
= \ldots
\]
\[
= T [K + xe_n^T - yf^T J]^i.
\]
Therefore,
\[
K^i e_1 = K^i T y = T [K + xe_n^T - yf^T J]^i y.
\]
Let
\[
t_i = [K + xe_n^T - yf^T J]^i y \quad \text{and} \quad \hat{T} = (t_1, t_2, \ldots, t_n).
\]
Then
\[
T t_i = T [K + xe_n^T - yf^T J]^i y = K^{i-1} e_1 = e_i,
\]
\[
T \hat{T} = T (t_1, t_2, \ldots, t_n) = (e_1, e_2, \ldots, e_n) = I_n.
\]
So the matrix \( T \) is invertible, and the inverse of \( T \) is the matrix \( \hat{T} = T^{-1} \).

For (b): First of all, it is easy to see that
\[
t_1 = y, \quad t_i = [K + xe_n^T - yf^T J] t_{i-1} \quad (i = 1, 2, \ldots, n),
\]
\[
t_i = T^{-1} e_i, \quad Je_i = e_{n-i+1},
\]
\[
JTJ = T^T, \quad JJ = I, \quad J^T = J.
\]
Then, for \( i > 1 \)
\[
t_i = K t_{i-1} + xe_n^T t_{i-1} - yf^T J t_{i-1}
\]
\[
= K t_{i-1} + xe_n^T T^{-1} e_{i-1} - yf^T J T^{-1} e_{i-1}
\]
\[
= K t_{i-1} + xe_n^T J J T^{-1} J e_{n-i+2} - yf^T T^{-1} J e_{i-1}
\]
\[
= K t_{i-1} + xe_n^T T^{-1} e_{n-i+2} - yf^T T^{-1} e_{n-i+2}
\]
\[
= K t_{i-1} + y e_{n-i+2} - yx^T e_{n-i+2}
\]
\[
= K t_{i-1} + y_{n-i+2} x - x_{n-i+2} y.
\]
So we have
\[
t_1 = y, \quad t_2 = Ky + y_n x - x_n y,
\]
\[
\ldots,
\]
\[
t_n = K^{n-1} y + K^{n-2} y y_n - K^{n-2} y x_n + \cdots + y x_2 - y x_2.
\]
\[
T^{-1} = (t_1, t_2, \ldots, t_n)
\]
\[
= (y, Ky, \ldots, K^{n-1} y)
\]
\[
= \left[ \begin{array}{cccc}
1 & -x_n & \cdots & -x_2 \\
-1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & 1
\end{array} \right]
\]
\[
+ (x, Kx, \ldots, K^{n-1} x)
\]
\[
= \left[ \begin{array}{cccc}
y_1 & y_n & \cdots & y_2 \\
y_2 & y_1 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
y_n & \cdots & y_2 & y_1
\end{array} \right]
\]
\[
= \left[ \begin{array}{cccc}
1 & -x_n & \cdots & -x_2 \\
-1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & 1
\end{array} \right]
\]
\[
+ (x, Kx, \ldots, K^{n-1} x)
\]
\[
= \left[ \begin{array}{cccc}
x_1 & x_n & \cdots & x_2 \\
x_2 & x_1 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
x_n & \cdots & x_2 & x_1
\end{array} \right]
\]
\[
= \left[ \begin{array}{cccc}
o & y_n & \cdots & y_2 \\
o & \ddots & \ddots & \ddots \\
o & \ddots & \ddots & \ddots \\
o & \ddots & \ddots & y_n
\end{array} \right].
\]
Remark. In Theorem 1, let the Toeplitz matrix $T = (a_{p-q})_{p,q=1}^n$ be a circulant Toeplitz matrix. That is to say, the elements of the matrix $T = (a_{p-q})_{p,q=1}^n$ satisfy $a_i = a_{i-n}$ for all $i = 1, \ldots, n-1$. It is easy to see that $f = 0$. Thus, $x = T^{-1}f = 0$. From (b) of Theorem 1, we get

$$T^{-1} = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ y_n & \cdots & y_2 & y_1 \end{bmatrix}.$$  

We conclude: The inverses of the circulant Toeplitz matrices are circulant Toeplitz matrices.

3. Stability analysis

In this section, we will show that the Toeplitz inversion formula presented in Section 2 is evaluation forward stable. An algorithm is called forward stable if for all well conditioned problems, the computed solution $\tilde{x}$ is close to the true solution $x$ in the sense that the relative error $\|x - \tilde{x}\|_2 / \|x\|_2$ is small. In the matrix computation, round-off errors occur. Let $A, B \in C^{n,n}$ and $\alpha \in C$. If we neglect the $O(\epsilon^2)$ terms, then for any floating-point arithmetic with machine precision $\epsilon$, then (cf. [9])

$$\text{fl}(\alpha A) = \alpha A + E, \quad \|E\|_F \leq \epsilon |\alpha| \|A\|_F \leq \epsilon \sqrt{n} |\alpha| \|A\|_2,$$

$$\text{fl}(A + B) = A + B + E, \quad \|E\|_F \leq \epsilon \|A + B\|_F \leq \epsilon \sqrt{n} \|A + B\|_2,$$

$$\text{fl}(AB) = AB + E, \quad \|E\|_F \leq \epsilon n \|A\|_F \|B\|_F.$$

According to the floating-point arithmetic, we have the following bound.

