New Necessary Conditions of Optimality for Control Problems
with State-Variable Inequality Constraints

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Necessary conditions of optimality for state-variable inequality constrained problems are derived which differ from those of Bryson, Denham, and Speyer with regard to the behavior of the adjoint variables at junctions of interior and boundary arcs. In particular, it is shown that the earlier conditions under-specify the behavior of the adjoint variables at the junctions. An example is used to demonstrate that the earlier conditions may yield non-stationary trajectories. For a certain class of problems, it is shown that only boundary points, as opposed to boundary arcs, are possible. An analytic example illustrates this behavior.

1. Introduction

Necessary conditions for the optimality of state-variable inequality constrained problems have been the subject of much research in the past ten years. Gamkrelidze [1], in 1960, approached the problem via the Pontryagin maximum principle. His results may be obtained formally by adjoining the first time-derivative of the constraint—which explicitly contains the control by his "regularity" assumption—to the cost functional and treating the resulting problem as one with a control constraint. Berkovitz [2], in 1962, derived the same conditions as in [1], by way of the classical calculus of variations.
variations. He too used the first time-derivative of the constraint, which by his constraint qualifications, contained the control.

Bryson, Denham and Dreyfus [3], obtained necessary conditions for cases where the state-variable constraint is of order $p \geq 1$. To ensure feasibility of the resulting trajectory they adjointed to the cost functional, a point equality constraint consisting of the state constraint and its $(p - 1)$ time-derivatives, at the time of entry of the trajectory onto the constraint boundary. Their results reduce to those of Gamkrelidze and Berkovitz for the case of a first order constraint.

Chang [4], in 1962, used an entirely different approach. He adjointed the constraint violation to the cost functional by a penalty parameter and used a limiting procedure to obtain the necessary conditions directly. His proofs were limited to the first order case ($p = 1$). Dreyfus, in his book [5], clarified this direct procedure of adjoining the state-variable constraint per se, and obtained the same necessary conditions. Speyer [6] pointed out that Dreyfus' arguments failed for constraints of order $p > 1$, as then the adjoining multiplier may exhibit impulsive behavior. Dreyfus suggested the resolution of this matter as a research problem.

Speyer [7] extended the direct approach to constraints of higher order, by adjoining directly the state-variable constraint to the cost functional, together with point equality constraints at junctions of boundary and interior arcs. He obtained necessary conditions which differed from, but were related to, those obtained in [3]. McIntyre and Paiewonsky [8] used a similar approach.

A third set of necessary conditions, differing considerably in form from those in [3] and [7] were obtained by Dreyfus in his Ph.D. thesis [9]. He used the constraint and its $p - 1$ time-derivatives to reduce, by $p$, the dimension of the state space along the constraint boundary. These results were related to those of [2] by Berkovitz and Dreyfus [10], for the case $p = 1$. Speyer has shown that these are related to the necessary conditions derived in [7]. Recently Neustadt [28] has developed an abstract theory of optimality for constrained control problems but has not related his results to those discussed above.

Concurrently with these theoretical investigations, research was progressing on numerical methods for the solution of state variable inequality constrained problems. Denham and Bryson [11] used the results of [3] for a steepest ascent algorithm. Speyer [7] proposed a second order sweep algorithm. In 1962, Kelley [12] contributed by extending a device of Courant [13] to obtain a penalty function technique for the numerical solution of such problems. Other penalty procedures have been investigated by Lasdon, Waren and

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1 The constraint is assumed to be of $p$-th order, i.e. the $p$-th time-derivative of the constraint is the first to contain the control variable explicitly.

2 However, see the Appendix of the present paper.
Kelley's procedure adjoins the square of the constraint violation to the cost by means of a penalty parameter; the resulting unconstrained problem is solved repeatedly for successively increasing penalty parameter values. The convergence of this type of procedure has been discussed by Butler and Martin [15], Russell [16], Lele and Jacobson [17], Cullum [18], and Beltrami [19]. Beltrami derived the generalized Kuhn–Tucker optimality conditions in a Hilbert space by investigating further the limiting behavior of the penalty method.

In this paper, we derive necessary conditions of optimality by using a separating hyperplane theorem. These necessary conditions of optimality are similar to those of Speyer [7], except at the junction points of boundary and interior arcs. At these points the influence functions exhibit fewer discontinuities than predicted in [7]. We show the relationship of our results to those of [3] and [7]. In particular, we demonstrate that Speyer's necessary conditions are identical to ours provided that all, except possibly one, of the multipliers adjoining the point constraint at the junction arc zero. This leads us to the conclusion that, in addition to Speyer's stated conditions, it is necessary that all his entry and exit point adjoining multipliers be zero except, possibly, the first.

The necessary conditions of [3] can be derived directly, by integration by parts, from ours; this derivation indicates that it is necessary that certain relationships hold between the entry (or exit) point multipliers.

Note that for the case where \( p = 1 \), and for \( p = 2 \) if the Hamiltonian is regular\(^3\), our results are equivalent to the above [7] known necessary conditions, since there is then only one adjoining multiplier at entry and exit points. We use a fourth order constrained problem to illustrate that the existing necessary conditions of Bryson, Denham, and Dreyfus and Speyer, can be satisfied by a non-extremal (i.e. non-stationary trajectory) and thus can yield an incorrect answer. In the particular example considered, a trajectory consisting of boundary and interior arcs is pieced together; this satisfies the existing necessary conditions [3] and [7]. However, it turns out that the unconstrained optimal trajectory is at all times feasible and yields a lower value of the cost. The problem was deliberately chosen to be convex so that no stationary but non-optimal solutions exist. This confirms that the necessary conditions of Bryson, Denham and Dreyfus and Speyer can be satisfied by a non-extremal.

In our necessary conditions the influence functions may exhibit discontinuities at junction points of boundary and interior arcs only along the direction \( S_x \).\(^4\) For problems where the Hamiltonian is regular, yielding a continuous optimal control function of time [8], we are able to derive a

\[^3\] Defined in Section 5.2.

\[^4\] \( S(x(t)) < 0, t \in [0, T] \) is the state-variable inequality constraint. \( S_x = \partial S/\partial x(x(t)) \).
particularly simple expression for the magnitude of this discontinuity. The form of this expression leads us to conclude that, in certain cases, problems with state constraints of odd order \((p > 1)\) will not exhibit any boundary arcs over non-zero intervals of time; i.e. the trajectory will, at most, only touch the constraint boundary but will not lie along it. This behavior is illustrated by a third order example which is solved analytically. As predicted by the theory, the constrained trajectories do not remain on the boundary for non-zero intervals of time.

In summary we have obtained necessary conditions of optimality that are considerably simpler and "sharper" than those of [3–11]. Furthermore, using these necessary conditions, we are able to prove that under certain conditions, problems with constraints of odd order \((p > 1)\) cannot contain boundary arcs of non-zero length.

2. Preliminaries

We first state the basic problem to be considered and establishes some notation.

The following problem will be referred to as "the basic problem" or "problem (I)."

