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Diagonal Convexity Conditions for Problems in Convex Analysis and Quasi-Variational Inequalities

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Many theorems in convex analysis and quasi-variational inequalities can be derived by using a class of weaker convexity (concavity) conditions which require a functional $\phi(x, y)$ to be quasi-convex or convex for diagonal entries of certain type. In this paper, we discuss such conditions and use them to generalize several important theorems such as Ky Fan's inequality and saddle point theorem and some recent results in quasi-variational inequalities.

I. INTRODUCTION

Throughout this paper, we let E be a locally convex topological vector space with dual E', and let K be a nonempty convex subset of E. Let $F\colon K \to 2^K$ be a set-valued map with nonempty closed convex values.

Several recent works (Mosco [7], see also Aubin and Ekeland [2, Theorem 6.4.21, p. 348], Shih and Tan [10], etc.) have studied existence of equilibrium for quasi-variational inequalities motivated by problems in impulsive control and game theory. In those problems of quasi-variational inequalities and general convex analysis, an extended real-valued functional

$$\phi\colon (x, y) \in K \times K \to \phi(x, y) \in \mathbb{R} \cup \{\pm\infty\} \tag{1.1}$$

is involved. Researchers have made assumptions on ϕ such that either ϕ is (quasi-)concave in one of the variables (say y in (1.1)) [2, p.348], or that ϕ is defined from some monotone set-valued mapping [10, Theorem 1].

After our recent study on the existence of Nash equilibrium for N person non-zero sum differential games [3, 13] we have found a class of useful

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concavity (convexity) conditions for a functional of two variables ϕ, which are weaker than the concavity assumption in [2, Theorem 6.4.21], and which also encompasses the case studied in [10, Theorem 1] where some monotone set-valued operator property was assumed.

In Section II, we introduce the definitions of these general convexity conditions and some of their basic properties. These conditions enable us to generalize some important theorems in convex analysis such as Ky Fan's inequality and a specialized saddle point theorem. They also encompass several useful properties of set-valued mappings.

To illustrate the usefulness of these conditions, we prove the existence of solutions for two types of (generalized) quasi-variational inequalities under the weakened conditions: Theorem 6.4.21 in [21], Theorem 1 in [10], and Theorem 2 in [12].

It is quite clear that these conditions are rather general and are applicable to various nonlinear problems. Their additional features are yet to be exploited.

II. Diagonal Convexity Conditions and Properties: Some Convex Analysis Theorems

For any given set $A \subset E$, we use Conv A to denote the closed convex hull of A. Let ϕ be a functional as given in (1.1).

Definition 2.1. *$\phi(x, y)$ is said to be diagonally quasi-convex in y*, in short DQCX in y, if for any finite subset $\{y_1, ..., y_m\} \subset K$ and any $y_0 \in \operatorname{Conv}\{y_1, ..., y_m\}$, we have

$$\phi(y_0, y_0) \leqslant \max_{1 \leqslant i \leqslant m} \{\phi(y_0, y_i)\}. \tag{2.1}$$

We say that $\phi(x, y)$ is *strictly* DQCX in y if "$\leqslant$" above can be replaced by "$<$" for $m \geqslant 2$, for $y_0 \neq y_i$, $1 \leqslant i \leqslant m$.

We say that $\phi(x, y)$ is (strictly) *diagonally quasi-concave* (DQCV) *in* y if $-\phi(x, y)$ is (strictly) DQCX in y.

Remark 2.2. (i) In [3], we had called diagonal quasi-convexity the generalized-convexity (GCX) condition. It enabled us to generalize several theorems of Nikaido and Isoda for noncooperative N person games [8, 9].

(ii) The nomenclature "diagonal quasi-convexity" is due to the reason that in (2.1) the pair (y_0, y_0) is on the diagonal of $K \times K$. If, instead of (2.1),

$$\phi(x, y_0) \leqslant \max_{1 \leqslant i \leqslant m} \{\phi(x, y_i)\}$$

holds for any $x \in K$, then we say that $\phi(x, y)$ is quasi-convex in y for every given x.

