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Generic canonical form of pairs of matrices with zeros

Tat'yana N. Gaiduk [a], Vladimir V. Sergeichuk [b,*,1]

[a] Department of Physics and Mathematics, Chernigov Pedagogical University, Chernigov, Ukraine
[b] Institute of Mathematics, National Academy of Sciences, Tereshchenkivska 3, Kiev, Ukraine

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Abstract

We consider a family of pairs of $m \times p$ and $m \times q$ matrices, in which some entries are required to be zero and the others are arbitrary, with respect to transformations $(A, B) \mapsto (SAR_1, SBR_2)$ with nonsingular S, R_1, and R_2. We prove that almost all of these pairs reduce to the same pair (A_0, B_0) from this family, except for pairs whose arbitrary entries are zeros of a certain polynomial. The polynomial and the pair (A_0, B_0) are constructed by a combinatorial method based on properties of a certain graph.

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1. Introduction and main results

Let $\mathscr{A} : U_1 \to V$ and $\mathscr{B} : U_2 \to V$ be linear mappings of vector spaces over an arbitrary field $\mathbb{F}$. Changing the bases of the vector spaces, we may reduce the matrices A and B of these mappings by transformations

$$(A, B) \mapsto (SAR_1, SBR_2) \quad \text{with nonsingular } S,\ R_1,\ \text{and } R_2. \tag{1}$$

* Corresponding author.
E-mail address: sergeich@imath.kiev.ua (V.V. Sergeichuk).

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A canonical form of (A, B) for these transformations is

$$\left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_r & 0 \\ 0 & 0 & 0 \\ I_t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right), \tag{2}$$

where I_r denotes the r-by-r identity matrix and r, s, and t are determined by the equalities $r + s = \operatorname{rank} A$, $r + t = \operatorname{rank} B$, and $r + s + t = \operatorname{rank}[A|B]$ (see Lemma 3).

We consider a family of pairs (A, B), in which n entries $a_1, \dots, a_n$ are arbitrary and the others are required to be zero. We prove that there exists a nonzero polynomial $f(x_1, \dots, x_n)$ such that all pairs (A, B) with $f(a_1, \dots, a_n) \neq 0$ reduce to the same pair $(A_{\text{gen}}, B_{\text{gen}})$ from this family. The pair $(A_{\text{gen}}, B_{\text{gen}})$ has the form (2) up to permutations of columns and simultaneous permutations of rows in A and B. Following [6], we call $(A_{\text{gen}}, B_{\text{gen}})$ a *generic canonical form* of the family (this notion has no sense if $\mathbb{F}$ is a finite field). We give a combinatorial method of finding $f(x_1, \dots, x_n)$ and $(A_{\text{gen}}, B_{\text{gen}})$.

1.1. Generic canonical form of matrices with zeros

Since the rows of A and B in (1) are transformed by the same matrix S, we represent the pair (A, B) by the block matrix $M = [A|B]$, which will be called a *bipartite matrix*. A family of bipartite matrices, in which some entries are zero and the others are arbitrary, may be given by a matrix

$$M(x) = [A(x)|B(x)], \quad x = (x_1, \dots, x_n), \tag{3}$$

whose n entries are unknowns $x_1, \dots, x_n$ and the others are zero. For instance,

$$M(x) = \left[\begin{array}{cc|ccc} 0 & 0 & x_4 & x_7 & 0 \\ x_1 & 0 & x_5 & 0 & 0 \\ 0 & x_2 & 0 & 0 & x_9 \\ 0 & x_3 & x_6 & x_8 & 0 \end{array}\right] \tag{4}$$

gives the family $\{M(a) \mid a \in \mathbb{F}^9\}$.

Considering (3) as a matrix over the field

$$\mathbb{K} = \left\{ \frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} \;\middle|\; f, g \in \mathbb{F}[x_1, \dots, x_n] \text{ and } g \neq 0 \right\} \tag{5}$$

of rational functions (its elements are quotients of polynomials), we put

$$r_A = \operatorname{rank}_{\mathbb{K}} A(x), \quad r_B = \operatorname{rank}_{\mathbb{K}} B(x), \quad r_M = \operatorname{rank}_{\mathbb{K}} M(x). \tag{6}$$

The following theorem is proved in Section 2.