**Problem (I).** Minimize \(\Phi(x(T))\) subject to

\[
\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0,
\]

and the scalar state-variable inequality constraint

\[
S(x(t)) \leq 0, \quad t \in [0, T].
\]

Here,

- \(u(t)\) scalar control variable
- \(x(t)\) \(n\)-dimensional vector of state-variables
- \(f\) \(n\)-dimensional vector function
- \(S\) \(p\)-th order state-variable constraint
- \(\Phi\) scalar function of terminal value of state-variable
- \(x_0\) initial value of state vector, assumed known
- \(\frac{d}{dt}\) \(\frac{d^p}{dt^p}\)
- \(\| \cdot \|\) norm over the space under consideration
- \(\in\) belonging to
- \(\forall\) for all
- \(\exists\) there exists
Assumptions. 1. \( u \in U \), where \( U \triangleq \{ u(\cdot) : u(\cdot) \) is piecewise continuous in the interval \([0, T]\) and \( \| u(\cdot) \| < \infty \) where \( \| u(\cdot) \| \triangleq \sup_{0 \leq t \leq T} | u(t) | \).

2. \( f \) is continuously differentiable up to \((p + 1)\)-th order in both \( x \) and \( u \) on the interval \([0, T]\).

3. \( \Phi \) is a continuous and differentiable function of \( x(T) \).

4. \( S \) is \((p + 1)\)-times continuously differentiable in \( x \).

5. The basic problem has an optimal solution with finite cost \( v_0 \).

6. Along a boundary arc, \( S = 0 \), the control that maintains \((\bar{S})(x, u) = 0\) is \( p \)-times continuously differentiable with respect to time.

7. Along the optimal solution, \((\bar{S})_u(x, u) \neq 0\), for all \( t \in [0, T] \).

Notes. 1. The basic problem is of the form of Mayer. However there is no loss of generality, for a problem in the form of Lagrange or Bolza can always be cast into the form of Mayer by defining an additional state-variable.

2. \( u \) and \( S \) are assumed to be scalars.

3. \( f, S, \) and \( \Phi \) are assumed to be implicit functions of time, without any loss of generality.

3. Summary of Previous Results

We shall summarize only the results of [3] and [7], as these are the closest in form to ours.

3.1. Necessary conditions of Bryson, Denham and Dreyfus

In [3] Bryson, Denham and Dreyfus extended the approach of [1] and [2] to problems with state constraints of orders higher than the first.\(^5\) They differentiated

\[
S(x(t))
\]

\( p \) times with respect to time to obtain the mixed control-state inequality constraint

\[
(\bar{S})(x(t), u(t)) \leq 0. \tag{4}
\]

They then transformed problem (I) into a control-constrained problem by applying the constraint (4) along boundary arcs. To ensure feasibility of the

\(^5\) See also [27, pp. 117–119].
resulting trajectories they also imposed the following equality constraint at points of entry of the trajectory onto the constraint boundary

\[ \Psi(t_{\text{entry}}) = \begin{pmatrix} S(x(t)) \\ \dot{S}(x(t)) \\ \vdots \\ S(x(t)) \end{pmatrix} = 0. \]  

(5)

Thus the single state constraint (3) was replaced by the point constraint (5) at entry points and the control inequality constraint (4) along the boundary \((S = 0)\). The equivalent problem was

Minimize \( \Phi(x(T)) \)  

subject to

\[ \dot{x}(t) = f(x(t), u(t)); \quad x(0) = x_0 \]

and

\[ \Psi(t_{\text{entry}}) = 0 \]

\[ (\dot{S})(x(t), u(t)) \leq 0, \quad t \in [t_{\text{entry}}, t_{\text{exit}}]. \]

Adjoining (4) and (5) to (6) by a scalar multiplier function \( \gamma(\cdot) \) \((\geq 0)\) and vectors of multipliers, \( \nu_i(t_i) \;(\geq 0) \;i = 1, \ldots, N\)—corresponding to the \( N \) entry points—they obtained the following necessary conditions

\[ \frac{\partial H}{\partial u} = 0 = \gamma(S)_u + f_u^T \lambda \]  

(7)

and the adjoint equations

\[ \lambda = \frac{\partial H}{\partial x} \]

\[ \lambda(T) = \frac{\partial \Phi}{\partial x} \bigg|_{t=T} \]

\[ = f_x^T \lambda + \gamma(\dot{S})_x \]

(8)

where the Hamiltonian \( H \) was defined as

\[ H = \gamma(\dot{S}) + \lambda^T f. \]  

(9)

\(^6\) This equality constraint could equally well be imposed at the points of exit from the boundary.
Due to the point constraints (5), the influence functions $\lambda(\cdot)$ suffer discontinuities of the form:

$$\lambda(t_i^+) = \lambda(t_i^-) - v_b^T(t_i) \left( \frac{\partial \Psi}{\partial x} \right)_{t_i}$$

(10)

at $t = t_i (i = 1, \ldots, N)$, the entry points.

3.2. Speyer's necessary conditions

Speyer [7] adjoins the state-variable constraint directly to the cost functional with a multiplier function $\mu(\cdot)(\geq 0)$. To ensure feasibility, he also adjoins the point constraint (5) both at entry and exit points of boundary arcs with multipliers $v_s(t_i)(\geq 0)$. He obtains the following set of necessary conditions:

$$\frac{\partial H}{\partial u} = 0 = f_u^T \lambda$$

(11)

and

$$-\lambda = \frac{\partial H}{\partial x} \left\{ \lambda(T) = \frac{\partial \phi}{\partial x} \right|_{T} = f_x^T \lambda + \mu S_x$$

(12)

where the Hamiltonian $H$ is

$$H = \mu S + \lambda^T f.$$  

(13)

At junction points of interior and boundary arcs, the adjoint variables suffer discontinuities; the boundary conditions are

$$\lambda(t_i^+) = \lambda(t_i^-) - v_s^T(t_i) \left( \frac{\partial \Psi}{\partial x} \right)_{t_i} \quad i = 1, \ldots, M.$$  

(14)

Speyer notes that, in going from an interior arc to a boundary arc, the jumps in $\lambda(\cdot)$ can be obtained as functions of $\lambda(\cdot)$ and $x(\cdot)$ immediately prior to the junction. Thus there are no more unknowns in his procedure than in the previous scheme [3].

4. The New Necessary Conditions

We derive the generalized Kuhn–Tucker conditions [20] in a Banach space and relate these to the conditions derived in [3, 7]. Russell [21] has previously derived necessary conditions in a general topological space and
applied them to a state constrained problem as have Neustadt [28] and Luenberger [22]. However, these researchers have not related their work to that of [3, 7]. Our proof, in Section 5.1, is based upon that in [22, pp. 247–250].

4.1. A generalized Kuhn–Tucker Theorem

For the purposes of this section we write the basic problem in the form

$$\begin{align*}
\text{Min } & \Phi(u) \\
\text{subject to } & S(u) \leq \theta \\
& u \in U
\end{align*}$$

Here $S$ maps $U$ into $C[0, T]$ and $\theta$ defines the null vector in $C[0, T]$. We norm $C[0, T]$ by $\|y\|_C = \sup_{t \in [0, T]} |y(t)|$. We will now consider necessary conditions of optimality for this problem.

All differentials and derivatives will be in the sense of Fréchet. The symbol $\langle y, T \rangle$ will be used to denote the value of the linear functional $T(y)$ at a point $y \in Y$ (cf. [23, p. 21]).