**Theorem 2.** Let $T = (a_{p-q})_{p,q=1}^n$ be a nonsingular Toeplitz matrix and be well conditioned; then the formula in Theorem 1 is forward stable.

**Proof.** Assume that we have computed the solutions $\tilde{x}, \tilde{y}$ in Theorem 1 which are perturbed by the normwise relative errors bounded by $\tilde{\epsilon}$,

$$\|\tilde{x}\|_2 \leq \|x\|_2 (1 + \tilde{\epsilon}), \quad \|\tilde{y}\|_2 \leq \|y\|_2 (1 + \tilde{\epsilon}).$$

Therefore, we have

$$\|T_1\|_F = \sqrt{n} \|y\|_2, \quad \|T_2\|_F = \sqrt{n} \|x\|_2,$$

$$\|U_1\|_F \leq \sqrt{n} \sqrt{1 + \|x\|_2^2}, \quad \|U_2\|_F \leq \sqrt{n} \|y\|_2.$$

Using the perturbed solutions $\tilde{x}, \tilde{y}$, the inversion formula in Theorem 1 can be expressed as

$$\tilde{T}^{-1} = \text{fl} \left( \tilde{T}_1 \tilde{U}_1 + \tilde{T}_2 \tilde{U}_2 \right)$$

$$= \text{fl} ((T_1 + \Delta T_1)(U_1 + \Delta U_1) + (T_2 + \Delta T_2)(U_2 + \Delta U_2))$$

$$= T^{-1} + \Delta T_1 + \Delta T_1 U_1 + \Delta T_2 U_1 + \Delta T_2 U_2 + \Delta U_2 + E + F.$$

Here, $E$ is the matrix containing the error which results from computing the matrix products, and $F$ contains the error from subtracting the matrices. For the error matrices $\Delta T_1, \Delta U_1, \Delta T_2$ and $\Delta U_2$, we have

$$\|\Delta T_1\|_F \leq \tilde{\epsilon} \|T_1\|_F = \tilde{\epsilon} \sqrt{n} \|y\|_2,$$

$$\|\Delta T_2\|_F \leq \tilde{\epsilon} \|T_2\|_F = \tilde{\epsilon} \sqrt{n} \|x\|_2,$$

$$\|\Delta U_1\|_F \leq \tilde{\epsilon} \|U_1\|_F \leq \tilde{\epsilon} \sqrt{n} \sqrt{1 + \|x\|_2^2},$$

$$\|\Delta U_2\|_F \leq \tilde{\epsilon} \|U_2\|_F \leq \tilde{\epsilon} \sqrt{n} \|y\|_2.$$
It follows that
\[
\| E \|_2 \leq \| E \|_F \\
\leq \varepsilon n (\| T_1 \|_F \| U_1 \|_F + \| T_2 \|_F \| U_2 \|_F ) \\
\leq \varepsilon n^2 \| y \|_2 (\sqrt{1 + \| x \|_2^2} + \| x \|_2) \\
\leq \varepsilon n^2 \| y \|_2 (1 + 2 \| x \|_2), \quad \| F \|_2 \leq \varepsilon \sqrt{n} \| T^{-1} \|_2.
\]

Adding all these error bounds, we have
\[
\| \tilde{T}^{-1} - T^{-1} \|_2 \leq n(2\varepsilon + n\varepsilon) \| y \|_2 (1 + 2 \| x \|_2) + \varepsilon \sqrt{n} \| T^{-1} \|_2.
\]

Note that \( Ty = e_1 \) and \( Tx = f \); then
\[
\| y \|_2 \leq \| T^{-1} \|_2 \quad \text{and} \quad \| x \|_2 \leq \| T^{-1} \|_2 \| f \|_2.
\]

Thus, the relative error is
\[
\frac{\| \tilde{T}^{-1} - T^{-1} \|_2}{\| T^{-1} \|_2} \leq n(2\varepsilon + n\varepsilon) \left(1 + 2 \| T^{-1} \|_2 \| f \|_2\right) + \varepsilon \sqrt{n}.
\]

As \( T \) is well conditioned, thus, \( \| T^{-1} \|_2 \) is finite. Obviously, \( \| f \|_2 \) is finite. Therefore, the formula presented in Theorem 1 is forward stable. \( \square \)

References