Theorem 1. *Let $M(x) = [A(x)|B(x)]$ be a matrix whose n entries are unknowns $x_1, \dots, x_n$ and the others are zero. Then there exists a nonzero polynomial*

$$f(x) = \sum c_i x_1^{m_{i1}} \cdots x_n^{m_{in}} \tag{7}$$

such that all matrices of the family

$$\mathscr{M}_f = \left\{M(a) \mid a \in \mathbb{F}^n \text{ and } f(a) \neq 0\right\}$$

reduce by transformations $[A|B] \mapsto [SAR_1|SBR_2]$ *with nonsingular* S, R_1, *and* R_2 *to the same matrix*

$$M_{\text{gen}} = [A_{\text{gen}}|B_{\text{gen}}] \in \mathscr{M}_f. \tag{8}$$

Up to a permutation of columns within A_{gen} *and* B_{gen} *and a permutation of rows, the matrix* M_{gen} *has the form*

$$\left[\begin{array}{ccc|ccc} I_r & 0 & 0 & 0 & I_r & 0 \\ 0 & I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{9}$$

which is uniquely determined by $M(x)$ *due to the equalities*

$$r + s = r_A, \quad r + t = r_B, \quad r + s + t = r_M \quad (\textit{see } (6)). \tag{10}$$

We call M_{gen} a *generic canonical form* of the family $\{M(a) \mid a \in \mathbb{F}^n\}$ because $M(a)$ reduces to M_{gen} for all $a \in \mathbb{F}^n$ except for those in the proper algebraic variety $\{a \in \mathbb{F}^n \mid f(a) = 0\}$.

1.2. A combinatorial method

The polynomial $f(x)$ and the matrix M_{gen} can be constructed by a combinatorial method: we represent the matrix $M(x) = [A(x)|B(x)]$ by a graph and study its subgraphs. Similar methods were applied in [2,4–6] to square matrices up to similarity and to pencils of matrices.

The graph is defined as follows. Its vertices are

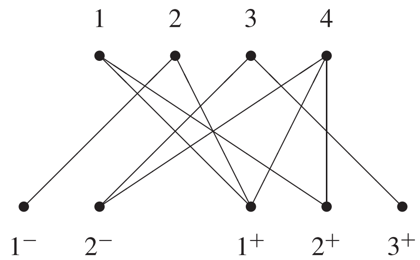
$$1, \ldots, m, \ 1^-, \ldots, p^-, \ 1^-, \ldots, q^+,$$

where $m \times p$ and $m \times q$ are the sizes of $A(x)$ and $B(x)$. Its edges

$$\alpha_1, \ldots, \alpha_n \tag{11}$$

are determined by the unknowns $x_1, \ldots, x_n$: if x_l is the (i, j) entry of $A(x)$ then $\alpha_l : i — j^-$ (that is, α_l links the vertices i and j^-), and if x_l is the (i, j) entry of $B(x)$ then $\alpha_l : i — j^+$. The edges between $\{1, \ldots, m\}$ and $\{1^-, \ldots, p^-\}$ are called *left edges*, and the edges between $\{1, \ldots, m\}$ and $\{1^+, \ldots, q^+\}$ are called *right edges*.

For example, the matrix (4) is represented by the graph



(12)

with the left edges $\alpha_1, \alpha_2, \alpha_3$ and the right edges $\alpha_4, \alpha_5, \ldots, \alpha_9$.

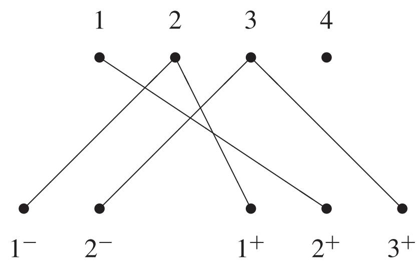
Each subset $\mathscr{S}$ in the set of edges (11) can be given by the *characteristic vector*

$$\varepsilon_{\mathscr{S}} = (e_1, \ldots, e_n), \quad e_l = \begin{cases} 1 & \text{if } \alpha_l \in \mathscr{S}, \\ 0 & \text{otherwise.} \end{cases}$$