**Theorem 1.** Let $\Phi$ be a real-valued Fréchet differentiable function on $U$ and $S : U \to C[0, T]$ a Fréchet differentiable mapping. Suppose $u^* \in U$ minimizes $\Phi$ subject to $S(u^*) \leq \theta$. Then there exists $r_0 \geq 0$, $\eta^* \in C^*[0, T]$, $\eta^* \geq \theta$ and non-decreasing such that the Lagrangian

$$r_0\Phi(u) + \langle \eta^*, S(u) \rangle$$

is stationary at $u^*$. Further

$$\langle \eta^*, S(u^*) \rangle = 0.\quad (17)$$

**Proof.** Define the following sets, $A$ and $B$, on $W = R \times C[0, T]$.

$$A = \{r, z \mid r \geq \delta \Phi(u^*; \delta u), z \geq S(u^*) + \delta S(u^*; \delta u) \text{ for some } \delta u \in U\}$$

and

$$B = \{r, z \mid r \leq 0, z \leq \theta\}.\quad (19)$$

The sets $A$ and $B$ are convex; $B$ contains interior points as $C[0, T]$ has an interior (since $\|y\|_C = \sup_{t \in [0, T]} |y(t)|$).

Denote by $\text{Int}(B)$ the interior of $B$. Then

$$A \cap \text{Int}(B) = \emptyset$$

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the empty set. For, if \((r, z) \in A \ni r < 0, z < \theta\), then \(\exists \delta u \in U \ni \delta \Phi(u^*; \delta u) < 0, S(u^*) + \delta S(u^*; \delta u) < 0\) \((21)\)

Then \(\exists\) a sphere of radius \(\rho\) centered on \(S(u^*) + \delta S(u^*; \delta u) \subseteq N\) (the negative cone in \(C[0, T]\)). For \(0 < \alpha < 1\) the point \(\alpha[S(u^*) + \delta S(u^*; \delta u)]\) is the center of an open sphere of radius \(\alpha \cdot \rho\) contained in \(N\); hence so is the point \((1 - \alpha) S(u^*) + \alpha[S(u^*) + \delta S(u^*; \delta u)] = S(u^*) + \alpha \cdot \delta S(u^*; \delta u)\). As for fixed \(\delta u\)

\[
\| S(u^* + \alpha \delta u) - S(u^*) - \alpha \cdot \delta S(u^*; \delta u) \| = o(\alpha) \quad (22)
\]

it follows that for sufficiently small \(\alpha\), \(S(u^* + \alpha \delta u) < \theta\). A similar argument shows that \(\Phi(u^* + \alpha \delta u) < \Phi(u^*)\) for sufficiently small \(\alpha\). This contradicts the optimality of \(u^*\). Therefore

\[A \cap \text{Int}(B) = \emptyset.\]

So \(A\) and \(B\) are two convex sets in the normed space \(R \times C[0, T]\) such that \(A \cap \text{Int}(B) = \emptyset\) and \(\text{Int}(B) \neq \emptyset\). Therefore \(\exists\) a closed hyperplane separating \(A\) and \(B\), \([24]\). Hence \(\exists r_0, \eta^*, \delta\) such that

\[r_0 \cdot r + \langle z, \eta^* \rangle \geq \delta, \quad \forall (r, z) \in A, \quad (23)\]

and

\[r_0 \cdot r + \langle z, \eta^* \rangle \leq \delta, \quad \forall (r, z) \in B. \quad (24)\]

As \((0, \theta) \in A \cap B, \delta = 0\); and from the nature of \(B\) it follows that \(r_0 \geq 0, \eta^* \geq \theta\). From the separation property

\[r_0 \cdot \delta \Phi(u^*; \delta u) + \langle S(u^*) + \delta S(u^*; \delta u), \eta^* \rangle \geq 0 \quad (25)\]

\(\forall \delta u \in U.\)

From the above inequality, \(\delta u = \theta \Rightarrow \langle S(u^*), \eta^* \rangle \geq 0\); but \(S(u^*) \leq \theta\) and \(\eta^* \geq \theta \Rightarrow \langle S(u^*), \eta^* \rangle \leq 0\), hence

\[\langle S(u^*), \eta^* \rangle = 0. \quad (26)\]

It then follows that \(\forall \delta u \in U\)

\[r_0 \delta \Phi(u^*; \delta u) + \langle \delta S(u^*; \delta u), \eta^* \rangle = 0. \quad (27)\]

From Riesz's theorem ([23], p. 119) we have

\[\langle S(u^*), \eta^* \rangle = \int_0^T S(u^*) \, d\eta^*, \quad \eta^* \text{ a function of bounded variation}. \quad (28)\]

where the integral is in the Stieltjes sense; and \((24)\) and \((28)\) imply that \(\eta^*\) is nondecreasing on the interval \([0, T]\).
We will now translate these results into the more familiar state space form.

4.2. The Stationarity Conditions in State Space

In state space, the equivalent of the Lagrangian (16) is, formally, the adjoined cost functional

$$J = r_0 \Phi(x(T)) + \int_0^T \lambda^T (f - \dot{x}) \, dt + \int_0^T S(x) \, d\eta^*$$

(29)

where we have adjoined the dynamics (2) with a vector of multiplier functions \( \lambda(\cdot) \). Assume that \( \lambda \in BV^n[0, T] \). Integrating by parts and considering variations in \( x \) and \( u \), we have

$$\delta J = \left[ r_0 \frac{\partial \Phi}{\partial x} - \lambda(T) \right] \delta x(T) + \int_0^T \left[ S_x \, d\eta^* + f_x^T \lambda \, dt + d\lambda \right] \delta x$$

$$+ \int_0^T f_u^T \lambda \, dt \, \delta u.$$  

(30)

Formally choosing \( \lambda \) such that

$$-d\lambda = f_x^T \lambda \, dt + S_x \, d\eta^* ; \quad \lambda(T) = r_0 \Phi_x$$

(31)

eliminates the \( \delta x \) terms. The necessary condition for stationarity of the Lagrangian is that

$$f_u^T \lambda(t) = 0 \quad \text{a.e.} \quad t \in [0, T],$$

(32)

which holds for all \( t \) because \( f_u^T S_x \equiv 0 \), \((S \text{ is } p\text{-th order})\).

Before proceeding to establish whether or not \( r_0 \) can be set to unity we will obtain a function representation for \( \eta^* \). From (17) we have that

$$\int_0^T S(x(t)) \, d\eta^*(t) = 0.$$

Assume (for simplicity, and without loss of generality) that the \( x(\cdot) \) trajectory consists of two interior arcs and a boundary arc. Denote the times

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7 The constraint \( \dot{x} = f \) could be included in the formulation (15) represented by some equality constraint \( C(u) = 0 \), and a more general multiplier rule than (27) could be obtained. However, this would not add substantially to the contribution of the paper and so, for simplicity we make this formal step. Refs. [21, 28] indicate other, rigorous, methods of treating the equality system constraints \( \dot{x} = f \).

8 The linearity of (31) justifies our assumption that \( \lambda \in BV^n[0, T] \). Here \( d\eta^* \triangleq \eta^*(t + dt) - \eta^*(t) \) and \( dt \) is an infinitesimal increment in the independent variable \( t \).
of entry and exit from the boundary \( S = 0 \) by \( t_1 \) and \( t_2 \) respectively. Then splitting up (17) as

\[
\int_0^{t_1} S \, d\eta^* + \int_{t_1}^{t_2} S \, d\eta^* + \int_{t_2}^{T} S \, d\eta^* = 0
\]

we see that

\( \eta^* \) is constant in \([0, t_1)\), and \((t_2, T]\),

since \( S < 0 \) in these intervals and \( \eta^* \) is nondecreasing.