(iii) According to Remark 3.3 in [3], if $\phi(x, y)$ is quasi-convex in y for every given x, then $\phi(x, y)$ is diagonally quasi-convex in y. But the converse does not hold. See [3, Remark 3.6] for a counterexample. So DQCX is indeed more general than quasi-convexity.

(iv) Since it is known that in general the sum of two quasi-convex functions does not remain quasi-convex, the same holds for the DQCX property.

DEFINITION 2.3. $\phi(x, y)$ is said to be *γ-diagonally quasi-convex* in y, in short γ-DQCX, for some $\gamma \in \mathbb{R} \cup \{\pm\infty\}$, if for any finite subset $\{y_1, ..., y_m\} \subset K$ and any $y_0 \in \operatorname{Conv}\{y_1, ..., y_m\}$, we have

$$\gamma \leqslant \max_{1 \leqslant i \leqslant m} \{\phi(y_0, y_i)\}. \tag{2.2}$$

Similarly, we say that $\phi(x, y)$ is γ-diagonally quasi-concave in y (γ-DQCV in y), provided that $-\phi(x, y)$ is γ-DQCX in y.

DEFINITION 2.4. $\phi(x, y)$ is said to be *γ-diagonally convex* in y, in short γ-DCX in y, for some $\gamma \in \mathbb{R} \cup \{\pm\infty\}$ if for any finite subset $\{y_1, ..., y_m\} \subset K$ and any $y_0 = \sum_{i=1}^m \alpha_i y_i$ ($\alpha_i \geqslant 0$, $\sum_{i=1}^m \alpha_i = 1$), we have

$$\gamma \leqslant \sum_{i=1}^{m} \alpha_i \phi(y_0, y_i). \tag{2.3}$$

Similarly, $\phi(x, y)$ is said to be *γ-diagonally concave* in y (γ-DCV in y) provided that $-\phi(x, y)$ is γ-DCX in y.

DEFINITION 2.5. $\phi(x, y)$ is said to be *diagonally convex* (DCX) in y if for any finite subset $\{y_1, ..., y_m\} \subset K$ and any $y_0 = \sum_{i=1}^m \alpha_i y_i$ ($\alpha_i \geqslant 0$, $\sum_{i=1}^m \alpha_i = 1$), we have

$$\phi(y_0, y_0) \leqslant \sum_{i=1}^{m} \alpha_i \phi(y_0, y_i). \tag{2.4}$$

Similarly, $\phi(x, y)$ is said to be *diagonally concave* (DCV) in y if $-\phi(x, y)$ is DCX in y.

Remark 2.6. (i) If $\phi(x, y)$ is DCX (resp. γ-DCX, DCV, γ-DCV) in y, then $\phi(x, y)$ is DQCX (resp. γ-DQCX, DQCV, γ-DQCV) in y.

(ii) We can similarly define *strict* DCX, γ-DCX, DCV, γ-DCV, DQCX, γ-DQCX, DQCV, γ-DQCV as in Definition 2.1.

(iii) If $\phi(x, y)$ is convex in y for each given $x \in K$, then $\phi(x, y)$ is DCX in y. The converse does not hold in general.

(iv) For N-person games (cf. [3], e.g.), in particular zero-sum games, there often exists $\phi(x, y)$ satisfying

$$\phi(x, x) = 0 \qquad \forall x \in K. \tag{2.5}$$

Under this condition, we see that if $\phi(x, y)$ is 0-DCX (resp. $0-\text{DCV}$) in y, then $\phi(x, y)$ is also DCX (resp. DCV) in y.

(v) Let $\phi(x, y)$ be γ-DQCX, γ-DQCV, γ-DCX, or γ-DCV in y. Then generally it is not true that $\phi(x, y)$ is, respectively, DQCX, DQCV, DCX, or DCV in y.

DEFINITION 2.7. An operator $A: K \to E'$ is said to be *monotone* if

$$\langle A(x) - A(y), x - y \rangle \geqslant 0 \qquad \forall x, y \in K.$$

EXAMPLE 2.8. Here we show that a function $\phi(x, y) = \langle A(y), x - y \rangle$ (constructed from some operator A) is both DQCX and DQCV in y, but not quasi-convex or quasi-concave in y (for some given x), and the operator A is not monotone.