By a *matchbox* we mean a set of edges (=*matches*) that have no common vertices. The *size* of a matchbox $\mathscr{S}$ is the number of its matches; since each row and each column of $M(\varepsilon_{\mathscr{S}})$ have at most one 1 and the other entries are zero,

$$\operatorname{size} \mathscr{S} = \operatorname{rank} M(\varepsilon_{\mathscr{S}}). \tag{13}$$

A matchbox is *left* (*right*) if all its matches are left (right). Such a matchbox is said to be *largest* if it has the maximal size among all left (right) matchboxes. For example, the subgraph



of (12) is formed by the largest left and right matchboxes

$$\mathscr{A} = \{2—1^-, 3—2^-\} \quad \text{and} \quad \mathscr{B} = \{1—2^+, 2—1^+, 3—3^+\}. \tag{14}$$

For a left matchbox $\mathscr{A}$ and a right matchbox $\mathscr{B}$, we denote by $\mathscr{A} \Cup \mathscr{B}$ the matchbox obtained from $\mathscr{A} \cup \mathscr{B}$ by removing all matches of $\mathscr{B}$ that have common vertices with matches of $\mathscr{A}$. For example,

$$\mathscr{A} \Cup \mathscr{B} = \{2—1^-, 3—2^-, 1—2^+\} \tag{15}$$

for the matchboxes (14).

For every matchbox

$$\mathscr{S} = \left\{i_1—j_1{}^-, \ldots, i_\alpha—j_\alpha^-, i_{\alpha+1}—k_1^+, \ldots, i_{\alpha+\beta}—k_\beta^+\right\},$$

we denote by $\mu_{\mathscr{S}}(x)$ the minor of order $\alpha+\beta$ in $M(x)=[A(x)|B(x)]$ whose matrix belongs to the rows numbered $i_1,\dots,i_{\alpha+\beta}$, to the columns of $A(x)$ numbered $j_1,\dots,j_\alpha$, and to the columns of $B(x)$ numbered $k_1,\dots,k_\beta$. For example, the matchbox (15) determines the minor

$$\mu_{\mathscr{A}\Cup\mathscr{B}}(x)=\begin{vmatrix}0&0&x_7\\x_1&0&0\\0&x_2&0\end{vmatrix}=x_1x_2x_7 \quad \text{in (4).}$$

The next theorem will be proved in Section 2.

Theorem 2. *The generic canonical form M_{gen} and the polynomial $f(x)$ from Theorem* 1 *may be constructed as follows. We represent $M(x)$ by the graph. Among pairs consisting of a largest left matchbox and a largest right matchbox, we choose a pair $(\mathscr{A},\mathscr{B})$ with the minimal number $v(\mathscr{A},\mathscr{B})$ of common vertices, and take*

$$M_{\text{gen}}=M(\varepsilon_{\mathscr{A}\cup\mathscr{B}}),\quad f(x)=f_{\mathscr{A}\mathscr{B}}(x), \tag{16}$$

where $f_{\mathscr{A}\mathscr{B}}(x)$ is the lowest common multiple of $\mu_{\mathscr{A}}(x)$, $\mu_{\mathscr{B}}(x)$, and $\mu_{\mathscr{A}\Cup\mathscr{B}}(x)$:

$$f_{\mathscr{A}\mathscr{B}}(x)=\text{LCM}\big\{\mu_{\mathscr{A}}(x),\mu_{\mathscr{B}}(x),\mu_{\mathscr{A}\Cup\mathscr{B}}(x)\big\}. \tag{17}$$

Up to permutations of columns within A_{gen} and B_{gen} and a permutation of rows, the matrix $M(\varepsilon_{\mathscr{A}\cup\mathscr{B}})$ has the form (9) *with*

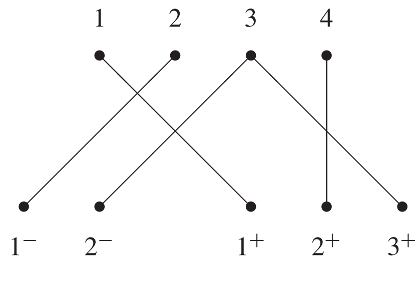
$$r=v(\mathscr{A},\mathscr{B}),\quad s=\text{size}\,\mathscr{A}-r,\quad \textit{and}\quad t=\text{size}\,\mathscr{B}-r. \tag{18}$$