To obtain a representation for \( d\eta^* \) along the boundary we turn to (32). As

\[
f_u^T \lambda = 0,
\]

it certainly holds along \( S = 0 \). Denote \( f_u^T \lambda \) by \( H_u \) where \( H = \lambda^T f \). We have, formally

\[
dH_u = \frac{d}{dt} (f_u^T) \lambda \, dt + f_u^T \, d\lambda = 0 \quad \text{a.e.}
\]

whence on substituting for \( d\lambda \) and noting that \( S_u f_u = (S)_u = 0 \) (as \( S \) is of order \( p \)),

\[
dH_u = [(f_u^T) \lambda - f_u^T f_x^T \lambda] \, dt = 0. \quad \text{a.e.}
\]

As the term \( (f_u^T) \lambda - f_u^T f_x^T \lambda \) is of bounded variation we may write

\[
(\dot{H}_u) = \frac{dH_u}{dt} = 0 \quad \text{a.e.}
\]

which holds for all \( t \) because \( S \) is \( p \)-th order.

Similarly

\[
d(\dot{H}_u) = [(f_u^T) \lambda - 2(f_u^T f_x^T \lambda - f_u^T (f_x^T) \lambda + f_u^T f_x^T f_x^T \lambda)] \, dt.
\]

So,

\[
(\dot{H}_u) = \frac{d}{dt} (\dot{H}_u) = 0 \quad \text{(37)}
\]

Proceeding in this fashion, we see that

\[
(\ddot{H}_u) = 0
\]

\[
(\ddot{H}_u) = 0
\]

and so on up to

\[
(\dddot{H}_u) = 0
\]

* From here on we use assumption 6.
Then
\[ d(H_u) = [(\frac{\partial}{\partial u} + \text{lower order terms}) \lambda \, dt - d\eta^*(S_u) = 0 \quad \text{a.e.} \quad (38) \]
which holds for all \( t \) because \( \eta^* \) is nondecreasing and \( (S_u) \) is absolutely continuous.

We now use assumption 7, (Constraint Qualification), so that
\[ d\eta^* = \frac{[f(u) + \text{lower order terms}] \lambda \, dt}{(S_u)} \quad (39) \]

The functions in the numerator of (39) are of bounded variation; so along the boundary the derivative of \( \eta^* \) is defined, i.e.
\[ \frac{d\eta^*}{dt} = \frac{[f(u) + \text{lower order terms}] \lambda(t)}{(S_u)} = BV(t_1, t_2) \quad (40) \]

Finally, from (31), we have that
\[ \lambda(t_1^+) = \lambda(t_1^-) - \int_{t_1^-}^{t_1^+} (S_u) \, d\eta^* \]
where \( t_1^- \in [0, t_2) \) and \( t_1^+ \in (t_1, t_2) \). Integration by parts yields
\[ \lambda(t_1^+) = \lambda(t_1^-) - [\eta^*(t_1^+) - \eta^*(t_1^-)] S_u(t_1) \]
\[ = \lambda(t_1^-) - \nu(t_1) S_u(t_1) \quad (41) \]
where
\[ \nu(t_1) = [\eta^*(t_1^+) - \eta^*(t_1^-)] \geq 0 \quad (42) \]
as \( \eta^* \) is nondecreasing. Similarly
\[ \lambda(t_2^+) = \lambda(t_2^-) - \nu(t_2) S_u(t_2). \quad (43) \]

So from (31), (40), (41)–(43) the functions \( \lambda(\cdot) \) are piecewise continuous with possible discontinuities at junction points of boundary and interior arcs and \( \eta^* \) is given by
\[ \eta^* = \begin{cases} \text{constant} & t \in [0, t_1), \\ \text{a continuous differentiable function} & t \in (t_1, t_2), \\ \text{constant} & t \in (t_2, T]. \end{cases} \quad (44) \]

This gives
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THEOREM 2. The necessary condition in state space is

\[ \frac{\partial H}{\partial u} = 0 = f_u^T \lambda(t) \quad \text{on} \quad [0, T] \]  

(45)

where the adjoint variables \( \lambda(\cdot) \) are given by

\[ -\dot{\lambda} = f_x^T \lambda + \dot{\eta} S_x \quad \text{on} \quad [0, T]; \quad \lambda(t_f) = r_0 \Phi_x \]  

(46)

and

\[ \dot{\eta} = \begin{cases} 0, & S < 0, \\ \geq 0, & S = 0, \quad \dot{\eta} = \frac{d\eta^*}{dt}, \end{cases} \]  

(47)

with boundary conditions

\[ \lambda(t_+^i) = \lambda(t_-^i) - \nu(t_i) S_x(t_i) \]  

(48)

\[ \nu(t_i) \geq 0. \]

4.3. Proof that \( r_0 \neq 0 \)

THEOREM 3. If

\[ \frac{\partial}{\partial u(t)} [(S)(x^*(t), u^*(t))] \neq 0 \quad \forall t \in [0, T] \]

then \( r_0 \neq 0. \)

Proof. For clarity, and without any loss of generality we will assume that the optimal trajectory \( x^* \) has only one boundary arc and two interior arcs. Suppose \( r_0 = 0 \). Then \( \lambda(T) = 0 \) (from (4)); hence

\[ \lambda(t) = 0, \quad t \in (t_{ex}, T] \]  

(49)

where \( t_{ex} = \) time of exit from boundary arc. From (45), \( \forall t \in [0, T] \)

\[ H_u = 0 \]

hence

\[ (H_u) = 0 \]

\[ \vdots \]

\[ (H_u)^{T} = 0 \]  

(50)
and similarly for higher derivatives of $H_u$. In particular, at $t = t_{ex}$

\[(H_u)^- = (H_u)^+ \]

\[(\dot{H}_u)^- = (\dot{H}_u)^+ \]

\[(\dddot{H}_u)^- = (\dddot{H}_u)^+ \]

where the $-$ and $+$ superscripts denote instants just prior to, and just after $t_{ex}$, respectively. From (48), we have that

\[\lambda(t_{ex}) = \lambda(t_{ex}^+) + \nu(t_{ex}) S_x(t_{ex})\]

which substituted into

\[(\dddot{H}_u)^- = (\dddot{H}_u)^+\]

yields, after some manipulation (since $\lambda(t) = 0; \ t \in (t_{ex}, T])$

\[\nu(t_{ex}) (S)'u(t_{ex}) = 0. \quad (51)\]

As, by assumption, $(S)'u \neq 0, \nu(t_{ex}) = 0$. This gives $\lambda(t_{ex}) = 0$. Now, from

\[(\dot{H}_u) = 0\]

along $S = 0$ we have

\[\dot{\lambda}(t) = [\text{terms involving } (f_u) \text{ etc}] \lambda(t) \quad (S)'u \]

This gives a linear homogeneous differential equation for $\lambda(\cdot)$ along $S = 0$, with the initial condition $\lambda(t_{ex}) = 0$. Hence $\lambda(\cdot) = 0$ along the boundary. Thus,

\[\dot{\lambda}(\cdot) = \theta \quad (53)\]

as well as $r_0 = 0$. But $r_0 = 0, \dot{\lambda}(\cdot) = \theta, \nu = 0$ contradicts the fact that $\exists$ a nontrivial separating hyperplane (Section 4.1). Thus $r_0 \neq 0$.

For convenience we set $r_0 = 1$. Then we have the final form of the necessary conditions in

**Theorem 4.** *The necessary condition for optimality of problem (1) is*

\[\frac{\partial H}{\partial u} = 0 = f_u^T \lambda \quad (54)\]
where the adjoint variables $\lambda(\cdot)$ are given by

$$
-\lambda = \frac{\partial H}{\partial x} + \frac{\partial \Phi}{\partial x} \\
= f_x^T \lambda + \tilde{q} S_x
$$

where

$$
\tilde{q}(t) = \begin{cases} 
  0 & S(x(t)) = 0 \\
  \geq 0 & S(x(t)) < 0 
\end{cases}
$$

is a bounded function for $t \in [0, T]$.

