JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **138**, 249–279 (1989)

The Stefan Problem for a Hyperbolic Heat Equation*

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Received October 7, 1987

In this paper we consider the 1-phase, 1-dimensional Stefan problem corresponding to the hyperbolic heat equation obtained by relaxing the Fourier law, taking $\tau q_t + q = -k\theta_x$ where q is the flux, θ the temperature, k the conductivity, and $\tau > 0$ the relaxation coefficient. We prove existence and uniqueness of a smooth solution with smooth free boundary. We also study the limiting solution as $\tau \to 0$, showing (under some conditions) that it converges to the solution of the classical Stefan problem.

1. THE PHYSICAL PROBLEM

Denote by $\theta(x, t)$ the absolute temperature in a material with conductivity k and by $q(x, t)$ the flux, $-\infty < x < \infty$. In deriving the heat equation one usually assumes the Fourier law

$$q = -k\theta_x. \tag{1.1}$$

Combining it with the conservation of energy law it follows that $\theta_t = (k\theta_x)_x$, and this implies infinite speed of propagation for the heat. In a number of physical situations the Fourier law is not suitable (see [1–4, 6, 11, 15]) and has been replaced by

$$q(x, t+\tau) = -(k\theta_x)(x, t) \qquad (\tau > 0)$$

* This work is partially supported by National Science Foundation Grant DMS-8612880.

0022-247X/89 $3.00

or by its approximation

$$\tau q_t + q = -k\theta_x. \tag{1.2}$$

The resulting heat equation is then hyperbolic and yields a finite speed of propagation for θ. In order to derive the heat equation we have to combine (1.2) with the conservation of energy law

$$\frac{\partial e}{\partial t} + q_x = 0 \qquad \text{in the weak sense;} \tag{1.3}$$

here e denotes the energy, given by

$$e = e_0(\theta) + A(\theta)\, q^2, \tag{1.4}$$

where $e_0(\theta)$ is the classical internal energy based on the assumption (1.1), whereas the term $A(\theta)\, q^2$, where

$$A(\theta) = -\frac{\theta^2}{2}\frac{d}{d\theta}\left(\frac{Z(\theta)}{\theta^2}\right), \qquad Z(\theta) = \frac{\tau(\theta)}{k(\theta)}, \tag{1.5}$$

is a consequence of the second law of thermodynamics combined with (1.2); the coefficients τ, k are generally dependent on θ and on the material. The derivation of (1.4), (1.5) is due to Coleman, Fabrizio, and Owen [7] (see also [5, 10] for related derivations for deformable media).

We are interested here in a melting problem with latent heat 1; then

$$e_0(\theta) = \begin{cases} 1+\theta & \text{in the fluid} \\ \theta & \text{in the solid.} \end{cases} \tag{1.6}$$

The coefficients $k(\theta)$, $\tau(\theta)$ have jump discontinuity, in general, across the interface, and thus the assertion (1.4), (1.5) does not seem to make sense across the interface unless $Z(\theta)$ is differentiable in a neighborhood of $\theta = \theta_f$, where θ_f is the temperature of the interface. Let us therefore assume that

$$\frac{\tau(\theta)}{k(\theta)} \text{ is differentiable in } \theta, \qquad \forall\theta, \tag{1.7}$$

and then (1.4), (1.5) hold globally. From (1.3) and (1.4), (1.5) we get

$$\frac{\partial e_0}{\partial t} + q_x = -\frac{\partial}{\partial t}\left\{\left[\frac{\tau(\theta)}{\theta k(\theta)} - \frac{1}{2}\frac{d}{d\theta}\left(\frac{\tau(\theta)}{k(\theta)}\right)\right] q^2\right\} \quad \text{in the weak sense.} \tag{1.8}$$

We shall henceforth assume that the interface is given by a curve $x = s(t)$ with the fluid on the side $\{x < s(t)\}$, and that the solid is held at temperature $\theta \equiv \theta_f$; this corresponds to a 1-phase Stefan problem. Then (1.8) yields the jump relation

$$s'(t)(1 + \theta - \theta_f) = q \qquad \text{on} \quad x = s(t), \tag{1.9}$$

where θ and q are the left limits, i.e., $\theta = \theta(s(t) - 0, t)$, $q = q(s(t) - 0, t)$.

The derivation of the second condition on the free boundary $\{x = s(t)\}$ is more problematic, since it is not clear whether (1.2) should be understood as a physical law, valid weakly throughout space, or, perhaps, as a constitutive law which holds separately in the fluid and separately in the solid. Under the first setting, we obtain the jump relation

$$\tau \frac{ds}{dt} q = \theta - \theta_f \qquad \text{on} \quad x = s(t). \tag{1.10}$$

If, on the other hand, we view (1.2) as a local constitutive law, then it does not yield any conditions on the free boundary. It is then natural to assume that the temperature is continuous across the free boundary, i.e.,

$$\theta(s(t) - 0, t) = \theta_f. \tag{1.11}$$

In the sequel we shall assume that $k(\theta)$ and $\tau(\theta)$ are constants in the fluid as well as in the solid, i.e., by (1.7),

$$\frac{\tau(\theta)}{k(\theta)} = \text{const.}, \qquad \forall \theta. \tag{1.12}$$

For simplicity we may also take

$$k(\theta) = 1 \qquad \text{in the fluid}, \tag{1.13}$$

and set $\tau(\theta) = \tau$ in the fluid. Then (1.8) becomes

$$\theta_t + q_x = \tau \frac{\partial}{\partial t} \frac{q^2}{\theta} \qquad \text{in the fluid.} \tag{1.14}$$

Since the parameter τ is very small and θ_f is also a large number (so that $1/\theta$ is small), we approximate the right-hand side of (1.14) by 0; this amounts to replacing (1.8) by

$$\frac{\partial e_0}{\partial t} + q_x = 0. \tag{1.15}$$

Combining (1.15) with (1.2), we get the hyperbolic equation

$$\tau\theta_{tt} + \theta_t - \theta_{xx} = 0. \tag{1.16}$$

Showalter and Walkington [12] have considered the 2-phase Stefan problem, also making the assumption (1.12). However, they use a different expression for e and their resulting free boundary conditions are quite different from ours. Although they proved existence and uniqueness of a weak solution for the 2-phase problem, they give an example whereby their 1-phase model has no solutions.

The formulations of the 1-phase Stefan problem based on (1.16), (1.9), (1.10) were studied by Solomon, Alexiades, Wilson, and Drake [13] (see also [14]). Greenberg [9] assumed the data

$$\theta(x, 0) \equiv \theta_f, \qquad \theta(0, t) \equiv \text{const.} > \theta_f \tag{1.17}$$

and established the existence of a weak solution (with θ Lipschitz continuous).

In this paper we shall be working with the same model (1.16), (1.9), (1.10), taking the initial and boundary data to be either

$$\theta(x, 0) = \theta_0(x), \qquad q(0, t) = g(t), \tag{1.18}$$

or

$$\theta(x, 0) = \theta_0(x), \qquad \theta(0, t) = h(t) \tag{1.19}$$

with $\theta_0 \geqslant \theta_f$, $g \geqslant 0$ in case (1.18), and $\theta_0 \geqslant \theta_f$, $\theta_{0,x} \leqslant 0$, $h' \geqslant 0$ in case (1.19). We shall prove the existence of a smooth solution provided some compatibility conditions hold at $(s(0), 0)$. We shall also prove uniqueness of the solution. Finally, in case (1.19), we let $\tau \to 0$ and prove that the solutions converge to the solution of the classical Stefan problem.

2. Mathematical Formulation

Setting $T = \theta - \theta_f$, the relations (1.15), (1.2) and (1.9), (1.10) become

$$T_t + q_x = 0 \qquad \text{if} \quad 0 < x < s(t),\ t > 0, \tag{2.1}$$

$$\tau q_t + T_x + q = 0 \qquad \text{if} \quad 0 < x < s(t),\ t > 0, \tag{2.2}$$

and

$$s'(t)(1+T(s(t),t))=q(s(t),t) \qquad \text{if} \quad t>0, \tag{2.3}$$

$$\tau s'(t)q(s(t),t)=T(s(t),t) \qquad \text{if} \quad t>0, \tag{2.4}$$

where $x=s(t)$ is the free boundary.

The first initial-boundary conditions are

$$\begin{aligned} T(x,0)=\phi(x)>0, \qquad q(x,0)=\psi(x) \qquad &\text{if} \quad 0\leqslant x<s(0),\\ T(0,t)=f(t)>0 \qquad &\text{if} \quad t\geqslant 0, \end{aligned} \tag{2.5}$$

where $s(0)$ is given, and the second initial-boundary conditions are

$$\begin{aligned} T(x,0)=\phi(x)>0, \qquad q(x,0)=\psi(x) \qquad &\text{if} \quad 0\leqslant x<s(0),\\ q(0,t)=f(t)>0 \qquad &\text{if} \quad t\geqslant 0. \end{aligned} \tag{2.6}$$

As in [9] it will be convenient to work with

$$A=T+\sqrt{\tau}\,q, \qquad B=T-\sqrt{\tau}\,q. \tag{2.7}$$

Then

$$\sqrt{\tau}\,A_t+A_x+\frac{A-B}{2\sqrt{\tau}}=0 \qquad \text{if} \quad 0<x<s(t),\ t>0, \tag{2.8}$$

$$\sqrt{\tau}\,B_t-B_x+\frac{B-A}{2\sqrt{\tau}}=0 \qquad \text{if} \quad 0<x<s(t),\ t>0, \tag{2.9}$$

$$B(s(t),t)=-\frac{A}{1+2A}(s(t),t) \qquad \text{if} \quad t>0, \tag{2.10}$$

$$\frac{ds}{dt}=\frac{1}{\sqrt{\tau}}\frac{A}{1+A}(s(t),t) \qquad \text{if} \quad t>0. \tag{2.11}$$

For the Neumann problem (2.6)

$$A(x,0)=A_0(x), \qquad B(x,0)=B_0(x) \qquad (0\leqslant x<s(0)), \tag{2.12}$$

$$A(0,t)=B(0,t)+2\sqrt{\tau}\,f(t) \qquad (t>0). \tag{2.13}$$

In Sections 3–5 we study the problem (2.8)–(2.13). In Section 3 we establish a priori estimates. Existence and regularity of the solution are proved in Section 4, and uniqueness in Section 5. In Section 6 we study the corresponding Dirichlet problem (2.5). In Section 7 we let $\tau\to 0$ and show that the solutions for the Dirichlet problem converge to the solution of the classical Stefan problem.

