A note on inversion of Toeplitz matrices

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Abstract

It is shown that the invertibility of a Toeplitz matrix can be determined through the solvability of two standard equations. The inverse matrix can be denoted as a sum of products of circulant matrices and upper triangular Toeplitz matrices. The stability of the inversion formula for a Toeplitz matrix is also considered.

Keywords: Toeplitz matrix; Circulant matrix; Inversion; Algorithm

1. Introduction

Let $T$ be an $n$-by-$n$ Toeplitz matrix:

$$
T = \begin{bmatrix}
a_0 & a_{-1} & a_{-2} & \cdots & a_{1-n} \\
a_1 & a_0 & a_{-1} & \cdots & a_{2-n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0
\end{bmatrix},
$$

where $a_{-(n-1)}, \ldots, a_{n-1}$ are complex numbers. We use the shorthand

$$
T = (a_{p-q})_{p,q=1}^n
$$

for a Toeplitz matrix.

The inversion of a Toeplitz matrix is usually not a Toeplitz matrix. A very important step is to answer the question of how to reconstruct the inversion of a Toeplitz matrix by a low number of its columns and the entries of the original Toeplitz matrix. It was first observed by Trench [1] and rediscovered by Gohberg and Semencul [2] that $T^{-1}$ can be reconstructed from its first and last columns provided that the first component of the first column does not vanish. Gohberg and Krupnik [3] observed that $T^{-1}$ can be recovered from its first and second columns if the last component of the first column does not vanish.

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In Heinig and Rost [4], an inversion formula was exhibited for every nonsingular Toeplitz matrix. The method requires the solution of linear systems of equations (the so-called fundamental equations), where the right-hand side of one of them is a shifted column of the Toeplitz matrix $T$. In [5], Ben-Artzi and Shalom proved that three columns of the inverse of a Toeplitz matrix, when properly chosen, are always enough to reconstruct the inverse. Labahn and Shalom [6], and Ng, Rost and Wen [7] presented modifications of this result. In [8], Georg Heinig discussed the problem of the reconstruction of Toeplitz matrix inverses from columns.

In this work, we give a new Toeplitz matrix inversion formula. The inverse matrix $T^{-1}$ can be denoted as a sum of products of circulant matrices and upper triangular Toeplitz matrices. The results obtained show that this formula is numerically forward stable.

2. Toeplitz inversion formula

Lemma 1. Let $T = (a_{p-q})_{p,q=1}^n$ be a $n \times n$ Toeplitz matrix; then it satisfies the formula

$$KT - TK = fe_n^T - e_1f^T J,$$

where

$$K = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 1 & \cdots & \cdots & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix},$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$f = \begin{bmatrix} a_{n-1} - a_{-1} \\ \vdots \\ a_2 - a_{-n+2} \\ a_1 - a_{-n+1} \end{bmatrix}.$$

Theorem 1. Let $T = (a_{p-q})_{p,q=1}^n$ be a Toeplitz matrix. If each of the systems of equations $Tx = f, Ty = e_1$ is solvable, $x = (x_1, x_2, \ldots, x_n)^T, y = (y_1, y_2, \ldots, y_n)^T$, then

(a) $T$ is invertible;
(b) $T^{-1} = T_1U_1 + T_2U_2$, where

$$T_1 = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ y_n & \cdots & y_2 & y_1 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 1 & -x_n & \cdots & -x_2 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & -x_n \\ \vdots & \ddots & \ddots & \vdots \end{bmatrix},$$

$$T_2 = \begin{bmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ x_n & \cdots & x_2 & x_1 \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} o & y_n & \cdots & y_2 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & y_n \end{bmatrix}.$$

Proof. From Lemma 1 and $Tx = f, Ty = e_1$, we have

$$KT = TK + fe_n^T - e_1f^T J = T[K + xe_n^T - yf^T J].$$
then,
\[
K^i T = K^{i-1}T[K + xe_n^T - yf^T J] \\
= \cdots \\
= T[K + xe_n^T - yf^T J]^i.
\]

Therefore,
\[
K^i e_1 = K^i T y = T[K + xe_n^T - yf^T J]^i y.
\]

Let
\[
t_i = [K + xe_n^T - yf^T J]^i y \quad \text{and} \quad \hat{T} = (t_1, t_2, \ldots, t_n).
\]

Then
\[
T t_i = T[K + xe_n^T - yf^T J]^{i-1} y = K^{i-1} e_1 = e_i,
\]
\[
T \hat{T} = T(t_1, t_2, \ldots, t_n) = (e_1, e_2, \ldots, e_n) = I_n.
\]

So the matrix \(T\) is invertible, and the inverse of \(T\) is the matrix \(\hat{T} = T^{-1}\).