Let $p \in E'$, $p \neq 0$. Define

$$A: K \to E'$$

$$A(x) = \|x\|\, p, \qquad \forall x \in K.$$

To show that $\phi(x, y)$ is DQCX in y, we note

$$\phi(x, x) = 0, \qquad \forall x \in E.$$

For if

$$y_0 = \sum_{i=1}^{m} \alpha_i y_i \qquad \left(\alpha_i \geqslant 0,\ \sum_{i=1}^{m} \alpha_i = 1\right)$$

and

$$0 > \max_{1 \leqslant i \leqslant m} \{\phi(y_0, y_i)\}, \tag{2.6}$$

it follows that

$$\langle p, y_0 - y_i \rangle < 0 \qquad \text{for all} \quad i = 1, \dots, m.$$

Then we have

$$0 > \sum_{i=1}^{m} \alpha_i \langle p, y_0 - y_i \rangle = \langle p, y_0 - y_0 \rangle = 0,$$

a contradiction. So $\phi(x, y)$ is DQCX in y.

Similarly, if we change max to min in (2.6) and reverse the directions of the last three inequalities above, we see that $\phi(x, y)$ is also DQCV in y.

To show that $\phi(x, y)$ is not quasi-concave in y for some x, choose $\hat{x} \in E$ such that

$$\langle p, \hat{x} \rangle > 0.$$

Set

$$y_1 = \tfrac{1}{2}\hat{x}, \qquad y_2 = -\tfrac{1}{2}\hat{x}.$$

Then we have

$$\phi(\hat{x}, y_1) = \tfrac{1}{4} \|\hat{x}\| \langle p, \hat{x} \rangle > 0,$$
$$\phi(\hat{x}, y_2) = \tfrac{3}{4} \|\hat{x}\| \langle p \, \hat{x} \rangle > 0.$$

But for $y_0 = \frac{1}{2}(y_1 + y_2) = 0$, we have

$$\phi(\hat{x}, y_0) = \|y_0\| \langle p, \hat{x} - y_0 \rangle = 0.$$

Thus

$$\phi(\hat{x}, \tfrac{1}{2}(y_1 + y_2)) < \min\{\phi(\hat{x}, y_1), \phi(\hat{x}, y_2)\}.$$

So $\phi(x, y)$ is not quasi-concave in y. Similarly, choosing $\hat{x} \in E$ with $\langle p, \hat{x} \rangle < 0$, by the same argument we can see that $\phi(x, y)$ is not quasi-convex in y either.

Finally we show that $A(x) = \|x\| \, p$ is not monotone. Choose $x \in E$ such that $\langle p, x \rangle \neq 0$. Take $y = 0$. We have

$$\langle A(x) - A(y), x - y \rangle = \|x\| \langle p, x \rangle,$$
$$\langle A(-x) - A(y), -x - y \rangle = -\|x\| \langle p, x \rangle.$$

One of the above must be negative, thus $A(x) = \|x\| \, p$ is not monotone.

The above example shows that the DQCX is indeed weaker than quasi-convexity or the condition defined from a monotone operator.

The next proposition tells the relationship between a DCV function and a monotone operator.

PROPOSITION 2.9. *If*

$$\phi(x, y) = \langle A(y), x - y \rangle$$

for some operator $A: E \to E'$. *Then* $\phi(x, y)$ *is* DCV *in* y *if and only if* A *is monotone.*

Proof. If A is monotone, i.e.,

$$\langle A(x) - A(y), x - y \rangle \geqslant 0 \qquad \forall x, y \in E,$$

then for any

$$y_0 = \sum_{i=1}^{m} \alpha_i y_i \qquad \left(\alpha_i \geqslant 0, \sum_{i=1}^{m} \alpha_i = 1, y_i \in E\right),$$

we have

$$\begin{aligned} \sum_{i=1}^{m} \alpha_i \phi(y_0, y_i) &= \sum_{i=1}^{m} \alpha_i \langle A(y_i), y_0 - y_i \rangle \\ &\leqslant \sum_{i=1}^{m} \alpha_i \langle A(y_0), y_0 - y_1 \rangle \\ &= \langle A(y_0), y_0 - y_0 \rangle \\ &= \phi(y_0, y_0) = 0. \end{aligned}$$

So $\phi(x, y)$ is DCV (0-DCV) in y.