1.3. An example

Let us apply Theorems 1 and 2 to the family given by the matrix (4) with the graph (12). The matchboxes (14) do not satisfy the conditions of Theorem 2 because they have two common vertices '2' and '3'. This number is not minimal since the largest matchboxes

$$\mathscr{A}=\{2—1^-,3—2^-\},\quad \mathscr{B}=\{1—1^+,3—3^+,\ 4—2^+\} \tag{19}$$

forming the graph

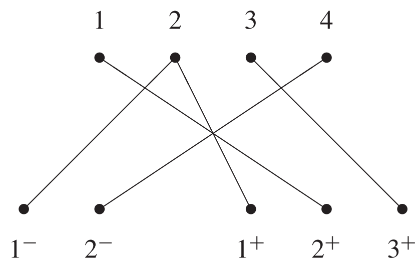


have a single common vertex '3'. The matchboxes (19) satisfy the conditions of Theorem 2 since there is no pair of largest matchboxes without common vertices.

The conditions of Theorem 2 also hold for the largest matchboxes

$$\mathscr{A}' = \{2—1^-, 4—2^-\}, \quad \mathscr{B}' = \{1—2^+, 2—1^+, 3—3^+\}$$

forming the graph



since they have a single common vertex too.

For these pairs of matchboxes, we have

$$\mathscr{A} \Cup \mathscr{B} = \left\{2—1^-, 3—2^-, 1—1^+, 4—2^+\right\},$$
$$f_{\mathscr{A}\mathscr{B}}(x) = \mathrm{LCM}\left\{x_1x_2,\, x_9(x_6x_7 - x_4x_8),\, x_1x_2(x_4x_8 - x_6x_7)\right\}$$

and

$$\mathscr{A}' \Cup \mathscr{B}' = \left\{2—1^-, 4—2^-, 1—2^+, 3—3^+\right\},$$
$$f_{\mathscr{A}'\mathscr{B}'}(x) = \mathrm{LCM}\left\{x_1x_3,\, -x_5x_7x_9,\, -x_1x_3x_7x_9\right\}.$$

By Theorems 1 and 2,

$$\left[\begin{array}{cc|ccc} 0 & 0 & a_4 & a_7 & 0 \\ a_1 & 0 & a_5 & 0 & 0 \\ 0 & a_2 & 0 & 0 & a_9 \\ 0 & a_3 & a_6 & a_8 & 0 \end{array}\right] \quad \text{with } a_1, \ldots, a_9 \in \mathbb{F}$$

(see (4)) reduces to the matrix

$$M(\varepsilon_{\mathscr{A}\cup\mathscr{B}}) = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \text{if } f_{\mathscr{A}\mathscr{B}}(a) = a_1a_2a_9(a_4a_8 - a_6a_7) \neq 0$$

and to the matrix

$$M(\varepsilon_{\mathscr{A}'\cup\mathscr{B}'}) = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \quad \text{if } f_{\mathscr{A}'\mathscr{B}'}(a) = a_1a_3a_5a_7a_9 \neq 0.$$

Up to permutations of columns within vertical strips and permutations of rows, these matrices have the form

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \text{(see (9)).}$$

2. Proof of Theorems 1 and 2

2.1. Bipartite matrices

The canonical form of a pair for transformations (1) is well known, see [1, Section 1.2]. We recall it since we will use it in the proof of Theorems 1 and 2.

Clearly, (A, B) reduces to (A', B') by transformations (1) if and only if $[A|B]$ reduces to $[A'|B']$ by a sequence of

(i) elementary row-transformations in $[A|B]$,
(ii) elementary column-transformations in A, and
(iii) elementary column-transformations in B.

Lemma 3. *Every bipartite matrix $M = [A|B]$ over a field $\mathbb{F}$ reduces by transformations* (i)–(iii) *to the form*

$$\left[\begin{array}{ccc|ccc} I_r & 0 & 0 & 0 & I_r & 0 \\ 0 & I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{20}$$

determined by the equalities

$$r + s = \operatorname{rank} A, \quad r + t = \operatorname{rank} B, \quad r + s + t = \operatorname{rank} M. \tag{21}$$