3. A Priori Estimates for the Neumann Problem

Throughout Sections 3–5 we assume that

$$\begin{aligned} &f\in C^2[0,\infty], && \phi\in C^2[0,s(0)], && \psi\in C^2[0,s(0)];\\ &\phi(x)>-\sqrt{\tau}\,\psi(x), && \phi(x)>\sqrt{\tau}\,\psi(x)-\tfrac{1}{4} && \text{if}\quad 0\leqslant x<s(0). \end{aligned} \tag{3.1}$$

If there exists a C^1 solution of (2.8)–(2.13), then the following relations hold,

$$\begin{aligned} A_0(0)&=B_0(0)+2\sqrt{\tau}\,f(0),\\ A_{0,x}(0)+B_{0,x}(0)&=-2(\tau f'(0)+f(0)), \end{aligned} \tag{3.2}$$

and

$$s(0)=s_0,\qquad s'(0)=s_1,\qquad s''(0)=s_2, \tag{3.3}$$

where $s_0>0$ is given,

$$\begin{aligned} s_1&=\frac{1}{\sqrt{\tau}}\frac{A_0(s_0)}{1+A_0(s_0)},\\ s_2&=\frac{1}{\tau}\frac{(\sqrt{\tau}\,s_1-1)\,A_{0,x}(s_0)-\dfrac{1}{2\sqrt{\tau}}(A_0(s_0)-B_0(s_0))}{(1+A_0(s_0))^2} \end{aligned} \tag{3.4}$$

and

$$B_0(s_0)=-\frac{A_0(s_0)}{1+2A_0(s_0)}, \tag{3.5}$$

$$B_{0,x}(s_0)=\frac{A_{0,x}(s_0)-\dfrac{4}{\sqrt{\tau}}\dfrac{A_0^2(s_0)(1+A_0(s_0))^3}{1+2A_0(s_0)}}{(1+2A_0(s_0))^3}. \tag{3.6}$$

The relations (3.2) and (3.5), (3.6) are compatibility conditions which will be assumed in the sequel.

In this section we assume that A, B, s form a smooth solution of (2.8)–(2.13) and we proceed to derive a priori estimates. We shall be integrating (2.8), (2.9) along characteristics, and obtain different formulas in the four different regions described in Fig. 1.

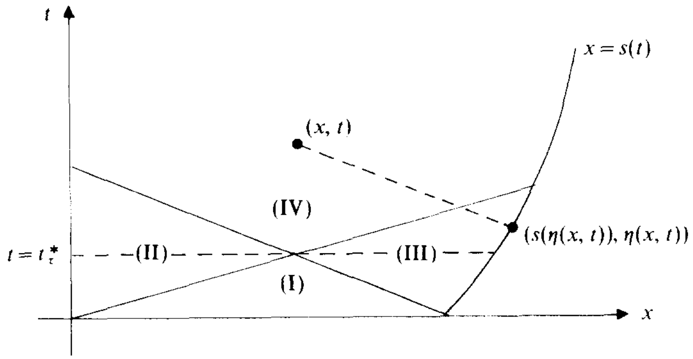

FIGURE 1

By integration we have, in $(\mathrm{I}) \cup (\mathrm{II})$,

$$B(x, t) = e^{-t/2\tau} B\left(x + \frac{t}{\sqrt{\tau}}, 0\right) + \frac{1}{2\tau} \int_0^t e^{-t/2\tau + r/2\tau} A\left(\frac{t-r}{\sqrt{\tau}} + x, r\right) dr \tag{3.7}$$

and, in $(\mathrm{III}) \cup (\mathrm{IV})$,

$$B(x, t) = e^{-(t-\eta(x,t))/2\tau} B(s(\eta(x, t)), \eta(x, t)) + \frac{1}{2\tau} \int_{\eta(x,t)}^t e^{-t/2\eta + r/2\tau} A\left(\frac{t-r}{\sqrt{2\tau}} + x, r\right) dr, \tag{3.8}$$

where

$$\frac{\eta(x, t)}{\sqrt{\tau}} + s(\eta(x, t)) = \frac{t}{\sqrt{\tau}} + x. \tag{3.9}$$

Similarly, in $(\mathrm{I}) \cup (\mathrm{III})$,

$$A(x, t) = e^{-t/2\tau} A\left(x - \frac{t}{\sqrt{\tau}}, 0\right) + \frac{1}{2\tau} \int_0^t e^{-t/2\tau + r/2\tau} B\left(\frac{r-t}{\sqrt{\tau}} + x, r\right) dr \tag{3.10}$$

and, in $(\mathrm{II}) \cup (\mathrm{IV})$,

$$A(x, t) = e^{-x/2\sqrt{\tau}} A(0, t - \sqrt{\tau}\, x) + \frac{1}{2\tau} \int_{t-\sqrt{\tau}\, x}^t e^{-t/2\tau + r/2\tau} B\left(\frac{r-t}{\sqrt{\tau}} + x, r\right) dr. \tag{3.11}$$

LEMMA 3.1. *Let (A, B) be a C^1 solution of* (2.8)–(2.10), (2.12), (2.3) *with $s'(t) < 1/\sqrt{\tau}$. Then*

$$A(x, t) > -\tfrac{1}{2}, \qquad B(x, t) > -\tfrac{1}{2} \qquad \textit{for} \quad 0 \leqslant x < s(t),\ t > 0. \tag{3.12}$$

Proof. By (3.1),

$$A_0(x) > -\tfrac{1}{2}, \qquad B_0(x) > -\tfrac{1}{2} \qquad (0 \leqslant x \leqslant s_0).$$

If the assertion is not true then there is a smallest value t_0 such that

$$A(x, t) > -\tfrac{1}{2}, \qquad B(x, t) > -\tfrac{1}{2} \qquad \text{if} \quad 0 \leqslant x \leqslant s(t),\ 0 \leqslant t < t_0$$

and

$$A(x_0, t_0) = -\tfrac{1}{2} \qquad \text{or} \qquad B(x_0, t_0) = -\tfrac{1}{2} \qquad \text{for some} \quad 0 \leqslant x_0 \leqslant s(t_0). \tag{3.13}$$

If the first equality holds then, since

$$A(0, t_0) = B(0, t_0) + 2\sqrt{\tau}\, f(t_0) \geqslant -\tfrac{1}{2} + 2\sqrt{\tau}\, f(t_0) > -\tfrac{1}{2},$$

we must have $x_0 > 0$. But, for any small $\varepsilon > 0$,

$$\begin{aligned} A(x_0, t_0) &= e^{-(t_0 - \varepsilon)/2\tau} A\left(x_0 - \frac{\varepsilon}{\sqrt{\tau}},\ t_0 - \varepsilon\right) \\ &\quad + \frac{1}{2\tau} \int_{t_0 - \varepsilon}^{t_0} e^{-(t_0 - x)/2\tau} B\left(\frac{r - t_0}{\sqrt{\tau}} + x_0,\ r\right) dr \\ &> -\frac{1}{2}\left(e^{-(t_0 - \varepsilon)/2\tau} + \left[e^{-(t_0 - r)/2\tau}\right]_{t_0 - \varepsilon}^{t_0}\right) = -\frac{1}{2}, \end{aligned}$$

a contradiction. Next, if the second equality holds in (3.13) then, since

$$B(s(t_0), t_0) > -\tfrac{1}{2} \qquad (\text{by (2.10) and } A(x, t_0) > -\tfrac{1}{2}),$$

we must have $x_0 < s(t_0)$. But then for any small $\varepsilon > 0$

$$\begin{aligned} B(x_0, t_0) &= e^{-(t_0 - \varepsilon)/2\tau} B\left(x_0 + \frac{\varepsilon}{\sqrt{\tau}},\ t_0 - \varepsilon\right) \\ &\quad + \frac{1}{2\tau} \int_{t_0 - \varepsilon}^{t_0} e^{-(t_0 - r)/2\tau} A\left(\frac{t_0 - r}{\sqrt{\tau}} + x_0,\ r\right) dr > -\frac{1}{2}, \end{aligned}$$

a contradiction.

LEMMA 3.2. *Let the assumptions of Lemma* 3.1 *hold and assume that, for some* $t_1 > 0$, $s(t) \leqslant M_\tau$ *if* $0 \leqslant t \leqslant t_1$. *Then there exists an* $\varepsilon_\tau > 0$ *depending only on* τ, t_1, M_τ, *such that*

$$A(s(t), t) \geqslant -\tfrac{1}{2} + \varepsilon_\tau \qquad \text{if} \quad 0 \leqslant t \leqslant t_1. \tag{3.14}$$

Proof. Clearly

$$f(t) \geqslant c > 0, \qquad A(0, t) = B(0, t) + 2\sqrt{\tau}\, f(t) \geqslant -\tfrac{1}{2} + 2\sqrt{\tau}\, c$$

for some $c > 0$ and $0 \leqslant t \leqslant t_1$. Further,

$$A(x, 0) > -\tfrac{1}{2} + 2\sqrt{\tau}\, c$$

if c is chosen small enough. If $(s(t), t) \in (\mathrm{III})$ then, by (3.10),

$$A(s(t), t) \geqslant e^{-t/2\tau}(-\tfrac{1}{2} + 2\sqrt{\tau}\, c) + (-\tfrac{1}{2})(1 - e^{-t/2\tau}) = -\tfrac{1}{2} + 2\sqrt{\tau}\, ce^{-t/2\tau}$$

whereas if $(s(t), t) \in (\mathrm{IV})$ then, by (3.11),

$$\begin{aligned} A(s(t), t) &\geqslant e^{-s(t)/2\sqrt{\tau}}(-\tfrac{1}{2} + 2\sqrt{\tau}\, c) + (-\tfrac{1}{2})(1 - e^{-s(t)/2\sqrt{\tau}}) \\ &\geqslant -\tfrac{1}{2} + 2\sqrt{\tau}\, ce^{-M_\tau/2\sqrt{\tau}}, \end{aligned}$$

and the lemma follows.