Conversely, let $\phi(x, y)$ be DCV in y. In this case, it is also 0-DCV in y. For any $v_1, v_2 \in E$, set $v_0 = \frac{1}{2}(v_1 + v_2)$. Then

$$\begin{aligned} 0 &\leqslant -\left[\tfrac{1}{2}\phi(v_0, v_1) + \tfrac{1}{2}\phi(v_0, v_2)\right] \\ &= -\left[\tfrac{1}{2}\langle A(v_1), v_0 - v_1 \rangle + \tfrac{1}{2}\langle A(v_2), v_0 - v_2 \rangle\right] \\ &= -\left[\tfrac{1}{2}\langle A(v_1), \tfrac{1}{2}(v_2 - v_1) \rangle + \tfrac{1}{2}\langle A(v_2), \tfrac{1}{2}(v_1 - v_2) \rangle\right] \\ &= \tfrac{1}{4}\langle A(v_1) - A(v_2), v_1 - v_2 \rangle. \end{aligned}$$

Since $v_1 v_2 \in E$ are arbitrary, A is a monotone operator. ▌

The advantage of γ-DCX or DCX functionals over γ-DQCX or DQCX ones are due to

PROPOSITION 2.10. *Let* $\phi_i(x, y)$, $1 \leqslant i \leqslant n$, *be a set of functionals, each of them is* γ-DCX (resp. DCX, γ-DCV, DCV) *in* y. *Then*

$$\phi(x, y) \equiv \sum_{i=1}^{n} a_i(x)\, \phi_i(x, y);$$

$$a_i: K \to \mathbb{R}, \qquad a_i(x) \geqslant 0, \qquad \sum_{i=1}^{m} a_i(x) = 1$$

remains as γ-DCX (resp. DCX, γ-DCV, DCV) *in* y.

For later use, we now generalize Ky Fan's inequality ([5], see also [2, Theorem 6.3.5, p. 330]) by relaxing the concavity assumption to γ-DQCV.

THEOREM 2.11. *Let K be compact convex in E and let $\phi: K \times K \to \mathbb{R} \cup \{\pm\infty\}$ satisfy*

$$\forall y \in K, \qquad x \to \phi(x, y) \text{ is lower semicontinuous}; \tag{2.7}$$

$$\phi(x, y) \text{ is } \gamma\text{-DQCV in } y. \tag{2.8}$$

Then there exists $\bar{x} \in K$ such that

$$\sup_{y \in K} \phi(\bar{x}, y) \leqslant \gamma. \tag{2.9}$$

Proof. If $\gamma = \infty$, then (2.9) is trivial. So we assume that $\gamma \neq \infty$.

Assume that (2.9) is false. Then for all $x \in K$, there exists $y_x \in K$ such that

$$\phi(x, y_x) > \gamma. \tag{2.10}$$

For each $x \in K$, define

$$N(y) = \{x \in K \mid \phi(x, y) > \gamma\}.$$

Every set $N(y)$ is nonempty and open in K. By (2.10)

$$K \subset \bigcup_{y \in K} N(y).$$

Since K is compact, the above has a sub-covering

$$K \subset \bigcup_{i=1}^{m} N(y_i)$$

for some $\{y_1, ..., y_m\} \subset K$. Choose a partition of unity $\mu_i(x)$, $\mu_i: K \to \mathbb{R}$, subordinate to $\{N(y_i)\}_{i=1}^m$. Define a map