At junction points $t_i$ of boundary and interior arcs, the influence functions $\lambda(\cdot)$ may be discontinuous. The boundary conditions are

$$
\lambda(t_i^+) = \lambda(t_i^-) - \nu(t_i) \left( \frac{\partial S}{\partial x} \right)_{t_i} ; \quad \nu(t_i) \geq 0
$$

and in addition

$$
H(t_i^+)^+ = H(t_i^-)^- \quad \text{(see [3], [7]).}
$$

The Hamiltonian $H$, used above, is defined by

$$
H = \tilde{q} S + \lambda^T f.
$$

4.4. Generalization

If terminal equality constraints ($\psi(x(t_f), t_f) = 0$) are present, then the necessary conditions take the following form:

**Theorem 5. (Equality Terminal Constraints)** If in addition to the assumptions of Section 4, the following are true

i) $\delta x = f_x(x^*(t), u^*(t)) \delta x + f_u(x^*(t), u^*(t)) \delta u$ is completely controllable

ii) $\psi(x^*(t_f), t_f)$ has rank $q$ ($\psi$ is a $q$-vector function) then necessary conditions of optimality are:

$$
\frac{\partial H}{\partial u} = 0 = f_u^T \lambda
$$

$$
-\lambda = \frac{\partial H}{\partial x} - f_x^T \lambda + \tilde{q} S_x ; \quad \lambda(T) = \frac{\partial \Phi}{\partial x} \bigg|_{x^T} \sigma^T \psi_x \bigg|_{T},
$$
\( \sigma \) is a \( q \)-vector of constant Lagrange multipliers and
\[
\dot{\lambda}(t) = \begin{cases} 
\geq 0, & S(x(t)) = 0, \\
0, & S(x(t)) < 0.
\end{cases}
\]

At junction points of boundary and interior arcs:
\[
\lambda(t_i^+) = \lambda(t_i^-) - \nu(t_i) \left( \frac{\partial S}{\partial x} \right)_{t_i}, \quad i = 1, \ldots, N, \\
H(t_i^+) = H(t_i^-), \\
\nu(t_i) \geq 0.
\]

**Proof.** We only sketch the proof here. The above problem may be considered in the following nonlinear programming formulation
\[
\text{Min } \Phi(u) \\
\text{subject to } S(u) \leq 0, \quad \psi(u) = 0, \quad u \in U.
\]

Define the set
\[
U_1 = \{ u : \psi(u(u^*)) \delta u = 0, u^* + \delta u \in U \}.
\]

The set \( A \) (Theorem 1) is now defined as
\[
A = \{ (r, z) : r \geq \delta \Phi(u^*; \delta u), z \geq S(u^*) + \delta S(u^*; \delta u); \text{ for some } \delta u \in U_1 \}.
\]

Using similar though more involved arguments than those of Theorem 1 it follows, because of condition ii) of Theorem 5, that
\[
\langle r_0 \Phi_u + \eta^* S_u, \delta u \rangle = 0 \quad \delta u \in U_1.
\]

Translating back into state space and using the assumption
\[
\frac{\partial}{\partial \nu(t)} (S) \neq 0; \quad t \in [0, T]
\]
yields the necessary conditions (59)–(62).

**5. Relation to Previous Results**

5.1. **Bryson, Denham and Dreyfus**

Let us rewrite (29) as the following (noting that \( r_0 = 1 \))
\[
J = \Phi(x(T)) + \sum_i \nu(t_i) S(t_i) + \int_0^T d\tau \dot{\eta}(\tau) S(x(\tau))
\]
subject to
\[ x(t) = f(x(t), u(t)); \quad x(0) = x_0. \] (68)

For simplicity, and with no loss of generality, we will assume that the optimal trajectory has only one constrained arc. Then integrating the cost functional by parts, equation (67) becomes
\[
\Phi(x(\tau)) + \nu(t_{\text{en}}) S(t_{\text{en}}) + \nu(t_{\text{ex}}) S(t_{\text{ex}}) + [\eta_1(\tau) S(x(\tau))]_{t_{\text{en}}}^{t_{\text{ex}}} \\
- \int_0^T d\tau \eta_1(\tau) \frac{d}{d\tau} (S(x(\tau)))
\] (69)

where
\[
\eta_1(t) = \int_0^t \dot{\eta}(\tau) d\tau.
\] (70)

Adding and subtracting \( \eta_1(t_{\text{ex}}) S(t_{\text{en}}) \) and \( \nu(t_{\text{ex}}) S(t_{\text{en}}) \), (69) becomes \(^{10}\)
\[
\Phi(x(\tau)) + \nu_1 S(x) \bigg|_{t-t_{\text{en}}} + \int_0^T \tilde{\eta}_1(\tau) \frac{d}{d\tau} (S(x(\tau)))
\] (71)

where
\[
\nu_1 = \nu(t_{\text{en}}) + \nu(t_{\text{ex}}) + \eta_1(t_{\text{ex}}) - \eta_1(t_{\text{en}}) \geq 0
\] (72)

and
\[
\tilde{\eta}_1(t) = \nu(t_{\text{ex}}) + \eta_1(t_{\text{ex}}) - \eta_1(t) \geq 0 \quad \forall t \in [t_{\text{en}}, t_{\text{ex}}].
\] (73)

We can carry on this process of integration by parts until finally we obtain
\[
J = \Phi(x(\tau)) + \nu^T \Psi + \int_0^T \tilde{\eta}_v(\tau) \frac{d}{d\tau} [S(x(\tau))] d\tau
\] (74)

where
\[
\Psi^T = (S, S, S, \ldots, (S^3))
\] (75)

and
\[
\nu^T = [\nu_1, \nu_2, \ldots, \nu_p]
\] (76)

and
\[
\nu_1 = \nu(t_{\text{en}}) + \nu(t_{\text{ex}}) + (\eta_1(t_{\text{ex}}) - \eta_1(t_{\text{en}}))
\]
\[
\nu_i = \eta_1(t_{\text{ex}}) - \eta_1(t_{\text{en}}) \geq 0; \quad i > 1
\] (77)

\(^{10}\) Note that \( \int_0^T \dot{\eta}(\tau) S(x(\tau)) d\tau = \int_{t_{\text{en}}}^{t_{\text{ex}}} \dot{\eta} S d\tau. \)
and
\[ \tilde{\eta}^p(t) = \eta_p(t_{ex}) - \eta_p(t) \]  
and where
\[ \eta_i(t) = \int_0^t \tilde{\eta}_{i-1}(\tau) \, d\tau \]  
and
\[ \tilde{\eta}_1 = \nu(t_{ex}) + \eta_1(t_{ex}) - \eta_1(t) \]
\[ \tilde{\eta}_i(t) = \eta_i(t_{ex}) - \eta_i(t) \geq 0 \quad i > 1. \]

Identifying \( \tilde{\eta}_i(\cdot) \) with \( \gamma(\cdot) \) we see that (74) is equivalent to the adjoined cost functional of [3]. If we now use (74) as the functional to be minimized subject to equation (68) we obtain the same stationarity conditions as those given by equations (8)-(10). The noteworthy point is the set of equations (77)-(80). These indicate that the \( \nu \)'s and \( \gamma(t) \) of [3] are related along the optimal trajectory.