If we assume that (2.11) also holds then we can clearly estimate M_τ by $t_1/\sqrt{\tau}$. But we can even derive a bound which is independent of τ:

LEMMA 3.3. *If in Lemma* 3.1 *we also assume that* (2.11) *is satisfied then, for any* $t_1 > 0$,

$$s(t) \leqslant M \qquad \text{if} \quad 0 \leqslant t \leqslant t_1 \tag{3.15}$$

where M *is a constant independent of* τ.

Proof. Integrating (2.1) in x we obtain an identity which can be written in the form

$$\frac{d}{dt}\left[\int_0^{s(t)} T(x, t)\, dx + s(t)\right] = f(t).$$

It follows that

$$\int_0^{s(t)} T(x, t)\, dx + s(t) = \int_0^t f(r)\, dr + \int_0^{s(0)} T(x, 0)\, dx + s(0). \tag{3.16}$$

Since $T = (A + B)/2 > -\tfrac{1}{2}$ by Lemma 3.1, the assertion follows.

Lemma 3.3 will not be needed for the existence proof in Section 4.

LEMMA 3.4. *Let the assumptions of Lemma* 3.1 *hold and assume, in addition, that*

$$A(s(t), t) \geqslant -\tfrac{1}{2} + \eta \qquad \text{if} \quad 0 \leqslant t \leqslant t_1 \tag{3.17}$$

for some $t_1 > 0$, $\eta > 0$. *Then, for* $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant t_1$,

$$A(x, t) \leqslant C\left(1 + \frac{1}{\eta}\right), \qquad B(x, t) \leqslant C\left(1 + \frac{1}{\eta}\right), \tag{3.18}$$

where C *is a constant independent of* τ, η; C *may depend on* t_1.

Proof. Extend $f(t)$ smoothly to $(-\infty, 0)$ so that $f \geqslant 0$, and $f(t) = 0$ if $t \leqslant -\frac{1}{2}$. Let T_0 be the solution of

$$\begin{aligned} \frac{\partial T_0}{\partial t} &= \frac{\partial^2 T_0}{\partial x^2} && \text{for} \quad 0 < x < \infty,\ t > -1, \\ -\frac{\partial T_0}{\partial x} &= f(t) && \text{for} \quad x = 0,\ t > -1, \\ T_0(x, -1) &= 0 && \text{for} \quad 0 < x < \infty \end{aligned}$$

which is bounded in every strip $\{-1 \leqslant t \leqslant t_1\}$, $t_1 < \infty$. Let $q_0 = -\partial T_0/\partial x$ and define

$$\begin{aligned} A_0 &= T_0 + \sqrt{\tau}\, q_0, \qquad & B_0 &= T_0 - \sqrt{\tau}\, q_0, \\ \tilde{A} &= A - A_0, & \tilde{B} &= B - B_0. \end{aligned}$$

Then $\tilde{A}(0, t) = \tilde{B}(0, t)$ and

$$\begin{aligned} \sqrt{\tau}\, \tilde{A}_t + \tilde{A}_x + \frac{\tilde{A} - \tilde{B}}{2\sqrt{\tau}} &= -\tau q_{0,t}, \\ \sqrt{\tau}\, \tilde{B}_t - \tilde{B}_x + \frac{\tilde{B} - \tilde{A}}{2\sqrt{\tau}} &= -\tau q_{0,t}. \end{aligned}$$

Obviously

$$\begin{aligned} |\tilde{A}(x, 0)| &\leqslant \tilde{C}, \qquad & |\tilde{B}(x, 0)| &\leqslant \tilde{C}, \\ |q_0| + |q_{0,t}| &\leqslant Q & \text{if} \quad 0 \leqslant x &\leqslant s(0),\ 0 \leqslant t \leqslant t_1, \end{aligned}$$

where $\tilde{C}$, Q are constants independent of τ, η. Let

$$I(t) = \max_{0 \leqslant x \leqslant s(t)} \tilde{B}(x, t), \qquad J(t) = \max_{0 \leqslant x \leqslant s(t)} \tilde{A}(x, t), \qquad K(t) = \max\{I(t), J(t)\}.$$

We first estimate $I(t)$. Take a point $x^* = x^*(t)$ such that $I(t) = \tilde{B}(x^*, t)$. If (x^*, t) belongs to $(\mathrm{I}) \cup (\mathrm{II})$ then

$$\begin{aligned} I(t) &\leqslant \tilde{C}e^{-t/2\tau} + \frac{1}{2\tau}\int_0^t e^{-t/2\tau + r/2\tau}(K(r) + 2\tau\sqrt{\tau}\,Q)\,dr \\ &= \tilde{C}e^{-t/2\tau} + 2\tau\sqrt{\tau}\,Q(1 - e^{-t/2\tau}) + \frac{1}{2\tau}\int_0^t e^{-t/2\tau + r/2\tau}K(r)\,dr \\ &\leqslant \tilde{C}e^{-t/2\tau} + 2\tau\sqrt{\tau}\,Q + \frac{1}{2\tau}\int_0^t e^{-t/2\tau + r/2\tau}K(r)\,dr. \end{aligned} \tag{3.19}$$

On the other hand if $(x^*, t) \in (\mathrm{III}) \cup (\mathrm{IV})$ then

$$\begin{aligned} I(t) \leqslant{} & \tilde{B}(s(\eta(x^*, t)), \eta(x^*, t))\, e^{-(t - \eta(x^*, t))/2\tau} \\ & + \frac{1}{2\tau}\int_{\eta(x^*, t)}^t e^{-t/2\tau + r/2\tau}(K(r) + 2\tau\sqrt{\tau}\,Q)\,dr. \end{aligned}$$

Since, by (2.10), (3.17),

$$\tilde{B}(s(\eta(x^*, t)), \eta(x^*, t)) \leqslant \frac{\frac{1}{2} - \eta}{2\eta} - B_0(s(\eta(x^*, t)), (\eta(x^*, t)) \leqslant \frac{1}{3\eta}$$

if η is sufficiently small, we get

$$I(t) \leqslant \frac{1}{3\eta} e^{-(t - \eta(x^*, t))/2\tau} + 2\tau\sqrt{\tau}\,Q + \frac{1}{2\tau}\int_{\eta(x^*, t)}^t e^{-t/2\tau + r/2\tau}K(r)\,dr.$$

Combining this with (3.19) we find that

$$\begin{aligned} I(t) \leqslant 2\tau\sqrt{\tau}\,Q + \max_{0 \leqslant \sigma \leqslant t} \Bigg[& \left(\frac{1}{3\eta} + \tilde{C}\right) e^{-(t - \sigma)/2\tau} \\ & + \frac{1}{2\tau}\int_\sigma^t e^{-t/2\tau + r/2\tau}K(r)\,dr \Bigg]. \end{aligned} \tag{3.20}$$

We next estimate $J(t)$ in a similar way, choosing a point (x_0, t) such that $J(t) = A(x_0, t)$. If $(x_0, t) \in (\mathrm{I}) \cup (\mathrm{III})$ then

$$J(t) \leqslant \tilde{C}e^{-t/2\tau} + 2\tau\sqrt{\tau}\,Q + \frac{1}{2\tau}\int_0^t e^{-t/2\tau + r/2\tau}K(r)\,dr \tag{3.21}$$

whereas if $(x_0, t) \in (\mathrm{II}) \cup (\mathrm{IV})$ then

$$\begin{aligned} J(t) \leqslant{} & e^{-x_0/2\sqrt{\tau}}\tilde{A}(0, t - \sqrt{\tau}\,x_0) + 2\tau\sqrt{\tau}\,Q \\ & + \frac{1}{2\tau}\int_{t - \sqrt{\tau}\,x_0}^t e^{-t/2\tau + r/2\tau}K(r)\,dr. \end{aligned} \tag{3.22}$$

By (3.20),

$$\tilde{A}(0, t-\sqrt{\tau}\, x_0) = \tilde{B}(0, t-\sqrt{\tau}\, x_0) \leqslant I(t-\sqrt{\tau}\, x_0)$$

$$\leqslant 2\tau\sqrt{\tau}\, Q + \max_{0\leqslant\sigma\leqslant t-\sqrt{\tau}x_0}\left[\left(\frac{1}{3\eta}+\tilde{C}\right)e^{-(t-\sqrt{\tau}x_0-\sigma)/2\tau}\right.$$

$$\left.+\frac{1}{2\tau}\int_\sigma^{t-\sqrt{\tau}x_0} e^{-(t-\sqrt{\tau}x_0)/2\tau+r/2\tau}K(r)\,dr\right].$$

Using this estimate in (3.22), we get

$$J(t) \leqslant 4\tau\sqrt{\tau}\, Q + \max_{0\leqslant\sigma\leqslant t}\left[\left(\frac{1}{3\eta}+\tilde{C}\right)e^{-(t-\sigma)/2\tau}\right.$$

$$\left.+\frac{1}{2\tau}\int_\sigma^{t} e^{-t/2\tau+r/2\tau}K(r)\,dr\right].$$

Combining this with (3.21) and recalling also the estimate (3.20), we conclude that

$$K(t) \leqslant 4\tau\sqrt{\tau}\, Q + \max_{0\leqslant\sigma\leqslant t}\left[\left(\frac{1}{3\eta}+\tilde{C}\right)e^{-(t-\sigma)/2\tau}\right.$$

$$\left.+\frac{1}{2\tau}\int_\sigma^{t} e^{-t/2\tau+r/2\tau}K(r)\,dr\right].$$

This implies that

$$K(t) \leqslant 4\tau\sqrt{\tau}\, Q + 2\sqrt{\tau}\, Qt + \frac{1}{3\eta} + \tilde{C};$$

hence $A(x, t) \leqslant K(t) + A_0(x, t) \leqslant C(1+1/\eta)$, and similarly $B(x, t) \leqslant C(1+1/\eta)$.