$$B: K \to K$$

$$B(y) = \sum_{i=1}^{m} \mu_i(y)\, y_i,$$

which is continuous and maps K into $S \equiv \text{Conv}\{x_1, ..., x_m\}$. In particular, B maps S into itself. By Brouwer's fixed point theorem, there exists $\hat{y} \in S$ such that

$$B(\hat{y}) = \hat{y}.$$

Let $I=\{i \mid 1 \leqslant i \leqslant m, \mu_i(\hat{y})>0\}$. Then for $i \in I$, $\hat{y} \in N(y_i)$, so

$$\phi(\hat{y}, y_i)>\gamma \qquad \text{for all} \qquad i \in I,$$

and

$$\hat{y} \in \operatorname{Conv}\{y_i \mid i \in I\}.$$

But by (2.8), there exists $i_0 \in I$ such that

$$\phi(\hat{y}, y_{i_0}) \leqslant \gamma,$$

a contradiction. ∎

COROLLARY 2.12 (Generalized Ky Fan's inequality). *Let the hypotheses of Theorem* 2.11 *hold for*

$$\gamma=\sup_{y \in K} \phi(y, y).$$

Then there exists $\bar{x} \in K$ *such that*

$$\sup_{y \in K} \phi(\bar{x}, y) \leqslant \sup_{y \in K} \phi(y, y).$$

Remark 2.13. The compactness of K in Corollary 2.12 can be removed by assuming the following:

> There exists a nonempty compact convex set $C \subset K$ such that for each $x \in K \backslash C$, there exists $y \in C$ with $\phi(x, y)>0$.

Cf. G. Allen [1, Theorem 2].

As a further illustration of diagonal convexity conditions, we give a special form of saddle point theorems as below, which has potential applications to N person nonzero sum games (cf. [3], e.g.). Compare [4, 6].

THEOREM 2.14. *Assume that* K *is a compact convex subset of a Banach space* E *and* $\phi: K \times K \rightarrow \mathbb{R}$ *satisfies*

(i) $\phi(x, x)=0, \forall x \in K$;

(ii) $\phi(x, y)$ *is* l.s.c. *in* x *for each* $y \in K$, *and* $\phi(x, y)$ *is* DQCX *in* x,

(iii) $\phi(x, y)$ *is* u.s.c. *in* y *for each* $x \in K$, *and* $\phi(x\ y)$ *is* DQCV *in* y.

Then

$$\max_{x \in K} \inf_{y \in K} \phi(x, y)=\min_{y \in K} \sup_{x \in K} \phi(x, y)=0.$$

Proof. Apply Theorem 2.11 to $\phi(y, x)$ with $\gamma = 0$, we have the existence of some $\hat{x} \in K$ such that

$$\phi(\hat{x}, y) \leqslant 0, \qquad \forall y \in K.$$

Apply Theorem 2.11 again to $\psi(x, y) \equiv -\phi(y, x)$, we have some $\tilde{x} \in K$ such that

$$\phi(y, \tilde{x}) \geqslant 0, \qquad \forall y \in K.$$

Thus $(\hat{x}, \tilde{x})$, is a saddle point satisfying

$$\phi(z, \tilde{x}) \leqslant \phi(\hat{x}, \tilde{x}) = 0 \leqslant \phi(\hat{x}, y), \qquad \forall (z, y) \in K \times K. \quad \blacksquare$$

Remark 2.15. Assume that the Banach space E is reflexive. The compactness of K in Theorem 2.14 can be removed by assuming the following:

There exist $(x_0, y_0) \in K \times K$ and $M > 0$ such that

$$\phi(x, y_0) > 0 \qquad \forall x \in K \text{ with } \|x\| > M$$

$$\phi(x_0, y) < 0 \qquad \forall y \in K \text{ with } \|y\| > M.$$

The proof can be carried out in the same way as in [1, Theorem 2] so it is omitted.

The uniqueness of the saddle point in Theorem 2.14 can be obtained if some strictness assumption is added.

COROLLARY 2.16. *Assume that condition* (iii) *in Theorem* 2.14 *is replaced by*

(iii′) $\phi(x, y)$ *is* u.s.c. *in* y *for each* $x \in K$, *and* $\phi(x, y)$ *is strictly quasi-concave in* y *for each* $x \in K$.