Proof. By transformations (i) and (ii), we reduce M to the form

$$\left[\begin{array}{cc|c} I_h & 0 & B_1 \\ 0 & 0 & B_2 \end{array}\right],$$

and then by elementary row-transformations within the second horizontal strip and by transformations (iii) to the form

$$\left[\begin{array}{cc|cc} I_h & 0 & B_3 & B_4 \\ 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Adding linear combinations of rows of I_t to rows of B_3 by transformations (i), we "kill" all nonzero entries of B_3:

$$\left[\begin{array}{cc|cc} I_h & 0 & 0 & B_4 \\ 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

At last, we reduce B_4 to $I_r \oplus 0$ by elementary transformations. The row-transformations with B_4 have "spoiled" the block I_h, but we restore it by the inverse column-transformations (ii) and obtain the matrix (20) with $r + s = h$.

Since the transformations (i)–(iii) with $M = [A|B]$ preserve the ranks of M, A, and B, we have the equalities (21). This implies the uniqueness of (20) since $s = \operatorname{rank} M - \operatorname{rank} B$, $t = \operatorname{rank} M - \operatorname{rank} A$, and $r = \operatorname{rank} A + \operatorname{rank} B - \operatorname{rank} M$. □

2.2. *Reduction of bipartite matrices by permutations of rows and columns*

In this section we consider a bipartite matrix $M = [A|B]$ with respect to permutations of rows and columns.

Lemma 4. *Every bipartite matrix $[A|B]$ with linearly independent columns reduces by a permutation of rows to the form*

$$\left[\begin{array}{c|c} A' & \cdot \\ \cdot & B' \\ \cdot & \cdot \end{array}\right], \tag{22}$$

where A' and B' are nonsingular square blocks and the points denote unspecified blocks.

Proof. By permutations of rows we reduce $[A|B]$ to the form

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \cdot & \cdot \end{array}\right]$$

with a nonsingular square matrix $[A_1|B_1]$. Laplace's theorem (see [3, Theorem 2.4.1]) states that the determinant of $[A_1|B_1]$ is equal to the sum of products of the minors whose matrices belong to the rows of A_1 by their cofactors (belonging to B_1). One of these summands is nonzero since $[A_1|B_1]$ is nonsingular. We collect the rows of the minor from this summand at the top and obtain the matrix (22). □

Lemma 5. *Every bipartite matrix $[A|B]$ reduces by permutations of rows and permutations of columns in A and B to the form*

$$\left[\begin{array}{ccc|ccc} X_r & \cdot & \cdot & \cdot & Y_r & \cdot \\ \cdot & Z_s & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & T_t & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right], \tag{23}$$

where X_r, Y_r, Z_s, and T_t are nonsingular $r \times r$, $r \times r$, $s \times s$, and $t \times t$ blocks in which all diagonal entries are nonzero and

$$r + s = \operatorname{rank} A, \quad r + t = \operatorname{rank} B, \quad r + s + t = \operatorname{rank}\,[A|B]. \tag{24}$$

Proof. Denote

$$\rho_A = \operatorname{rank} A, \quad \rho_B = \operatorname{rank} B, \quad \rho_M = \operatorname{rank}\,[A|B].$$

We first reduce $[A|B]$ by a permutation of columns to the form $[\cdot\ A_1|B]$, where A_1 has ρ_A columns and they are linearly independent. Then we reduce it to the form $[\cdot\ A_1|B_1\ \cdot]$, where $[A_1|B_1]$ has ρ_M columns and they are linearly independent.

Lemma 4 to $[A_1|B_1]$ ensures that the matrix $[\cdot\ A_1|B_1\ \cdot]$ reduces by a permutation of rows to the form

$$\rho_A \text{ rows}\{\left[\begin{array}{cc|cc} \cdot & A_2 & \cdot & \cdot \\ \cdot & \cdot & B_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\}\rho_M \text{ rows} \\ \\ \end{array} \tag{25}$$

with nonsingular square matrices A_2 and B_2.