Remark 3.1. Lemmas 3.1, 3.2, 3.4 give a bound (assuming $s_\tau(t) \leqslant M_\tau$)

$$|A(x, t)| + |B(x, t)| \leqslant N_\tau. \tag{3.23}$$

If we assume that (2.11) holds then the assumption $s_\tau(t) \leqslant M_\tau$ is satisfied; further (3.16) is valid and we deduce (since $|T| \leqslant 2N_\tau$) that

$$s(t) \geqslant c_\tau > 0 \qquad \text{if} \quad 0 \leqslant t \leqslant t_1; \tag{3.24}$$

further, by (3.23), (3.14),

$$|s'(t)| \leqslant \frac{1}{\sqrt{\tau}}(1-\tilde{\varepsilon}_\tau), \tag{3.25}$$

where

$$\tilde{\varepsilon}_\tau = \min\left\{\frac{1}{1+N_\tau}, \frac{4\varepsilon_\tau}{1+2\varepsilon_\tau}\right\}. \tag{3.26}$$

Later on we shall have to consider a hyperbolic system

$$\begin{aligned} \sqrt{\tau}\, A_t + A_x + \frac{A-B}{2\sqrt{\tau}} &= 0, \\ \sqrt{\tau}\, B_t - B_x + \frac{B-A}{2\sqrt{\tau}} &= 0 \end{aligned} \tag{3.27}$$

with more general boundary conditions

$$B(s(t), t) = b(t, A(s(t), t)), \tag{3.28}$$

$$A(0, t) = a(t, B(0, t)) \tag{3.29}$$

and initial conditions

$$A(x, 0) = A_0(x), \qquad B(x, 0) = B_0(x). \tag{3.30}$$

LEMMA 3.5. *Assume that* (A, B) *is a* C^1 *solution of* (3.27)–(3.30) *with* $s \in C^1$, *and*

$$s(t) \geqslant c_\tau > 0, \qquad |s'(t)| < \frac{1}{\sqrt{\tau}} \qquad \text{for} \quad 0 < t < t_1. \tag{3.31}$$

Assume further that

$$\begin{aligned} &|A_0(x)| \leqslant M, && |B_0(x)| \leqslant M, \\ &|b(t, \lambda)| \leqslant L + K|\lambda|, && |a(t, \lambda)| \leqslant L + K|\lambda|. \end{aligned} \tag{3.32}$$

Then

$$|A(x, t)|,\ |B(x, t)| \leqslant C_\tau \qquad \text{for} \quad 0 \leqslant t \leqslant t_1, \tag{3.33}$$

where C_τ *is a constant depending only on* $t_1, \tau, M, L, K, c_\tau$.

Proof. It suffices to prove (3.33) for $t \leqslant t_\tau^*$ (see Fig. 1); for since $s(t) \geqslant c_\tau > 0$, we can continue step-by-step on t-intervals (of fixed length $\geqslant c_\tau^* > 0$). Let

$$I(t) = \max_{0 \leqslant x \leqslant s(t)} |B(x, t)|, \qquad J(t) = \max_{0 \leqslant x \leqslant s(t)} |A(x, t)|.$$

Proceeding by integration, as in (3.7)–(3.11), and using (3.28), (3.29), we obtain, very crudely,

$$I(t)+J(t)\leqslant 2(L+KM)+\frac{K+1}{2\tau}\int_0^t (I(r)+J(r))\,dr$$

for $0\leqslant t\leqslant t_\tau^*$, and (3.31) then follows.

The next lemma establishes Lipschitz bounds on the solution of (3.26)–(3.30).

LEMMA 3.6. *Assume that* (A, B) *is a* C^1 *solution of* (3.27)–(3.30) *with* $s\in C^1$, *and*

$$s(t)\geqslant c_\tau>0, \qquad |s'(t)|\leqslant\frac{1}{\sqrt{\tau}}(1-\varepsilon_\tau) \qquad \textit{for} \quad 0\leqslant t\leqslant t_1, \tag{3.34}$$

where $\varepsilon_\tau>0$. *Assume further that*

$$\begin{aligned} &|A(x,t)|\leqslant M_0, \qquad |B(x,t)|\leqslant M_0, \qquad \textit{for} \quad t\leqslant t_1,\\ &|A_{0,x}(x)|\leqslant M_1, \qquad |B_{0,x}(x)|\leqslant M_1 \end{aligned} \tag{3.35}$$

and that $a(t,\lambda)$, $b(t,\lambda)$ *are Lipschitz continuous and*

$$|a_t|,\ |a_\lambda|,\ |b_t|,\ |b_\lambda|\leqslant K. \tag{3.36}$$

Then

$$|A_x|,\ |B_x|,\ |A_t|,\ |B_t|\leqslant C_\tau \qquad \textit{for} \quad t\leqslant t_1, \tag{3.37}$$

where C_τ *is a constant depending only on* τ, c_τ, M_0, M_1, K, *and* t_1.

Proof. Take for simplicity $\tau=1$. Let h be any small positive number, and introduce the functions

$$I(t)=\max_x \max_{|\xi|\leqslant h,\,|\sigma|\leqslant h} |B(x+\xi,t+\sigma)-B(x,t)|,$$

$$J(t)=\max_x \max_{|\xi|\leqslant h,\,|\sigma|\leqslant h} |A(x+\xi,t+\sigma)-A(x,t)|,$$

where (x,t) and $(x+\xi,t+\sigma)$ are required to belong to the region $\{(x',t');0\leqslant x'\leqslant s(t'),\ 0\leqslant t'\leqslant t_1\}$. For any $t\in(0,t_\tau^*)$, choose x^*,ξ^*,σ^* such that

$$I(t)=|B(x^*+\xi^*,t+\sigma^*)-B(x^*,t)|.$$

Case 1. (x^*, t) and $(x^* + \xi^*, t + \sigma^*)$ belong to $(\mathrm{I}) \cup (\mathrm{II})$. Then

$$\begin{aligned}
I(t) &\leqslant |e^{-(t+\sigma^*)/2} B(x^* + \xi^* + t + \sigma^*, 0) - e^{-t/2} B(x^* + t, 0)| \\
&\quad + \frac{1}{2} \left| \int_0^{t+\sigma^*} e^{-(t+\sigma^*)/2 + r/2} A(t + \sigma^* + x^* + \xi^* - r, r)\, dr \right. \\
&\quad \left. - \int_0^t e^{-t/2 + r/2} A(t + x^* - r, r)\, dr \right| \\
&\leqslant M_0 h + M_1 (2h) + \frac{1}{2} \left| \int_{-\sigma^*}^t e^{-t/2 + r/2} A(t + x^* + \xi^* - r, r + \sigma^*)\, dr \right. \\
&\quad \left. - \int_0^t e^{-t/2 + r/2} A(t + x^* - r, r)\, dr \right| \\
&\leqslant 2(M_0 + M_1) h + \frac{1}{2} \int_0^t J(r)\, dr.
\end{aligned}$$

Case 2. One of the points (x^*, t), $(x^* + \xi^*, t + \sigma^*)$ lies in $(\mathrm{I}) \cup (\mathrm{II})$ and the other lies in (III). For definiteness we take

$$(x^*. t) \in (\mathrm{I}) \cup (\mathrm{II}), \qquad (x^* + \xi^*, t + \sigma^*) \in (\mathrm{III}). \tag{3.38}$$

Let $\eta = \eta(x^* + \xi^*, t + \sigma^*)$. From (3.38) and the second inequality in (3.34) we deduce that $\eta < 2h/\varepsilon_\tau$. Therefore, from

$$\begin{aligned}
B(x^* + \xi^*, t + \sigma^*) &= e^{-(t+\sigma^* - \eta)/2} B(s(\eta), \eta) \\
&\quad + \frac{1}{2} \int_\eta^{t+\sigma^*} e^{-(t+\sigma^*)/2 + r/2} A(t + x - r, r)\, dr
\end{aligned}$$

and from the corresponding expression for $B(x^*, t)$ we get

$$I(t) \leqslant Ch + \frac{1}{2} \int_0^t J(r)\, dr + |B(s(\eta), \eta) - B(x^* + t, 0)|. \tag{3.39}$$

Next

$$\begin{aligned}
&|B(s(\eta), \eta) - B(x^* + t, 0)| \\
&\quad \leqslant |B(s(\eta), \eta) - B(s(0), 0)| + |B(s(0), 0) - B(x^* + t, 0)| \\
&\quad \leqslant Ch + |b(\eta, A(s(\eta), \eta)) - b(0, A_0(s(0))| \qquad (\text{since } |x^* + t - s(0)| \leqslant Ch) \\
&\quad \leqslant Ch + K|A(s(\eta), \eta) - A_0(s(0))|.
\end{aligned} \tag{3.40}$$

Since $(s(\eta), \eta) \in (\text{III})$,

$$A(s(\eta), \eta) = e^{-\eta/2} A_0(s(\eta) - \eta) + \frac{1}{2} \int_0^{\eta} e^{-\eta/2 + r/2} B(r - \eta + x, r)\, dr$$

and thus

$$|A(s(\eta), \eta) - A_0(s(0))| \leqslant |e^{-\eta/2} A_0(s(\eta) - \eta) - A_0(s(0))| + Ch \leqslant C_1 h.$$

Using this in (3.40) and substituting the resulting estimate into (3.39), we get

$$I(t) \leqslant Ch + \frac{1}{2} \int_0^t J(r)\, dr.$$

Case 3. (x^*, t) and $(x^* + \xi^*, t + \sigma^*)$ belong to (III). Let $\eta_0 = \eta(x^*, t)$, $\eta_1 = \eta(x^* + \xi^*, t + \sigma^*)$. Then

$$\begin{aligned} B(x^* + \xi^*, t + \sigma^*) &= e^{-(t + \sigma^* - \eta_1)/2} B(s(\eta_1), \eta_1) \\ &\quad + \frac{1}{2} \int_{\eta_1}^{t + \sigma^*} e^{-(t + \sigma^*)/2 + r/2} A(t + x - r, r)\, dr, \\ B(x^*, t) &= e^{(t - \eta)/2} B(s(\eta_0), \eta_0) \\ &\quad + \frac{1}{2} \int_{\eta_0}^{t} e^{-t/2 + r/2} A(t + x - r, r)\, dr. \end{aligned}$$

From (3.19) and the second inequality in (3.34) we deduce that $|\eta_0 - \eta_1| \leqslant 2h/\varepsilon_\tau$. Therefore

$$I(t) \leqslant Ch + \frac{1}{2} \int_0^t J(r)\, dr + |B(s(\eta_1), \eta_1) - B(s(\eta_0), \eta_0)|.$$

Next

$$\begin{aligned} |B(s(\eta_1), \eta_1) - B(s(\eta_0), \eta_0)| &= |b(\eta_1, A(s(\eta_1), \eta_1)) - b(\eta_0, A(s(\eta_0), \eta_0))| \\ &\leqslant Ch + K|A(s(\eta_1), \eta_1) - A(s(\eta_0), \eta_0)|. \end{aligned}$$

Since $(s(\eta_1), \eta_1)$ and $(s(\eta_0), \eta_0)$ belong to (III), we can estimate

$$|A(s(\eta_1), \eta_1) - A(s(\eta_0), \eta_0)| \leqslant Ch + \frac{1}{2} \int_0^t I(r)\, dr.$$

Combining these estimates we find that

$$I(t) \leqslant Ch + \frac{1}{2}\int_0^t J(r)\,dr + \frac{K}{2}\int_0^t I(r)\,dr.$$

This estimate is valid in all three cases.