For (b): First of all, it is easy to see that
\[
t_1 = y, \quad t_i = [K + xe_n^T - yf^T J]t_{i-1} \quad (i = 1, 2, \ldots, n),
\]
\[
t_i = T^{-1} e_i, \quad J e_i = e_{n-i+1},
\]
\[
JTJ = T^T, \quad JJ = I, \quad J^T = J.
\]

Then, for \(i > 1\)
\[
t_i = Kt_{i-1} + xe_n^T t_{i-1} - yf^T J t_{i-1} \\
= Kt_{i-1} + xe_n^T T^{-1} e_{i-1} - yf^T J T^{-1} e_{i-1} \\
= Kt_{i-1} + xe_n^T J J T^{-1} J e_{n-i+2} - yf^T J J T^{-1} J e_{i-1} \\
= Kt_{i-1} + xe_n^T T^{-1} e_{n-i+2} - yf^T T^{-1} e_{n-i+2} \\
= Kt_{i-1} + xy^T e_{n-i+2} - yx^T e_{n-i+2} \\
= Kt_{i-1} + y_{n-i+2} - x_{n-i+2}.
\]

So we have
\[
t_1 = y, \quad t_2 = K y + y_n x - x_n y, \\
\ldots, \\
t_n = K^{n-1} y + K^{n-2} xy_n - K^{n-2} x y_n + \cdots + xy_2 - yx_2
\]
\[
T^{-1} = (t_1, t_2, \ldots, t_n)
\]
\[
= (y, Ky, \ldots, K^{n-1} y) \left[ \begin{array}{ccc} 1 & -x_n & \cdots & -x_2 \\
 & 1 & \ddots & \\
 & & \ddots & -x_n \\
 & & & 1 \\
y_1 & y_n & \cdots & y_2 \\
y_2 & y_1 & \ddots & \\
 & \ddots & \ddots & y_n \\
y_n & \cdots & y_2 & y_1 \end{array} \right] \left[ \begin{array}{c} o \\
 & 1 \\
 & x_1 & x_n & \cdots & x_2 \\
 & x_2 & x_1 & \ddots & \\
 & \ddots & \ddots & \ddots & \ddots \\
 & x_n & \cdots & x_2 & x_1 \end{array} \right] \left[ \begin{array}{c} y_2 \\
 & y_n \end{array} \right].
\]
Remark. In Theorem 1, let the Toeplitz matrix \( T = (a_{p-q})_{p,q=1}^n \) be a circulant Toeplitz matrix. That is to say, the elements of the matrix \( T = (a_{p-q})_{p,q=1}^n \) satisfy \( a_i = a_{i-n} \) for all \( i = 1, \ldots, n - 1 \). It is easy to see that \( f = 0 \). Thus, \( x = T^{-1} f = 0 \). From (b) of Theorem 1, we get

\[
T^{-1} = \begin{bmatrix}
y_1 & y_n & \cdots & y_2 \\
y_2 & y_1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
y_n & \cdots & y_2 & y_1
\end{bmatrix}.
\]

We conclude: The inverses of the circulant Toeplitz matrices are circulant Toeplitz matrices.

3. Stability analysis

In this section, we will show that the Toeplitz inversion formula presented in Section 2 is evaluation forward stable. An algorithm is called forward stable if for all well conditioned problems, the computed solution \( \tilde{x} \) is close to the true solution \( x \) in the sense that the relative error \( \|x - \tilde{x}\|_2 / \|x\|_2 \) is small. In the matrix computation, round-off errors occur. Let \( A, B \in \mathbb{C}^{n,n} \) and \( \alpha \in \mathbb{C} \). If we neglect the \( O(\varepsilon^2) \) terms, then for any floating-point arithmetic with machine precision \( \varepsilon \), then (cf. [9])

\[
\begin{align*}
\text{fl}(\alpha A) &= \alpha A + E, \quad \|E\|_F \leq \varepsilon \|\alpha\|_F, \\
\text{fl}(A + B) &= A + B + E, \quad \|E\|_F \leq \varepsilon \|A + B\|_F, \\
\text{fl}(AB) &= AB + E, \quad \|E\|_F \leq \varepsilon n \|A\|_F \|B\|_F.
\end{align*}
\]

According to the floating-point arithmetic, we have the following bound.