Let all the other hypotheses of Theorem 2.14 *remain valid. Then* $\phi(x, y)$ *has a unique saddle point* $(\hat{u}, \hat{u})$, *for some* $\hat{u}$.

The proof is omitted.

III. EXISTENCE OF SOLUTIONS TO GENERALIZED QUASI-VARIATIONAL INEQUALITIES UNDER DIAGONAL CONVEXITY CONDITIONS

In Theorem 2.11 and Corollary 2.12, we have already illustrated how to use diagonal convexity (concavity) conditions to generalize the useful Ky Fan inequality. In this section, we similarly generalize some recent results in quasi-variational inequalities.

We first generalize Theorem 6.4.21 in [2].

THEOREM 3.1. *Assume that*

(i) *K is compact convex*

(ii) *$F: K \to 2^K$ is upper hemicontinuous* (cf. [2]) *with nonempty closed convex values. Let $f: K \times K \to \mathbb{R}$ satisfy*

(iii) *$\forall y \in K$, $x \to f(x, y)$ is* l.s.c.

(iv) *$f(x, y)$ is* 0-DCV *in y.*

Finally, suppose that F and f are related by the property

$$\{x \in K \mid \sup_{y \in F(x)} f(x, y) \leqslant 0\} \text{ is closed.}$$

Then there exists a solution $\bar{x} \in K$ to the quasi-variational inequality

$$\bar{x} \in F(\bar{x}), \qquad \sup_{y \in F(\bar{x})} f(\bar{x}, y) \leqslant 0.$$

Proof. Since most of the arguments remain the same, let us adopt all the notations and procedures as given on [2, pp. 249–350] to avoid repetition.

We need only note that the functional $\phi: K \times K \to \mathbb{R}$ defined by

$$\phi(x, y) = a_0(x) f(x, y) + \sum_{i=1}^{n} a_i(x) \langle p_i, x - y \rangle,$$

as a sum of 0-DCV functionals in y, remains 0-DCV in y by our Proposition 2.10. Now, instead of quoting Ky Fan's inequality [2, Theorem 6.3.5; p. 330], we use our version of Ky Fan's inequality given in Corollary 2.11, with $\gamma = 0$. We still get inequality (64) in [2, p. 349]:

$$\sup_{y \in K} \phi(\bar{x}, y) \leqslant 0,$$

by Proposition 2.10 and (iv).

The rest of the argument remains the same and the proof is done. ∎

Following [2], a set-valued map $T: K \to 2^{E'}$ is said to be *monotone* if

$$\langle u - v, x - y \rangle \geqslant 0$$

for all $x, y \in K$ and for all $u \in T(x)$, $v \in T(y)$.

PROPOSITION 3.2. *If $T: K \to 2^{E'}$ is a monotone set-valued map, then the function*

$$\phi(x, y) = \sup_{v \in T(y)} \langle v, x - y \rangle$$

is 0-DCV *in y and continuous in x.*

Proof. The continuity of $\phi(x, y)$ in x is obvious; we need only prove the 0-DCV in y.

Assume that

$$y_0 = \sum_{i=1}^{m} \alpha_i y_i \qquad \left(\alpha_i \geqslant 0,\ \sum_{i=1}^{m} \alpha_i = 1,\ y_i \in K\right),$$

then $y_0 \in k$ and for any $\varepsilon > 0$, there exists $v_i \in T(y_i)$ such that

$$\sup_{v \in T(y_i)} \langle v, y_0 - y_i \rangle < \langle v_i, y_0 - y_i \rangle + \varepsilon.$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{m} \alpha_i \phi(y_0, y_i) &= \sum_{i=1}^{m} \alpha_i \sup_{v \in T(y_i)} \langle v, y_0 - y_i \rangle \\
&< \sum_{i=1}^{m} \alpha_i \langle v_i, y_0 - y_i \rangle + \varepsilon \\
&\leqslant \sum_{i=1}^{m} \alpha_i \langle v_0, y_0 - y_i \rangle + \varepsilon \qquad (v_0 \in T(y_0)) \\
&= \langle v_0, y_0 - y_0 \rangle + \varepsilon = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\sum_{i=1}^{m} \alpha_i \phi(y_0, y_i) \leqslant 0.$$