We can derive similar estimates for $J(t)$, and thus

$$I(t) + J(t) \leqslant Ch + \frac{1+K}{2}\int_0^t (I(r) + J(r))\,dr.$$

This implies that

$$I(t) + J(t) \leqslant \tilde{C}h \qquad \text{if} \quad 0 \leqslant t \leqslant t_\tau^*.$$

We can now proceed step-by-step to establish this inequality for all $t \leqslant t_1$, with a different $\tilde{C}$ (which will eventually depend on τ and t_1).

4. Existence and Regularity

Theorem 4.1. *If* (3.1), (3.2) *and* (3.5), (3.6) *hold, then there exists a solution* (A, B, s) *of* (2.8)–(2.13) *with* $s \in C^{2,1}$ *and* A, B *in* $C^{1,1}$ *up to the boundary.*

Proof. Given any $t_1 \in (0, \infty)$ we shall construct a solution for $t \leqslant t_1$. Let c_τ be defined as in (3.24) when

$$s(t) \leqslant M_\tau \equiv s_0 + \frac{t_1}{\sqrt{\tau}} \equiv c_0 \tag{4.1}$$

is assumed; defining $\tilde{\varepsilon}_\tau$ by (3.26) with N_τ determined as in (3.23) (N_τ depends on the choice M_τ in (4.1)), we set

$$c_1 = \frac{1}{\sqrt{\tau}}(1 - \tilde{\varepsilon}_\tau), \tag{4.2}$$

$$\sigma = \frac{c_\tau}{2c_1}. \tag{4.3}$$

Let

$$K = \{s \in C^2[0, \sigma];\ s(0) = s_0,\ s'(0) = s_1,\ s''(0) = s_2,$$
$$|s'|_{L^\infty(0,\sigma)} \leqslant c_1,\ |s''|_{L^\infty(0,\sigma)} \leqslant c_2\},$$

where s_1, s_2 are defined by (3.4), c_1 is as in (4.2), and c_2 is still to be determined. If $s \in K$ then

$$\frac{c_\tau}{2} < s_0 - c_1\sigma \leqslant s(t) \leqslant s_0 + c_1 t_1 < c_0. \tag{4.4}$$

Given $s \in K$ we can solve the system (2.8)–(2.10) with the initial-boundary conditions (2.12), (2.13). In fact, this can be done by successive integrations, with A^m, B^m defined by

$$\sqrt{\tau}\, A_t^{n+1} + A_x^{n+1} + \frac{A^n - B^n}{2\sqrt{\tau}} = 0,$$

$$\sqrt{\tau}\, B_t^{n+1} - B_x^{n+1} + \frac{B^n - A^n}{2\sqrt{\tau}} = 0.$$

Letting

$$I_n(t) = \max_x |B^{n+1}(x, t) - B^n(x, t)|, \qquad J_n(t) = \max_x |A^{n+1}(x, t) - A^n(x, t)|$$

one can estimate successively $I_n + J_n$ and thus show that the sequences A^m, B^m are uniformly convergent to a solution A, B which is Lipschitz continuous. By Lemmas 3.1, 3.2, 3.4 we have

$$|A(x(t)|, |B(x, t)| \leqslant N_\tau \tag{4.5 }$$

with N_τ determined as in (3.23); it is the same N_τ as used above in determining the $\tilde{\varepsilon}_\tau$ in (4.2) by means of (3.26). By formal differentiation

$$\begin{aligned} &\sqrt{\tau}\, A_{xt} + A_{xx} + \frac{A_x - B_x}{2\sqrt{\tau}} = 0, \\ &\sqrt{\tau}\, B_{xt} - B_{xx} + \frac{B_x - A_x}{2\sqrt{\tau}} = 0 \end{aligned} \tag{4.6}$$

and

$$A_x(0, t) = -B_x(0, t) - f(t) - \tau f'(t), \tag{4.7}$$

$$\begin{aligned} B_x(s(t), t) = {} & \frac{1 - \sqrt{\tau}\, s'(t)}{1 + \sqrt{\tau}\, s'(t)} \cdot \frac{A_x(s(t), t)}{(1 + 2A(s(t), t))^2} \\ & + \frac{1}{2\sqrt{\tau}\,(1 + \sqrt{\tau}\, s'(t))} \cdot \left(\frac{A - B}{1 + 2A} + B - A\right)(s(t), t), \end{aligned} \tag{4.8}$$

and

$$A_x(x,0)=A_{0,x}(x), \qquad B_x(x,0)=B_{0,x}(x). \tag{4.9}$$

In order to justify it, we consider (4.6)–(4.9) as a system for A_x, B_x and denote its solution by $\tilde{A}$, $\tilde{B}$. Next we finite difference the equations for A, B and subtract the resulting equations, for

$$\tilde{A}_h \equiv \frac{A(x+h,t)-A(x,t)}{h}, \qquad \tilde{B}_h \equiv \frac{B(x+h,t)-B(x,t)}{h},$$

from the corresponding equations for $\tilde{A}$, $\tilde{B}$. We then proceed to estimate $\tilde{A}-\tilde{A}_h$, $\tilde{B}-\tilde{B}_h$ similarly to the proof of Lemma 3.6, and thus conclude that

$$\tilde{A}_h \to \tilde{A} = A_x, \qquad \tilde{B}_h \to \tilde{B} = B_x.$$

Since we need only the integrated forms of the hyperbolic system in the proofs of the previous lemmas, we can now work with the integrated form of the differential system for A_{xx}, B_{xx}, i.e., with integral equations for A_x and B_x.

From (4.6)–(4.9) and Lemma 3.5 (with appropriate choice of $a(t,\lambda)$, $b(t,\lambda)$) we get

$$|A_x|, |A_t|, |B_x|, |B_t| \leqslant \tilde{M}_1, \tag{4.10}$$

where $\tilde{M}_1$ depends only on $c_\tau/2$, $\tilde{\varepsilon}_\tau$; applying Lemma 3.6 to the same system one gets, a.e.,

$$|D^2A|, |D^2B| \leqslant M_2 \tag{4.11}$$

for any second derivative D^2. We now define a mapping $G: K \to C^2[0,\sigma]$ by

$$Gs(t) \equiv (Gs)(t) = s_0 + \int_0^t \frac{1}{\sqrt{\tau}} \frac{A}{1+A}(s(r),r)\,dr.$$

Then, since A, B are in $C^{1,1}$,

$$Gs(0)=s_0, \qquad \frac{d}{dt}Gs(0)=s_1, \qquad \frac{d^2}{dt^2}Gs(0)=s_2$$

and

$$\left|\frac{d}{dt}Gs(t)\right| = \frac{1}{\sqrt{\tau}} \frac{|A(s(t),t)|}{1+A(s(t),t)} \leqslant c_1$$

by (3.23) (or (4.5)) and (3.26), (4.2). From the relation

$$\frac{d^2}{dt^2} Gs(t) = \frac{1}{\sqrt{\tau}} \frac{A_x(s(t), t)s'(t) + A_t(s(t), t)}{1 + A(s(t), t))^2}$$

we obtain

$$\left|\frac{d^2}{dt^2} Gs(t)\right| \leqslant \frac{4}{\sqrt{\tau}} \tilde{M}_1(1 + c_1) \qquad \text{by} \quad (4.11),$$

$$\left|\frac{d^2}{dt^2} Gs(t+h) - \frac{d^2}{dt^2} Gs(h)\right| \leqslant c_3|h| \qquad \text{by} \quad (4.11).$$

Choosing

$$c_2 = \frac{4}{\sqrt{\tau}} \tilde{M}_1(1 + c_1)$$

we conclude that G maps K into itself, it is continuous, and the set $G(K)$ is equicontinuous. Thus G has a fixed point, say $\bar{s}(t)$.

This proves the existence of a solution in $[0, \sigma]$. Since $\bar{s}$ is a fixed point, (2.11) is satisfied and hence (3.24) holds for this solution. But then we can extend the solution to $[\sigma, 2\sigma]$ by the same argument, using the same constant c_1; it is easily seen that at $t = \sigma$ the two solutions fit in a $C^{1,1}$ fashion.

Proceeding step-by-step we can now extend the solution to all of $\{0 < t \leqslant t_1\}$. Next we can extend the solution to $\{t_1 \leqslant t \leqslant 2t_1\}$ with c_1, $\tilde{\varepsilon}_\tau$ depending on a new constant N_τ (for which (4.1) is valid in $0 \leqslant t \leqslant 2t_1$), then to $2t_1 \leqslant t \leqslant 3t_1$, etc.

Remark 4.1. If we assume that f, A_0, B_0 are in C^3 and satisfy a compatibility condition involving $B_{0,xx}$, $A_{0,xx}$, then we can derive by the above method a solution A, B in $C^{2,1}$ with $s \in C^{3,1}$. Similarly we can establish higher regularity for the solution.