5.2. Relation to Speyer's necessary conditions

Speyer's [7] necessary conditions reduce to those given by us, if his multipliers \( \nu_{S_{2}} \) through \( \nu_{S_{p}} \) are zero. For, if that is the case, (14) and (57) are the same, and setting \( \mu(\cdot) \) equal to \( \tilde{\eta}(\cdot) \) completes the connection.

The fact that, along an extremal, Speyer’s multipliers \( \nu_{S_{2}} \) through \( \nu_{S_{p}} \) are zero leads to an interesting result. We introduce the following

**Definition.** The Hamiltonian \( H \) is said to be regular if along a given \( x(t), \lambda(t) \) trajectory (say \( x(t), \lambda(t) \)), \( H(u, x, \lambda) \) has a unique minimum in \( u, t \in [0, T] \).

Then Speyer [7] and McIntyre and Paiewonsky [8] have shown that \( u(\cdot) \) must be continuous across the junction and that \( \nu_{S_{p}} = 0 \). But if \( \nu_{S_{p}}, \ldots, \nu_{S_{2}} \) are all zero, then extending their reasoning, \( u \) and all its time derivatives up to \( (u) \) must be continuous. The result is easily obtained from [7] or Eq. (14) and (37).

6. A Further Consequence of the New Necessary Conditions

For the case of a regular Hamiltonian, where, from the preceding discussion, \( u \) and its \((p - 2)\) time derivatives are continuous we have:

**Theorem 6.** If the Hamiltonian \( H \) is regular, \( S \in B_{2p-1}[0, T]^{11} \) and the

\[ B_{2p-1} = \text{space of all functions whose } (2p - 1)\text{th derivatives exist.} \]
extremal path has an interior arc which joins a boundary arc of non-zero length, then

\[ \nu(t_1) = (-1)^p \frac{H_{uu}(u)^- - (u)^+}{(S)^-} \geq 0 \]  

where \((\quad)^-\) denotes \((\quad)\) on the interior arc at the junction time \(t_1\).

**Proof.** We use (57) and

\[ (H_u)^- = (H_u)^+ \]  

which holds across a junction; noting that

\[ (H_u) = \frac{d^{p-1}}{dt^{p-1}} (f_u T) \lambda \]

\[ = f_u T (u) + \text{terms of lower order time derivatives of } u + \text{terms in } f_u f_x \text{ etc.} \]

we have the following expression for \(\nu(t_1)\) (after simplifying it with the aid of (57) and the above expression for \((H_u)^+)\):

\[ \nu(t_1) = \frac{(-1)^p (f_u T)^+ \lambda^-[(u)^- - (u)^+]}{(S)^+} \]  

As \(S \in B_{2p-1}[0, T]\) by assumption, we have, from the general expression

\[ (S) = (S) + \text{lower order time derivatives of } u + \text{terms in } f, (f) \text{ etc.} \]

the relation

\[ (S)^- - (S)^+ = (S)^+ [(u)^- - (u)^+] \]

whence, \(\nu(t_1) = \frac{(-1)^p (H_{uu})^- [(u)^- - (u)^+]^2}{(S)^-} \).  

This expression for \(\nu(t_1)\) is very significant. Note that \(H_{uu} > 0\) (strengthened necessary condition for a minimum), \([(u)^- - (u)^+]^2 \geq 0\) and as \(S\) and its time derivatives up to \((S)\) are continuous (therefore zero)

\[ (S)^- > 0 \]

\( (S)^+ = 0 \) as \(S\) and all its time derivatives are zero along boundary.
for the trajectory to reach the boundary. This implies that
\[ \nu(t_1) \leq 0 \]  
(85)
for \( p \) odd. But \( \nu(t_1) \geq 0 \) and hence (85) implies that for odd order constraints the trajectory will, at most, only touch the boundary, if \( \frac{u}{2} \neq \frac{u}{2}^+ \). Note that for \( p = 1, u^- = u^+ \), so that, from (83) \( \nu(t_1) = 0 \); thus for the first order case boundary arcs are permitted.

This behavior and (84) are reminiscent of junction conditions in singular control problems. This provides a further hint of a close connection between state-constrained and singular control problems which has been suggested elsewhere [25].

7. A Third Order Problem

Third and fourth order state constrained problems are illustrated. The third order problem confirms the result of Section 6, as all optimal trajectories do not stay on the constraint boundary for any nonzero length of time.

Consider the following problem:

Minimize \( \int_0^1 \frac{u^2}{2} \, dt \)  
(86)
subject to

\[ \begin{align*}
\dot{x}_1 &= x_2, & x_1(0) &= 0 = x_3(1), \\
\dot{x}_2 &= x_3, & x_2(0) &= 1 = -x_2(1), \\
\dot{x}_3 &= u, & x_3(0) &= 2 = x_3(1),
\end{align*} \]  
(87)

and the constraint

\[ x_1(t) - t \leq 0, \quad t \in [0, 1], \]  
(88)
where \( t \) ranges as

\[ \frac{3}{8} \geq t \geq 0. \]  
(89)

The solution to the unconstrained problem is obtained first. The Hamiltonian \( H \) is

\[ H = \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 u. \]  
(90)
The adjoint equations are

\[ \begin{align*}
\lambda_1 &= 0 & \lambda_1(1) &= \text{constant} \\
\lambda_2 &= -\lambda_1 & \lambda_2(1) &= \text{constant} \\
\lambda_3 &= -\lambda_2 & \lambda_3(1) &= \text{constant}
\end{align*} \]  
(91)
Minimizing the Hamiltonian gives the optimal control

\[ H_u = 0 = u + \lambda_3 \Rightarrow u = -\lambda_3 \]  \hfill (92)

The solution to the problem is:

The optimal control is

\[ u^0 = 48(t - \frac{1}{2}) \]  \hfill (93)

The optimal trajectory is

\[ \begin{align*}
x_1^0 &= 2t^4 - 4t^3 + t^2 + t \\
x_2^0 &= 8t^3 - 12t^2 + 2t + 1 \\
x_3^0 &= 24t^2 - 24t + 2
\end{align*} \]  \hfill (94)

The adjoint variable histories are

\[ \begin{align*}
\lambda_1^0 &= 0 \\
\lambda_2^0 &= 48 \\
\lambda_3^0 &= 48\left(\frac{1}{2} - t\right)
\end{align*} \]  \hfill (95)

Note that the constraint is not effective for \( \ell > \frac{2}{3} \).

The solution to the constrained problem consists of two parts.