So $\phi(x, y)$ is 0-DCV in y. ∎

There is considerable interest (see Yan [12] and the references therein) in solving variational inequalities of the form

$$\hat{x} \in F(\hat{x})$$

$$\sup_{u \in T(\hat{x})} \langle u, \hat{x} - y \rangle + f(\hat{x}, y) \leqslant 0 \qquad \forall y \in F(\hat{x})$$

As a further application of our diagonal convexity argument, we prove the following theorem, which generalizes Theorem 3.2 and some key results in [10, 12].

THEOREM 3.3. *Let k be a compact convex subset of E. Assume that*

(i) $F: K \to 2^{E'}$ *is* u.s.c. (cf. [2; 10], e.g.), *with nonempty closed convex values*;

(ii) *$T: K \to 2^{E'}$ is a monotone set-value map with nonempty values such that for each one-dimensional flat* [10] $L \subset E$, $T|_{L \cap K}$ *is* l.s.c. *from the topology of E into the weak *-topology $\sigma(E', E)$ of E';*

(iii) *the map $f(x, y)$: $K \times K \to R$ is concave in y and* l.s.c. *in x such that $f(x, x) = 0$, $\forall x \in K$;*

(iv) *the maps F, T, and f are related by the property that the set*

$$\sum = \{x \in K \mid \sup_{y \in F(x)} \sup_{v \in T(y)} \langle v, x - y \rangle + f(x, y) \leqslant 0\}$$

is closed.

Then there exists a point $\hat{x} \in K$ such that

$$\hat{x} \in F(\hat{x})$$

$$\sup_{u \in T(\hat{x})} \langle u, \hat{x} - y \rangle + f(\hat{x}, y) \leqslant 0 \qquad \forall y \in F(\hat{x}).$$

Proof. Again we will try to be brief. We first note that the map

$$\eta: K \times K \to \mathbb{R} \cup \{\pm\infty\}$$

$$\eta(x, y) = \sup_{u \in T(y)} \langle u, x - y \rangle + f(x, y)$$

is 0-DCV in y, and l.s.c. in x. by Proposition 3.2 and 2.10. So we apply Theorem 3.1 and conclude that there exists $\hat{x} \in K$ such that

$$\begin{aligned} &\hat{x} \in F(\hat{x}) \\ &\eta(\hat{x}, y) = \sup_{u \in T(y)} \langle u, \hat{x} - y)\rangle + f(\hat{x}, y) \leqslant 0, \qquad \forall y \in F(\hat{x}). \end{aligned} \tag{3.1}$$

The remaining procedures are similar to those in Step 2 of the proof of Theorem 1 in [10]. Let $y \in F(\hat{x})$ be arbitrarily fixed and let $z_t = ty + (1 - t)\hat{x} = \hat{x} - t(\hat{x} - y)$ for $t \in [0, 1]$. By (3.1), we have

$$\sup_{v \in T(zt)} \langle v, \hat{x} - z_t \rangle + f(\hat{x}, z_t) \leqslant 0, \qquad \forall t \in [0, 1]. \tag{3.2}$$

By assumption (iv), for the second term in (3.2) we have

$$f(\hat{x}, z_t) = f(\hat{x}, ty + (1 - t)\hat{x}) \geqslant tf(\hat{x}, y) + (1 - t) f(\hat{x}, \hat{x}) = tf(\hat{x}, y). \tag{3.3}$$

The first term $\sup\langle v, \hat{x} - z_t \rangle$ can be handled in the same way as in the proof of Theorem 3.2 or [10, Theorem 1]. So when it is combined with (3.3) we get

$$\sup_{u \in T(\hat{x})} \langle u, \hat{x} - y \rangle + f(\hat{x}, y) \leqslant 0$$

and the proof is complete. ∎

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