5. Uniqueness

THEOREM 5.1. *For any $t_1 > 0$ there exists at most one C^1 solution (A, B, s) of (2.8)–(2.13) for $0 \leqslant t \leqslant t_1$.*

Proof. Take for simplicity $\tau = 1$. Suppose there exist two solutions (A, B, s) and $(\tilde{A}, \tilde{B}, \tilde{s})$. Let M be any constant such that

$$|E|, |E_x|, |E_t| \leqslant M \qquad \text{for} \quad E = A, B, \tilde{A}, \tilde{B}. \tag{5.1}$$

By Lemma 3.2

$$A(s(t), t), \tilde{A}(\tilde{s}(t), t) \geqslant -\tfrac{1}{2} + \varepsilon_0 \tag{5.2}$$

for some $\varepsilon_0 > 0$, and then, by Remark 3.1,

$$\left|\frac{ds}{dt}\right|, \left|\frac{d\tilde{s}}{dt}\right| \leqslant 1 - \varepsilon_0 \tag{5.3}$$

for all $0 \leqslant t \leqslant t_1$. Let

$$\begin{aligned} s_0(t) &= \min\{s(t), \tilde{s}(t)\}, \qquad I(t) = |s'(t) - \tilde{s}'(t)|, \\ J(t) &= \max_{0 \leqslant x \leqslant s_0(t)} |A(x, t) - \tilde{A}(x, t)|, \\ K(t) &= \max_{0 \leqslant x \leqslant s_0(t)} |B(x, t) - \tilde{B}(x, t)| \end{aligned}$$

and set $(\mathrm{III})_0 = (\mathrm{III}) \cap (\widetilde{\mathrm{III}})$, where $(\widetilde{\mathrm{III}})$ is defined as (III) with respect to $\tilde{s}$ (see Fig. 1).

Let $0 < t < t_\tau^*$ $(\tau = 1)$ and choose x^* such that

$$J(t) = |A(x^*, t) - \tilde{A}(x^*, t)|, \qquad 0 \leqslant x^* \leqslant s_0(t).$$

If $(x^*, t) \in (\mathrm{I}) \cup (\mathrm{III})_0$ then, by (3.10),

$$J(t) \leqslant \frac{1}{2} \int_0^t |B(r - t + x^*, r) - \tilde{B}(r - t + x^*, r)| \, dr \leqslant \frac{1}{2} \int_0^t K(r) \, dr. \tag{5.4}$$

If $(x^*, t) \in (\mathrm{II})$ then, by (3.11),

$$J(t) \leqslant \frac{1}{2} \int_0^t K(r) \, dr + |A(0, t - x^*) - \tilde{A}(0, t - x^*)|.$$

By (2.13) we have

$$\begin{aligned} |A(0, t - x^*) - \tilde{A}(0, t - x^*)| &= |B(0, t - x^*) - \tilde{B}(0, t - x^*)| \\ &\leqslant \frac{1}{2} \int_0^{t - x^*} J(r) \, dr \leqslant \frac{1}{2} \int_0^t J(r) \, dr \end{aligned}$$

and therefore

$$J(t) \leqslant \frac{1}{2} \int_0^t (J(r) + K(r)) \, dr. \tag{5.5}$$

Recalling (5.4) we conclude that (5.4) holds in $(\mathrm{I}) \cup (\mathrm{II}) \cup (\mathrm{III})_0$.

Next

$$\max_{0\leqslant t'\leqslant t} |s(t') - \tilde{s}(t')| \leqslant \int_0^t I(r)\, dr. \tag{5.6}$$

Let (x_0, t) be such that

$$K(t) = |B(x_0, t) - \tilde{B}(x_0, t)|, \qquad 0 \leqslant x_0 \leqslant s_0(t).$$

If $(x_0, t) \in (\mathrm{I}) \cup (\mathrm{II})$ then, by (3.7),

$$K(t) \leqslant \frac{1}{2} \int_0^t J(r)\, dr. \tag{5.7}$$

If $(x_0, t) \in (\mathrm{III})_0$ then assume for definiteness that $\eta(x_0, t) \geqslant \tilde{\eta}(x_0, t)$, and set $\eta = \eta(x_0, t)$, $\tilde{\eta} = \tilde{\eta}(x_0, t)$. By (3.9)

$$0 \leqslant \eta - \tilde{\eta} = \tilde{s}(\tilde{\eta}) - s(\eta) \leqslant \tilde{s}(\tilde{\eta}) - s(\tilde{\eta}) + (1 - \varepsilon_0)(\eta - \tilde{\eta})$$

and therefore, by (5.6),

$$0 \leqslant \eta - \tilde{\eta} \leqslant \frac{1}{\varepsilon_0} \int_0^t I(r)\, dr. \tag{5.8}$$

Using (3.8) we get

$$\begin{aligned}
K(t) &\leqslant |e^{-(t-\eta)/2} B(s(\eta), \eta) - e^{-(t-\tilde{\eta})/2} \tilde{B}(\tilde{s}(\tilde{\eta}), \tilde{\eta})| \\
&\quad + \frac{1}{2} \int_{\tilde{\eta}}^{\eta} |\tilde{A}(t + x_0 - r, r)|\, dr + \frac{1}{2} \int_{\eta}^{t} J(r)\, dr \\
&\leqslant 2M(\eta - \tilde{\eta}) + \frac{1}{2} \int_0^t J(r)\, dr \\
&\quad + |B(s(\eta), \eta) - \tilde{B}(\tilde{s}(\tilde{\eta}), \tilde{\eta})| \\
&\leqslant \frac{2M}{\varepsilon_0} \int_0^t I(r)\, dr + \frac{1}{2} \int_0^t J(r)\, dr \\
&\quad + C_M |A(s(\eta), \eta) - \tilde{A}(\tilde{s}(\tilde{\eta}), \tilde{\eta})|,
\end{aligned} \tag{5.9}$$

where (2.10) was used both for A, B and for $\tilde{A}$, $\tilde{B}$.

We can clearly find a point $(\xi, \alpha) \in (\mathrm{III})_0$ with $\tilde{\eta} \leqslant \alpha \leqslant \eta$ such that

$$|\xi - s(\eta)| \leqslant \max_{\tilde{\eta} \leqslant r \leqslant \eta} |\tilde{s}(r) - s(r)| + \max_{\tilde{\eta} \leqslant r, \sigma \leqslant \eta} |s(r) - s(\sigma)|,$$

$$|\xi - \tilde{s}(\tilde{\eta})| \leqslant \max_{\tilde{\eta} \leqslant r \leqslant \eta} |\tilde{s}(r) - s(r)| + \max_{\tilde{\eta} \leqslant r, \sigma \leqslant \eta} |\tilde{s}(r) - \tilde{s}(\sigma)|.$$

Hence

$$|\xi - s(\eta)| \leqslant \int_{\tilde{\eta}}^{\eta} I(r)\,dr + (\eta - \tilde{\eta}) \leqslant \frac{C}{\varepsilon_0} \int_0^t I(r)\,dr, \tag{5.10}$$

where (5.8) was used and, similarly,

$$|\xi - \tilde{s}(\tilde{\eta})| \leqslant \frac{C}{\varepsilon_0} \int_0^t I(r)\,dr. \tag{5.11}$$

Using (5.10), (5.11) we can estimate the last term in (5.9) by

$$\begin{aligned} |A(s(\eta), \eta) - \tilde{A}(\tilde{s}(\tilde{\eta}), \tilde{\eta})| &\leqslant |A(s(\eta), \eta) - A(\xi, \alpha)| \\ &\quad + |A(\xi, \alpha) - \tilde{A}(\xi, \alpha)| + |\tilde{A}(\xi, \alpha) - \tilde{A}(\tilde{s}(\tilde{\eta}), \tilde{\eta})| \\ &\leqslant 2M(|s(\eta) - \xi| + |\tilde{s}(\tilde{\eta}) - \xi| + |\eta - \alpha| \\ &\quad + |\tilde{\eta} - \alpha|) + \frac{1}{2}\int_0^{\alpha} K(r)\,dr \\ &\leqslant \tilde{C} \int_0^t I(r)\,dr + \frac{1}{2}\int_0^t K(r)\,dr \end{aligned} \tag{5.12}$$

with an appropriate constant $\tilde{C}$. We then get, from (5.9),

$$K(t) \leqslant \frac{1}{2}\int_0^t J(r)\,dr + \tilde{C}\left[\int_0^t K(r)\,dr + \int_0^t I(r)\,dr\right] \tag{5.13}$$

with another constant $\tilde{C}$. Next, from (2.11) and (5.2),

$$\begin{aligned} I(t) = |s'(t) - \tilde{s}'(t)| &\leqslant C_M |A(s(t), t) - \tilde{A}(\tilde{s}(t), t)| \\ &\leqslant \tilde{C}\left[\int_0^t I(r)\,dr + \int_0^t K(r)\,dr\right] \end{aligned}$$

by the same estimate as in (5.12).

Combining this with the previous estimates on $J(t)$ and $K(t)$ we get

$$I(t) + J(t) + K(t) \leqslant \tilde{C} \int_0^t [I(r) + J(r) + K(r)]\,dr.$$

This implies $I(t) = J(t) = K(t) = 0$ for $0 \leqslant t \leqslant t_\tau^*$. Proceeding step-by-step we establish uniqueness for all $0 \leqslant t \leqslant t_1$.

6. The First Initial-Boundary Problem

In this section we extend the results of Sections 3–5 to the first initial-boundary conditions (2.5). Introducing A and B as in (2.7), these conditions consist of (2.12) and of

$$A(0, t) = 2f(t) - B(0, t) \qquad (t > 0). \tag{6.1}$$

Condition (2.10) was used in the proof of Lemma 3.1 (where we deduced that $A(0, t) > -\frac{1}{2}$). Under (6.1) we cannot extend Lemma 3.1. If however we make the a priori assumption that, for any smooth solution,

$$A(s(t), t) \geqslant -\tfrac{1}{2} + \varepsilon_\tau \qquad \text{for some} \quad \varepsilon_\tau > 0 \tag{6.2}$$

then we can extend Lemma 3.4 and Remark 3.1; thus we obtain bounds $|A| \leqslant N_\tau$, $|B| \leqslant N_\tau$ with N_τ depending on τ.