Rearranging rows of the first strip and breaking it into two substrips, we reduce (25) to the form

$$\begin{array}{r} \rho_A \text{ rows}\left\{\begin{array}{l} \\ \\ \end{array}\right. \\ \\ \\ \end{array} \left[\begin{array}{cc|cc} \cdot & A_3 & \cdot & \cdot \\ \cdot & A_4 & B_3 & \cdot \\ \cdot & \cdot & B_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right] \begin{array}{l} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\}\rho_B \text{ rows} \\ \\ \end{array} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\}\rho_M \text{ rows} \tag{26}$$

where the matrices

$$\left[\begin{array}{c} A_3 \\ \hline A_4 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc} B_3 & \cdot \\ \hline B_2 & \cdot \end{array}\right]$$

have linearly independent rows. Lemma 4 to their transposes insures that (26) reduces by permutations of columns to the form

$$\begin{array}{r} \rho_A\left\{\begin{array}{l} \\ \\ \end{array}\right. \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} \cdot & Z & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & X & Y & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & T & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] \begin{array}{l} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\}\rho_B \\ \\ \end{array} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\}\rho_M \tag{27}$$

with nonsingular X, Y, Z, and T. If an n-by-n matrix has a nonzero determinant, then one of its $n!$ summands is nonzero, and we may dispose the entries of this summand along the main diagonal by a permutation of columns. In this manner we make nonzero the diagonal entries of X, Y, Z, and T. At last, we reduce (27) to the form (23) by permutations of rows and columns. □

2.3. Proof of Theorems 1 and 2

In this section $M(x) = [A(x)|B(x)]$ is the matrix (3), $\mathscr{A}$ and $\mathscr{B}$ are the matchboxes from Theorem 2, and r_A, r_B, r_M are the numbers (6).

Lemma 6

$$\text{size}\,\mathscr{A} = r_A, \quad \text{size}\,\mathscr{B} = r_B, \quad \text{size}\,\mathscr{A} \Cup \mathscr{B} = r_M. \tag{28}$$

Proof. By Lemma 5, the matrix $M(x)$ over the field $\mathbb{K}$ of rational functions (5) reduces by permutations of rows and by permutations of columns within $A(x)$ and $B(x)$ to a matrix $N(x)$ of the form (23), in which by (24)

$$r + s = r_A, \quad r + t = r_B, \quad r + s + t = r_M. \tag{29}$$

The diagonal entries of X_r, Y_r, Z_s, and T_t are all nonzero, and hence they are independent unknowns; replacing them by 1 and the other unknowns by 0, we obtain the matrix

$$N(a) = \left[\begin{array}{ccc|ccc} I_r & 0 & 0 & 0 & I_r & 0 \\ 0 & I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad a \in \{0, 1\}^n. \tag{30}$$

The inverse permutations of rows and columns reduce $N(x)$ to $M(x)$, and hence $N(a)$ to $M(a)$. As follows from (30), $a = \varepsilon_{\mathscr{A}' \cup \mathscr{B}'}$, where $\mathscr{A}'$ is a left matchbox, $\mathscr{B}'$ is a right matchbox, and by (29)

$$\text{size}\,\mathscr{A}' = r_A, \quad \text{size}\,\mathscr{B}' = r_B, \quad \text{size}\,\mathscr{A}' \Cup \mathscr{B}' = r_M.$$

Since the matchboxes $\mathscr{A}$ and $\mathscr{B}$ are largest, size $\mathscr{A} \geqslant r_A$ and size $\mathscr{B} \geqslant r_B$. The minors $\mu_{\mathscr{A}}(x)$ of $A(x)$ and $\mu_{\mathscr{B}}(x)$ of $B(x)$ (defined in Section 1.2) are nonzero and their orders are equal to the sizes of $\mathscr{A}$ and $\mathscr{B}$, hence size $\mathscr{A} \leqslant r_A$ and size $\mathscr{B} \leqslant r_B$. We have

$$\text{size}\,\mathscr{A} = \text{size}\,\mathscr{A}' = r_A, \quad \text{size}\,\mathscr{B} = \text{size}\,\mathscr{B}' = r_B,$$

and so the matchboxes $\mathscr{A}'$ and $\mathscr{B}'$ are largest too. Because of the minimality of the number $v(\mathscr{A}, \mathscr{B})$ of common vertices and since

$$\text{size}\,\mathscr{A} \Cup \mathscr{B} = \text{size}\,\mathscr{A} + \text{size}\,\mathscr{B} - v(\mathscr{A}, \mathscr{B}), \tag{31}$$

we have

$$v(\mathscr{A}, \mathscr{B}) \leqslant v(\mathscr{A}', \mathscr{B}'), \quad \text{size}\,\mathscr{A} \Cup \mathscr{B} \geqslant \text{size}\,\mathscr{A}' \Cup \mathscr{B}' = r_M.$$