Theorem 2. Let \( T = (a_{p-q})_{p,q=1}^n \) be a nonsingular Toeplitz matrix and be well conditioned; then the formula in Theorem 1 is forward stable.

Proof. Assume that we have computed the solutions \( \tilde{x}, \tilde{y} \) in Theorem 1 which are perturbed by the normwise relative errors bounded by \( \tilde{\varepsilon} \),

\[
\|\tilde{x}\|_2 \leq \|x\|_2 (1 + \tilde{\varepsilon}), \quad \|\tilde{y}\|_2 \leq \|y\|_2 (1 + \tilde{\varepsilon}).
\]

Therefore, we have

\[
\begin{align*}
\|T_1\|_F &= \sqrt{n} \|y\|_2, \\
\|T_2\|_F &= \sqrt{n} \|x\|_2, \\
\|U_1\|_F &\leq \sqrt{n} \sqrt{1 + \|x\|_2^2}, \\
\|U_2\|_F &\leq \sqrt{n} \|y\|_2.
\end{align*}
\]

Using the perturbed solutions \( \tilde{x}, \tilde{y} \), the inversion formula in Theorem 1 can be expressed as

\[
\tilde{T}^{-1} = \text{fl} \left( \tilde{T}_1 \tilde{U}_1 + \tilde{T}_2 \tilde{U}_2 \right) = \text{fl} \left( (T_1 + \Delta T_1)(U_1 + \Delta U_1) + (T_2 + \Delta T_2)(U_2 + \Delta U_2) \right) = T^{-1} + \Delta T_1 U_1 + T_1 \Delta U_1 + \Delta T_2 U_2 + T_2 \Delta U_2 + E + F.
\]

Here, \( E \) is the matrix containing the error which results from computing the matrix products, and \( F \) contains the error from subtracting the matrices. For the error matrices \( \Delta T_1, \Delta U_1, \Delta T_2 \) and \( \Delta U_2 \), we have

\[
\begin{align*}
\|\Delta T_1\|_F &\leq \tilde{\varepsilon} \|T_1\|_F = \tilde{\varepsilon} \sqrt{n} \|y\|_2, \\
\|\Delta T_2\|_F &\leq \tilde{\varepsilon} \|T_2\|_F = \tilde{\varepsilon} \sqrt{n} \|x\|_2, \\
\|\Delta U_1\|_F &\leq \tilde{\varepsilon} \|U_1\|_F = \tilde{\varepsilon} \sqrt{n} \sqrt{1 + \|x\|_2^2}, \\
\|\Delta U_2\|_F &\leq \tilde{\varepsilon} \|U_2\|_F = \tilde{\varepsilon} \sqrt{n} \|y\|_2.
\end{align*}
\]
It follows that
\[
\|E\|_2 \leq \|E\|_F \\
\leq \varepsilon n (\|T_1\|_F \|U_1\|_F + \|T_2\|_F \|U_2\|_F) \\
\leq \varepsilon n^2 \|y\|_2 \left(\sqrt{1 + \|x\|_2^2} + \|x\|_2\right) \\
\leq \varepsilon n^2 \|y\|_2 (1 + 2 \|x\|_2), \quad \|F\|_2 \leq \varepsilon \sqrt{n} \left\|T^{-1}\right\|_2.
\]

Adding all these error bounds, we have
\[
\left\|\tilde{T}^{-1} - T^{-1}\right\|_2 \leq n (2\varepsilon + n\varepsilon) \|y\|_2 (1 + 2 \|x\|_2) + \varepsilon \sqrt{n} \left\|T^{-1}\right\|_2.
\]

Note that \(Ty = e_1\) and \(Tx = f\); then
\[
\|y\|_2 \leq \left\|T^{-1}\right\|_2 \|f\|_2 \\
\|x\|_2 \leq \left\|T^{-1}\right\|_2 \|f\|_2.
\]

Thus, the relative error is
\[
\left\|\tilde{T}^{-1} - T^{-1}\right\|_2 \leq n (2\varepsilon + n\varepsilon) \left(1 + 2 \left\|T^{-1}\right\|_2 \|f\|_2\right) + \varepsilon \sqrt{n}.
\]

As \(T\) is well conditioned, thus, \(\left\|T^{-1}\right\|_2\) is finite. Obviously, \(\|f\|_2\) is finite. Therefore, the formula presented in Theorem 1 is forward stable. □

References