The proofs of Lemmas 3.5, 3.6 remain valid for (6.1), provided condition (6.2) is assumed, so does the proof of Theorem 4.1. Hence:

Lemma 6.1. *If* (3.1), (3.2) *and* (3.5), (3.6) *hold, and if one can establish the a priori estimate* (6.2) *for* $0 \leqslant t \leqslant t_1$ *with any* $t_1 \in (0, \infty)$ (ε_τ *depends on* τ, t_1), *then there exists a solution* (A, B, s) *of* (2.8)–(2.12), (6.1) *with* $s \in C^{2,1}$ *and* A, B *in* $C^{1,1}$ *up to the boundary.*

We next prove:

Lemma 6.2. *Assume that the data in* (2.5) *satisfy*

$$\begin{aligned} &f'(t) \geqslant 0 && \text{for} \quad 0 \leqslant t < \infty, \\ &-\phi'(x) > \sqrt{\tau}\, |\psi'(x)| && \text{for} \quad 0 \leqslant x \leqslant s_0,\ \phi(s_0) + \sqrt{\tau}\, \psi(s_0) > 0. \end{aligned} \tag{6.3}$$

Then for any $C^{1,1}$ *solution of* (2.8)–(2.12), (6.1) *there holds*

$$A_x < 0, \qquad B_x < 0, \tag{6.4}$$

$$A(s(t), t) > 0, \tag{6.5}$$

$$T(x, t) > 0. \tag{6.6}$$

Notice that the last condition in (3.1) follows from (6.3).

Proof. We first prove that (6.4) and (6.5) hold for all t. Indeed, by (6.3) and continuity, (6.4) and (6.5) hold for all t sufficiently small. If either (6.4) or (6.5) is not true for all $t > 0$, then there exists a smallest t_0 such that (6.4) and (6.5) hold for $0 \leqslant x \leqslant s(t)$, $0 \leqslant t < t_0$, and

$$\text{either } A_x(x_0, t_0) = 0 \text{ or } B_x(x_0, t_0) = 0, \tag{6.7}$$

for some $x_0 \in [0, s(t_0)]$. The functions $\tilde{A} = A_x$, $\tilde{B} = B_x$ satisfy

$$\sqrt{\tau}\, \tilde{A}_t + \tilde{A}_x + \frac{\tilde{A} - \tilde{B}}{\sqrt{\tau}} = 0, \qquad \sqrt{\tau}\, \tilde{B}_t - \tilde{B}_x + \frac{\tilde{B} - \tilde{A}}{\sqrt{\tau}} = 0,$$
$$\tilde{A}(0, t) = \tilde{B}(0, t) - 2\sqrt{\tau}\, f'(t) \leqslant \tilde{B}(0, t)$$

and, on $x = s(t)$,

$$\tilde{B} = \frac{\tilde{A} - (4/\sqrt{\tau})\, A^2(1 + A)^3/(1 + 2A)}{(1 + 2A)^3} \qquad \text{by (2.10).}$$

We can now proceed as in the proof of Lemma 3.1 to establish that $\tilde{A}(x_0, t_0) < 0$ and $\tilde{B}(x_0, t_0) < 0$. Recalling (6.7) we conclude that $A(s(t_0), t_0) = 0$. Next

$$\begin{aligned} \frac{d}{dt} A(s(t), t)) &= A_x s' + A_x \\ &= A_x \frac{1}{\sqrt{\tau}} \frac{A}{1 + A} - \frac{1}{\sqrt{\tau}} \left(A_x + \frac{A - B}{2\sqrt{\tau}} \right) && \text{(by (2.8), (2.11))} \\ &= -\frac{A_x}{\sqrt{\tau}} \frac{1}{1 + A} - \frac{1}{2\tau} \left(A + \frac{A}{1 + 2A} \right) && \text{(by (2.10))} \\ &\geqslant -\frac{1}{2\tau} A \left(1 + \frac{1}{1 + 2A} \right) && \text{(by (6.4), (6.5)).} \end{aligned}$$

Hence, on $\{x = s(t)\}$,

$$\frac{d}{dt} A + cA \geqslant 0, \qquad c > 0$$

from which we conclude that $A(s(t_0), t_0) > 0$, a contradiction.

To prove (6.6) it suffices to notice that $T_x < 0$ by (6.4), whereas, on $x = s(t)$,

$$2T = A + B = A - \frac{A}{1 + 2A} = \frac{2A^2}{1 + 2A} \geqslant 0.$$

Combining Lemma 6.1 with 6.2 we obtain:

THEOREM 6.3. *If* (3.1), (3.2), (3.5), (3.6), *and* (6.3) *hold then there exists a solution* (A, B, s) *of* (2.8)–(2.12), (6.1) *with* $s \in C^{2,1}$ *and* A, B *in* $C^{1,1}$ *up to the boundary.*

The proof of Theorem 5.1 extends to the present case with minor changes. Thus:

THEOREM 6.4. *For any* $t_1 > 0$ *there exists at most one* C^1 *solution* (A, B, s) *of* (2.8)–(2.12), (6.1) *for* $0 \leqslant t \leqslant t_1$.

7. Asymptotic Behavior of The Solution as $\tau \to 0$

In this section we take

$$f(t) = f_\tau(t), \qquad \phi(x) = \phi_\tau(x), \qquad \psi(x) = \psi_\tau(x)$$

satisfying all the assumptions as in Theorem 6.3, and

$$f_\tau(t) \to f(t), \qquad \phi_\tau(x) \to \phi(x) \qquad \text{uniformly in } t, x \text{ as } \tau \to 0.$$

We shall denote the corresponding solution established in Section 6 by (A_τ, B_τ, s_τ) and prove that, as $\tau \to 0$, it converges to the solution of the Stefan problem corresponding to f, ϕ.

Recall that by the results of Section 6, $T_{\tau,x} < 0$, $T_\tau > 0$. Since for any $t_1 > 0$

$$T_\tau(0, t) = f_\tau(t) \leqslant C_1 \qquad \text{if} \quad 0 \leqslant t \leqslant t_1$$

(C_1 depends on t_1), it follows that $T \leqslant C_1$ if $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant t_1$. Next, since $A_{\tau,x} < 0$, $A_\tau(s(t), t)) > 0$ we have $A_\tau > 0$; similarly, $B_{\tau,x} < 0$ and $B_\tau(s(t), t) > -\frac{1}{2}$ (by (2.10)) so that $B_\tau > -\frac{1}{2}$. Using $A_\tau + B_\tau = 2T_\tau \leqslant C_1$, we get

$$0 < A_\tau < C_1 + \tfrac{1}{2}, \qquad -\tfrac{1}{2} < B_\tau < C_1. \tag{7.1}$$

Hence also

$$\sqrt{\tau}\, |q_\tau| \leqslant \frac{C_1 + 1}{2}. \tag{7.2}$$

We also note that by (2.11), (6.5),

$$\frac{ds_\tau}{dt} \geqslant 0. \tag{7.3}$$

Analogously to Lemma 3.3 we have:

LEMMA 7.1. *For any* $t_1 > 0$ *there exists a constant* M *independent of* τ *such that*

$$s_\tau(t) \leqslant M \qquad \text{if} \quad 0 \leqslant t \leqslant t_1. \tag{7.4}$$

Proof. Integrating (2.2) in x, $0 \leqslant x < s(t)$, we obtain an identity which can be written in the form

$$\tau \frac{d}{dt} \int_0^{s(t)} q(x, t)\, dx + \int_0^{s(t)} q(x, t)\, dx = f(t); \tag{7.5}$$

hence, for some constant C independent of τ,

$$\int_0^{s(t)} q(x, t)\, dx = \left(\int_0^{s(0)} q(x, 0)\, dx \right) e^{-t/\tau} + \int_0^t \frac{f(r)}{\tau} e^{-(t-r)/\tau}\, dr \leqslant C. \tag{7.6}$$

Next, multiplying (2.1) by x and integrating in x, $0 < x < s(t)$, we get an identity which can be written in the form

$$\frac{d}{dt} \left[\int_0^{s(t)} xT(x, t)\, dx + \frac{s^2(t)}{2} \right] = \int_0^{s(t)} q(x, t)\, dx.$$

Integrating in t and using (7.6) and the fact that $T(x, t) \geqslant 0$, the assertion (7.4) follows.

Let

$$Q_\tau = \{(x, t);\ 0 \leqslant x \leqslant s_\tau(t),\ 0 \leqslant t \leqslant t_1\},$$
$$Q = \{(x, t);\ 0 \leqslant x \leqslant M,\ 0 \leqslant t \leqslant t_1\}$$

and extend each T_τ into $Q \backslash Q_\tau$ by zero.