For \( \ell \) in the range \( \frac{9}{40} < \ell \leq \frac{3}{8} \) there is only one point of contact with the constraint boundary at \( t = \frac{1}{2} \). The complete solution is

\[ u^0 = \begin{cases} 
\frac{at^2 + bt + c}{\left[a(1 - t)^2 + b(1 - t) + c\right]} & 0 \leq t \leq \frac{1}{3} \\
\frac{at^3 + bt^4 + c(1 - t)^2 + t}{24} + t^2 + t & \frac{1}{3} \leq t \leq \frac{1}{2} \\
\frac{a(1 - t)^5}{60} + \frac{b(1 - t)^4}{24} + \frac{c(1 - t)^3}{6} + (1 - t)^2 + (1 - t) & \frac{1}{2} \leq t \leq 1
\end{cases} \]  \hfill (96)

\[ x_1^0 = \begin{cases} 
\frac{at^5}{12} + \frac{bt^3}{6} + \frac{ct^2}{2} + 2t + 1 & 0 \leq t \leq \frac{1}{2} \\
-\left[\frac{a(1 - t)^4}{12} + \frac{b(1 - t)^3}{6} + \frac{c(1 - t)^2}{2} + 2(1 - t) + 1\right] & \frac{1}{2} \leq t \leq 1
\end{cases} \]  \hfill (97)

\[ x_2^0 = \begin{cases} 
\frac{at^3}{3} + \frac{bt^2}{2} + ct + 2 & 0 \leq t \leq \frac{1}{2} \\
\frac{a(1 - t)^3}{3} + \frac{b(1 - t)^2}{2} + c(1 - t) + 2 & \frac{1}{2} \leq t \leq 1
\end{cases} \]  \hfill (98)
and the adjoint variable histories are

$$\lambda_1 = \begin{cases} -2a \\ 2a \end{cases} \quad 0 \leq t \leq \frac{1}{2}$$

$$\lambda_2 = \begin{cases} 2at + b \\ 2a(1 - t) + b \end{cases} \quad \frac{1}{2} \leq t \leq 1$$

and

$$\lambda_3 = \begin{cases} -at^2 - bt - c \\ -a(1 - t)^2 + b(1 - t) + c \end{cases} \quad \frac{1}{2} \leq t \leq 1$$

where

$$a = 5120(\xi - \frac{3}{10})$$

$$b = -3200(\xi - \frac{36}{100})$$

$$c = 320(\xi - \frac{9}{20})$$

When $\xi < \frac{9}{10}$, the single point of contact splits into two, instead of a boundary arc as would occur for the same constraint on $x_d(1)$ [27]. The solution is a trifle more complicated and consists of four parts: one prior to the first point of contact with the constraint boundary at time $t_1(\xi < \frac{9}{10})$, one for $t \in [t_1, 1]$, one for $t \in [\frac{1}{2}, 1 - t_1]$ and one for $t \in [1 - t_1, 1]$. The times of contact are symmetric about $t = \frac{1}{2}$.

The optimal control is:

$$u^0 = \begin{cases} at^2 + bt + c \\ dt + e \\ -a(1 - t)^2 - b(1 - t) - c \end{cases} \quad 0 \leq t \leq t_1 \quad t_1 \leq t \leq \frac{1}{2} \quad \frac{1}{2} \leq t \leq 1 - t_1 \quad 1 - t_1 \leq t \leq 1$$

The optimal trajectory is

$$x_1^0 = \begin{cases} \frac{at^5}{60} + \frac{bt^4}{24} + \frac{ct^3}{6} + t^2 + t \quad 0 \leq t \leq t_1 \\ \frac{dt^4}{24} + \frac{c^2}{6} + \frac{At^3}{2} + Bt + C \quad t_1 \leq t \leq \frac{1}{2} \\ \frac{d(1 - t)^4}{24} + \frac{c(1 - t)^3}{6} + \frac{A(1 - t)^2}{2} + B(1 - t) + C \quad \frac{1}{2} \leq t \leq 1 - t_1 \\ \frac{a(1 - t)^5}{60} + \frac{b(1 - t)^4}{24} + \frac{c(1 - t)^3}{6} + (1 - t)^2 + (1 - t) + (1 - t) \quad 1 - t_1 \leq t \leq 1 \end{cases}$$
NECESSARY CONDITIONS OF OPTIMALITY

\[
\begin{aligned}
    x_2^0 &= \begin{cases} 
        \frac{at^3}{12} + \frac{bt^2}{6} + \frac{ct^2}{2} + 2t + 1 & 0 \leq t \leq t_1 \\
        \frac{dt^3}{6} + \frac{et^2}{2} + At + B & t_1 \leq t \leq \frac{1}{2} \\
        - \frac{d(1 - t)^3}{6} - \frac{e(1 - t)^2}{2} - A(1 - t) - B & \frac{1}{2} \leq t \leq 1 - t_1 \\
        - \frac{a(1 - t)^4}{12} - \frac{b(1 - t)^3}{6} - \frac{c(1 - t)^2}{2} - 2(1 - t) - 1 & 1 - t_1 \leq t \leq 1 
    \end{cases} 
\end{aligned}
\]

and

\[
\begin{aligned}
    x_3^0 &= \begin{cases} 
        \frac{at^3}{3} + \frac{bt^2}{2} + ct + 2 & 0 \leq t \leq t_1 \\
        \frac{dt^2}{2} + et + A & t_1 \leq t \leq \frac{1}{2} \\
        \frac{d(1 - t)^2}{2} + e(1 - t) + A & \frac{1}{2} \leq t \leq 1 - t_1 \\
        \frac{a(1 - t)^3}{3} + \frac{b(1 - t)^2}{2} + c(1 - t) + 2 & 1 - t_1 \leq t \leq 1 
    \end{cases} 
\end{aligned}
\]

where

\[
A \equiv \frac{dt_1^3}{3} + \frac{bt_1^2}{2} + ct_1 + 2 - \frac{dt_1^2}{2} - et_1 \\
B \equiv \frac{dt_1^4}{12} + \frac{bt_1^3}{6} + \frac{ct_1^2}{2} + 2t_1 + 1 - \frac{dt_1^3}{6} - \frac{et_1^2}{2} - At_1 \\
C \equiv \frac{at_1^5}{60} + \frac{bt_1^4}{24} + \frac{ct_1^3}{6} + t_1^2 + t_1 - \frac{dt_1^4}{24} - \frac{et_1^3}{6} - \frac{At_1^2}{2} - Bt_1 
\]

The adjoint variable histories are

\[
\lambda_1 = \begin{cases} 
    -2a & 0 \leq t \leq t_1 \\
    0 & t_1 \leq t \leq 1 - t_1 \\
    2a & 1 - t_1 \leq t \leq 1 
\end{cases}
\]

\[
\lambda_2 = \begin{cases} 
    2at + b & 0 \leq t \leq t_1 \\
    d & t_1 \leq t \leq 1 - t_1 \\
    +2a(1 - t) + b & 1 - t_1 \leq t \leq 1 
\end{cases}
\]
and

\[
\lambda_3 = \begin{cases} 
- at^2 - bt - c & 0 \leq t \leq t_1 \\
- dt - e & t_1 \leq t \leq \frac{1}{2} \\
d(1 - t) + e & \frac{1}{2} \leq t \leq 1 - t_1 \\
(1 - t)^2 + b(1 - t) + c & 1 - t_1 \leq t \leq 1 
\end{cases}
\]  

(113)

The constants \(a, b, c, d, e\) and \(t_1\) are related to \(\ell\) by the following set of equations:

\[
d + 2e = 0 \\
3at_1^2 + bt_1 + e - dt_1 - e = 0 \\
12d + e + A + B = 0 \\
6at_1^4 + 2bt_1^3 + ct_1^2 + 2t_1 + 1 = 0 \\
60dt_1^4 + 24bt_1^3 + 6ct_1^2 + t_1^3 + t_1 = \ell \\
2at_1 + b - d = 0.
\]

(114)

The equations (114) were treated as a set of linear equations in \(a, b, c, d, e\) and \(\ell\), and solutions found for \(t_1\) in the range \([0, \frac{1}{2}]\). The times of contact are plotted vs. \(\ell\) in Fig. 1 which shows \(x_1(\cdot)\) for various values of \(\ell\). Fig. 2
shows the corresponding control history. It is worth noting that, as the Hamiltonian is regular, the control $u$ is continuous, as also is $\dot{u}$ but $\ddot{u}$ is discontinuous at $t_1$ and $1 - t_1$. For this problem, $\dot{\gamma}(\cdot) = \theta$.