In actual fact the last inequality is an equality since the size of $\mathscr{A} \Cup \mathscr{B}$ is equal to the order of the minor $\mu_{\mathscr{A} \Cup \mathscr{B}}(x)$. This minor is nonzero and hence its order is at most r_M. □

Lemma 7. *If* $a \in \mathbb{F}^n$ *and* $f_{\mathscr{A}\mathscr{B}}(a) \neq 0$, *then*

$$\text{rank}\,A(a) = r_A, \quad \text{rank}\,B(a) = r_B, \quad \text{rank}\,M(a) = r_M. \tag{32}$$

Proof. The matrix $A(a)$ has a nonzero minor $h(a)$, whose order is equal to the rank of $A(a)$. The corresponding minor $h(x)$ of $A(x)$ (belonging to the same rows and columns) is a nonzero polynomial, and so rank $A(a) \leqslant \text{rank}_{\mathbb{K}}\,A(x) = r_A$. Analogously rank $B(a) \leqslant r_B$ and rank $M(a) \leqslant r_M$.

By (17), the minors $\mu_{\mathscr{A}}(a)$ of $A(a)$, $\mu_{\mathscr{B}}(a)$ of $B(a)$, and $\mu_{\mathscr{A} \Cup \mathscr{B}}(a)$ of $M(a)$ are nonzero. Their orders are equal to the sizes of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{A} \Cup \mathscr{B}$, hence

$$\operatorname{rank} A(a) \geqslant \operatorname{size} \mathscr{A}, \quad \operatorname{rank} B(a) \geqslant \operatorname{size} \mathscr{B}, \quad \operatorname{rank} M(a) \geqslant \operatorname{size} \mathscr{A} \Cup \mathscr{B}.$$

This proves (32) due to (28). □

Let $a \in \mathbb{F}^n$ and $f_{\mathscr{A}\mathscr{B}}(a) \neq 0$. By Lemma 3, $M(a)$ reduces to the matrix (9), which is determined by (10) due to (21) and (32). The matrix $M(\varepsilon_{\mathscr{A}\cup\mathscr{B}})$ reduces by permutations of rows and columns to the same matrix (9) because (13) and (28) imply

$$\operatorname{rank} A(\varepsilon_{\mathscr{A}\cup\mathscr{B}}) = \operatorname{size} \mathscr{A} = r_A, \quad \operatorname{rank} B(\varepsilon_{\mathscr{A}\cup\mathscr{B}}) = \operatorname{size} \mathscr{B} = r_B, \tag{33}$$

$$\operatorname{rank} M(\varepsilon_{\mathscr{A}\cup\mathscr{B}}) = \operatorname{rank} M(\varepsilon_{\mathscr{A} \Cup \mathscr{B}}) = \operatorname{size} \mathscr{A} \Cup \mathscr{B} = r_M. \tag{34}$$

Hence $M(a)$ reduces to $M(\varepsilon_{\mathscr{A} \Cup \mathscr{B}})$. This proves Theorem 1: we can take M_{gen} and $f(x)$ as indicated in (16). This also proves Theorem 2; the equalities (18) follow from (33), (34), and (31).

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References

[1] P. Gabriel, A.V. Roiter, Representations of Finite-Dimensional Algebras, Springer-Verlag, 1997.
[2] D. Hershkowitz, The relation between the Jordan structure of a matrix and its graph, Linear Algebra Appl. 184 (1993) 55–69.
[3] V.V. Prasolov, Problems and Theorems in Linear Algebra, in: Translations of Mathematical Monographs, vol. 134, Amer. Math. Soc., Providence, RI, 1996.
[4] K. Röbenack, K.J. Reinschke, Graph-theoretically determined Jordan-block structure of regular matrix pencils, Linear Algebra Appl. 263 (1997) 333–348.
[5] K. Röbenack, K.J. Reinschke, Digraph-based determination of Jordan block size structure of singular matrix pencils, Linear Algebra Appl. 275–276 (1998) 495–507.
[6] J.W. van der Woude, The generic canonical form of a regular structured matrix pencil, Linear Algebra Appl. 353 (2002) 267–288.