In view of (7.4), there is a subsequence of τ_n (denoted again by τ_n) for which

$$T_{\tau_n} \to T_0 \text{ weakly in } (L^\infty(Q))^*, \qquad T_0 \in L^\infty(Q). \tag{7.7}$$

Since $s_{\tau_n}(t)$ is bounded (independently of τ_n) and monotone in t, by Helly's theorem we may assume that, for some monotone function $s_0(t)$,

$$s_{\tau_n}(t) \to s_0(t) \qquad \text{pointwise;}$$

hence, by the Lebesgue dominated convergence theorem,

$$s_{\tau_n} \to s_0 \text{ in } L^p(0, t_1), \qquad \forall 1 \leqslant p < \infty. \tag{7.8}$$

Finally, defining

$$g_\tau(x, t) = \begin{cases} 1 + T_\tau(x, t) & \text{if } 0 \leqslant x < s_\tau(t) \\ 0 & \text{if } s_\tau(t) \leqslant x \leqslant M \end{cases}$$

we may assume that, in addition to (7.7), (7.8),

$$g_{\tau_n} \to g_0 \text{ weakly in } (L^\infty(Q))^*. \tag{7.9}$$

LEMMA 7.2. *There hold*

$$T_0(x, t) = 0 \qquad \textit{if} \quad x > s_0(t), \tag{7.10}$$

$$g_0(x, t) = \begin{cases} 1 + T_0(x, t) & \textit{if} \quad x < s_0(t) \\ 0 & \textit{if} \quad x > s_0(t). \end{cases} \tag{7.11}$$

Proof. Set $Q_0 = \{(x, t);\ 0 \leqslant x \leqslant s_0(t),\ 0 \leqslant t \leqslant t_1)\}$, $\tilde{Q}_\tau = Q \backslash Q_\tau$, $\tilde{Q}_0 = Q \backslash Q_0$. Then, by (7.8),

$$|Q_0 \Delta Q_\tau| = |\tilde{Q}_0 \Delta \tilde{Q}_\tau| = \int_0^{t_1} |s_0(t) - s_\tau(t)| \to 0 \qquad \textit{if} \quad \tau = \tau_n \to 0. \tag{7.12}$$

For any $\psi \in L^1(Q)$

$$\int_{\tilde{Q}_0} T_0 \psi = \int_Q T \psi \chi_{\tilde{Q}_0} = \int_Q (T_0 - T_{\tau_n}) \psi \chi_{\tilde{Q}_0} + \int_Q T_{\tau_n} \psi \chi_{\tilde{Q}_0} = I_1 + I_2$$

and $I_1 \to 0$ by (7.7) whereas

$$|I_2| \leqslant \int_{\tilde{Q}_0 \cap \tilde{Q}_{\tau_n}} |T_{\tau_n}|\, |\psi| \to 0$$

by (7.7), (7.12). It follows that $T_0 = 0$ on $\tilde{Q}_0$.

To prove (7.11) we denote the right-hand side by $\tilde{g}_0$ and proceed to show that $g_0 = \tilde{g}_0$. For any $\psi \in L^1(Q)$,

$$\begin{aligned} \int_Q (g_{\tau_n} - \tilde{g}_0)\psi &= \int_{Q \cap Q_{\tau_n}} (T_{\tau_n} - T_0)\psi + \int_{Q_0 \Delta Q_{\tau_n}} (g_{\tau_n} - \tilde{g}_0)\psi \\ &= \int_{Q_0} (T_{\tau_n} - T_0)\psi - \int_{Q \backslash Q_{\tau_n}} (T_{\tau_n} - T_0)\psi \\ &\quad + \int_{Q_0 \Delta Q_{\tau_n}} (g_{\tau_n} - \tilde{g}_0)\psi = J_1 + J_2 + J_3. \end{aligned}$$

By (7.7), $J_1 \to 0$ and, by (7.7), (7.12), $J_2 \to 0$ and $J_3 \to 0$. Hence $\tilde{g}_0 = g_0$.

Consider now the solution (A, B, s) of (2.8)–(2.12), (6.1) constructed in Section 6, and the corresponding functions T_τ, q_τ.

Multiplying(2.2) by ψ_x and (2.1) by ψ where $\psi(x, t)$ is any smooth function and integrating over $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant t_1$, we easily obtain, after using (2.3), (2.4),

$$\begin{aligned}
&\int_{Q_\tau}\int [T_\tau \psi_{xx} + (1+T_\tau)\,\psi_t]\,dx\,dt + \int_0^{t_1} [g_\tau(0,t)\psi(0,t) + T_\tau(0,t)\,\psi_x(0,t)]\,dt \\
&\quad + \int_0^{s(0)} (1+T_\tau(x,0))\psi(x,0)\,dz - \int_0^{s(t_1)} (1+T_\tau(x,t_1))\psi(x,t_1)\,dx \\
&= \tau\left\{\int_0^{s(t_1)} q_\tau(x,t_1)\,\psi_x(x,t_1)\,dx - \int_0^{s(0)} q_\tau(x,0)\,\psi_x(x,0)\,dx \right.\\
&\quad \left. - \int_{Q_\tau} q_\tau(x,t)\,\psi_{xt}(x,t)\,dx\,dt\right\}.
\end{aligned} \tag{7.13}$$

Choose ψ such that

$$\psi(x, t_1) = 0, \qquad \psi(0, t) = 0. \tag{7.14}$$

Then, taking $\tau = \tau_n \to 0$ and using (7.2) and Lemma 7.2, we obtain

$$\int_Q\int [T_0\psi_{xx} + a(T_0)\,\psi_t] + \int_0^{t_1} f(t)\,\psi_x(0,t)\,dt + \int_0^{s(0)} a(\phi(x))\,\psi(x,0)\,dx = 0, \tag{7.15}$$

where

$$a(T_0(x,t)) = \begin{cases} 1 + T_0(x,t) & \text{if} \quad x < s_0(t) \\ 0 & \text{if} \quad x > s_0(t), \end{cases}$$

$$a(\phi(x)) = \begin{cases} 1 + \phi(x) & \text{if} \quad x < s(0) \\ 0 & \text{if} \quad x > s(0). \end{cases}$$

Since $T_0 \geqslant 0$ and T_0 satisfies the heat equation (in the distribution sense) in $\Omega^* \equiv \{0 < x < s_0(t-0),\ 0 < t < t_1\}$, the maximum principle shows that $T_0 > 0$ in Ω^* and thus, a.e.,

$$a(T_0(x,t)) = \begin{cases} 1 + T_0(x,t) & \text{if} \quad T_0(x,t) > 0 \\ 0 & \text{if} \quad T_0(x,t) = 0. \end{cases}$$

Let u be the solution of the 1-phase Stefan problem corresponding to the data $f(t)$, $\phi(x)$, $s(0)$. Writing the weak formulation for u and subtracting from (7.15), we get

$$\int_Q\int [(T_0-u)\psi_{xx}+(a(T_0)-a(u))\psi_t]=0$$

with ψ as in (7.14), or

$$\int_Q\int (a(T_0)-a(u))(\psi_t+e(x,t)\psi_{xx})=0,$$

where

$$e(x,t)=\begin{cases}(T_0-u)/(a(T_0)-a(u)) & \text{if } T_0\neq u\\ 0 & \text{if } T_0=u.\end{cases}$$

We can now proceed as in [8] to construct suitable ψ's and show that $a(T_0)=a(u)$. Consequently,

T_0 is uniquely determined as the solution of the 1-phase Stefan problem. (7.16)

This implies, of course, that the free boundary $x=s_0(t)$ is C^∞. We have thus proved the following result:

THEOREM 7.3. *Let* (3.1), (3.2), (3.5), (3.6), *and* (6.3) *hold. Then the solution* T_τ *of* (2.8)–(2.12), (6.1) (*extended by zero to* $\{x>s_\tau(t)\}$) *satisfies*: $T_\tau\to u$ *weakly in* $(L^\infty\{0\leqslant x\leqslant M,\ 0\leqslant t\leqslant t_1\})^*$ *for any* $M>0$, $t_1>0$, *where* u *is the solution of the Stefan problem* (*with free boundary* $x=s(t)$):

$$u_t-u_{xx}=0 \qquad \text{if } 0<x<s(t),\ t>0,$$

$$u(x,0)=f(t) \qquad \text{if } t>0,\ u(x,0)=\phi(x) \text{ if } 0<x<s(t),$$

$$u=0, \qquad u_x=-\frac{ds}{dt} \qquad \text{on } x=s(t).$$

Remark 7.1. Lemma 3.3 shows that for the Neumann problem the solution (T_τ, s_τ) satisfies $s_\tau(t)\leqslant C_1$ if $t\leqslant t_1$, where C_1 is a constant independent of τ. However, we were not able to extend Theorem 7.3 to this case.

Remark 7.2. The free boundary problem with (1.10) replaced by (1.11) leads to the free boundary conditions

$$B(s(t),t)=A(s(t),t), \qquad \frac{ds}{dt}=\frac{A(s(t),t)}{\sqrt{\tau}}$$

instead of (2.9), (2.10). Since in general A is not bounded above by 1, the system (2.8), (2.9) cannot be solved by integration along characteristics.

References

1. D. Bogy and P. Naghdi, On heat conduction and wave propagation in rigid solids, *J. Math. Phys.* **11** (1970), 917–923.
2. J. Breezel and E. Nolan, Non-Fourier effects in the transmission of heat, *in* "Proc. 6th Conf. on Thermal Conductivity, Dayton, October 1966," 237–254.
3. J. Brown, D. Chung, and P. Mattews, Heat pulses at low temperature, *Phys. Lett.* **21** (1966) 241–243.
4. C. Cattaneo, Sulla conduzione del calore, *Atti Sem. Mat. Univ. Modena* **3** (1948/49), 3–21.
5. P. J. Chen and M. E. Gurtin, On the second sound of materials with memory, *Z. Angew. Math. Phys.* **21** (1970), 232–241.
6. M. Chester, Second sound in solids, *Phys. Rev.* **131** (1963), 2013–2015.
7. B. C. Coleman, M. Fabrizio, and D. R. Owen, Thermodynamics and the constitutive relations for second sounds in crystals, *in* "Lecture Notes in Physics" (G. Grioli, Ed.), pp. 20–43, Springer-Verlag, Heidelberg, 1983.
8. A. Friedman, The Stefan problem in several space variables, *Trans. Amer. Math. Soc.* **133** (1968), 51–87.
9. J. M. Greenberg, A hyperbolic heat transfer problem with phase change, *IMA J. Appl. Math.* **38** (1987), 1–21.
10. M. E. Gurtin and A. C. Pipkin, A general theory of heat conduction with finite wave speeds, *Arch. Rational Mech. Anal.* **31** (1968), 113–126.
11. J. C. Maxwell, *Philos. Trans. Roy. Soc. London* **157** (1867), 69–88.
12. R. R. Showalter and N. J. Walkington, A hyperbolic Stefan problem, *Quart. Appl. Math.* **45** (1987), 769–781.
13. A. Solomon, V. Alexiades, D. Wilson, and J. Drake, The formulation of a hyperbolic Stefan problem, *Quart. Appl. Math.* **45** (1987), 469–481.
14. A. Solomon, V. Alexiades, D. Wilson, and J. Greenberg, "A Hyperbolic Stefan Problem with Discontinuous Temperatures," Oak Ridge Nat. Lab., March 1986.
15. P. Vernolte, Les paradoxes de la théorie continue de l'équation de la chaleur, *Comp. Rend. Paris* **246** (1958), 3154–3155.