**Fig. 2.** Third order problem $u(\cdot)$ vs. time.

### 8. A Fourth Order Problem

We have set up this problem to demonstrate that a nonextremal can satisfy the necessary conditions of Bryson, Denham and Dreyfus [3] and Speyer [7]. Consider the following fourth order problem

$$\begin{align*}
\text{Min}_u \int_0^{10} \frac{u^2}{2} \, dt \\
\text{subject to} & \\
\begin{align*}
\dot{x}_1 &= x_2 & x_1(0) &= 0 = x_1(10) \\
\dot{x}_2 &= x_3 & x_2(0) &= -x_2(10) = -\frac{15}{12} \\
\dot{x}_3 &= x_4 & x_3(0) &= x_3(10) = -\frac{15}{12} \\
\dot{x}_4 &= u & x_4(0) &= -x_4(10) = \frac{15}{16} \\
\end{align*}
\end{align*}$$

and the constraint

$$x_4(t) - 1 \leq 0 \quad t \in [0, 10].$$

The following trajectory, consisting of a boundary arc between $t = 4$...
and \( t = 6 \) and two interior arcs satisfies all the necessary conditions given in [7].

\[
\begin{align*}
\mathbf{u}_1(t) &= \begin{cases} 
\frac{-15}{128}(4 - t) & 0 \leq t \leq 4 \\
0 & 4 \leq t \leq 6 \\
\frac{15}{128}(6 - t) & 6 \leq t \leq 10 
\end{cases} 
\tag{118}
\end{align*}
\]

\[
\begin{align*}
\mathbf{x}_1(t) &= \begin{cases} 
\ell - \frac{\alpha(4 - t)^5}{120} & 0 \leq t \leq 4 \\
\ell & 4 \leq t \leq 6 \\
\ell + \frac{\alpha(6 - t)^5}{120} & 6 \leq t \leq 10 
\end{cases} 
\tag{119}
\end{align*}
\]

\[
\begin{align*}
\mathbf{x}_2(t) &= \begin{cases} 
\frac{\alpha(4 - t)^4}{24} & 0 \leq t \leq 4 \\
0 & 4 \leq t \leq 6 \\
\frac{-\alpha(6 - t)^4}{24} & 6 \leq t \leq 10 
\end{cases} 
\tag{120}
\end{align*}
\]

\[
\begin{align*}
\mathbf{x}_3(t) &= \begin{cases} 
\frac{-\alpha(4 - t)^3}{6} & 0 \leq t \leq 4 \\
0 & 4 \leq t \leq 6 \\
\frac{\alpha(6 - t)^3}{6} & 6 \leq t \leq 10 
\end{cases} 
\tag{121}
\end{align*}
\]

\[
\begin{align*}
\mathbf{x}_4(t) &= \begin{cases} 
\frac{\alpha(4 - t)^2}{2} & 0 \leq t \leq 4 \\
0 & 4 \leq t \leq 6 \\
\frac{-\alpha(6 - t)^2}{2} & 6 \leq t \leq 10 
\end{cases} 
\tag{122}
\end{align*}
\]

where \( \alpha = \frac{15}{128} \). The adjoint variables are

\[
\begin{align*}
\lambda_1 &= 0 \quad 0 \leq t \leq 10 \\
\lambda_2 &= 0 \quad 0 \leq t \leq 10 \\
\lambda_3 &= \begin{cases} 
\frac{15}{128} & 0 \leq t \leq 4 \\
0 & 4 \leq t \leq 6 \\
\frac{-15}{128} & 6 \leq t \leq 10 
\end{cases} 
\tag{125}
\end{align*}
\]
and
\[ \lambda_4 = -u. \]  
\hfill (126)

Here \( \mu(\cdot) = 0, v_{s_1} = 0, v_{s_2} = 0, v_{s_3} = \frac{15}{128}, \) and \( v_{s_4} = 0 \) at entry and exit. All of Speyer's optimality conditions are satisfied and the value of the cost functional is 0.293. Also, Bryson, Denham and Dreyfus' necessary conditions are satisfied with

\[ \begin{align*}
v_{b_1} &= 0, & v_{b_2} &= 0, & v_{b_3} &= \frac{30}{128}, & v_{b_4} &= \frac{30}{128}
\end{align*} \]

and

\[ \begin{align*}
\gamma(\cdot) &= 0 & 0 & \leq t & \leq 4, & 6 & \leq t & \leq 10 \\
\gamma(t) &= \frac{15}{128} (6 - t), & 4 & \leq t & \leq 6.
\end{align*} \]

However, the unconstrained optimal trajectory given below, turns out to be feasible, \(^{12}\) and gives a cost of 0.2897. Other stationary trajectories are ruled out as the problem is convex. This implies that the necessary conditions of Bryson, Denham and Dreyfus and Speyer have yielded a spurious extremal.

\[ \text{FIG. 3. Fourth order problem } x_1(\cdot) \text{ vs. time.} \]

\(^{12}\) Note that for certain initial conditions optimal trajectories may lie along the constraint boundary; expression (81) suggests this.
The unconstrained optimal trajectory is

\[ u^0 = bt^2 + ct + d \quad (127) \]

\[ x_1^0 = \frac{bt^6}{360} + \frac{ct^5}{120} + \frac{dt^4}{24} + \frac{15t^3}{96} - \frac{15t^2}{24} + \frac{15t}{12} \quad (128) \]

\[ x_2^0 = \frac{bt^5}{60} + \frac{ct^4}{24} + \frac{dt^3}{6} + \frac{15t^2}{32} - \frac{15t}{12} + \frac{15}{12} \quad (129) \]

\[ x_3^0 = \frac{bt^4}{12} + \frac{ct^3}{6} + \frac{dt^2}{2} + \frac{15t}{16} - \frac{15}{12} \quad (130) \]

\[ x_4^0 = \frac{bt^3}{3} + \frac{ct^2}{2} + dt + \frac{15}{16} \quad (131) \]

where \( b = -0.02025, c = 0.2025, d = -0.525 \). Fig. 3 shows \( x_1 \) for both cases.

9. Conclusions

We have considered the question of necessary conditions for optimality of state-constrained control problems. New necessary conditions were obtained which yield a considerable simplification in the junction conditions on the influence functions over those obtained by previous researchers. We do not imply that the necessary conditions obtained by previous workers are incorrect, but rather, that, inasmuch as they underspecify the conditions at the junction, there exists the possibility of non-stationary solutions satisfying these conditions as shown in Section 8. Thus misleading results may be obtained using the existing necessary conditions.

For the case of regular Hamiltonian, we have discovered that, if \( (u)^{-} \neq (u)^{+} \), problems with odd-ordered constraints do not have boundary arcs, (as opposed to boundary points). We feel that this result has a two-fold significance; first, it yields further insight into the structure of solutions of state constrained problems, and second, it provides one more clue towards the connection between state-constrained and singular problems, which has been speculated upon elsewhere [25]. Moreover, this result appears to be of importance in economic growth theory [29].

The comparatively simple form of the new necessary conditions should stimulate research into new, efficient techniques for solving state constrained optimal control problems.
NECESSARY CONDITIONS OF OPTIMALITY

APPENDIX

Professor L. W. Neustadt has suggested to the authors that an attempt be made to obtain rigorous proofs of the necessary conditions by combining the abstract conditions given in [28] with the necessary differentiability assumptions and the concept of \("p\)-th order constraint." It is felt that rigorous proofs along these lines should be possible.

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