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Uniform coverings of 2-paths with 4-cycles

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Abstract

Let G be a graph [a digraph] and H be a subgraph of G. A $D(G, H, \lambda)$ *design* is a multiset $\mathcal{D}$ of subgraphs of G each isomorphic to H so that every 2-path [directed 2-path] of G lies in exactly λ subgraphs in $\mathcal{D}$. In this paper, we show that there exists a $D(K_{n,n}, C_4, \lambda)$ design if and only if (i) n is even, or (ii) n is odd and λ is even. We also show that there exists a $D(K^*_{n,n}, \overrightarrow{C}_4, \lambda)$ design for every n and λ, where $K_{n,n}$ and $K^*_{n,n}$ are the complete bipartite graph and the complete bipartite digraph, respectively; C_4 and $\overrightarrow{C}_4$ are a 4-cycle and a directed 4-cycle, respectively.

Keywords: Dudeney design; Covering of 2-paths; Covering with 4-cycles

1. Introduction

Consider a graph G and a subgraph H of G. A $D(G, H, \lambda)$ *design* is a multiset $\mathcal{D}$ of subgraphs of G each isomorphic to H so that every 2-path (path of length 2) of G lies in exactly λ subgraphs in $\mathcal{D}$. Analogously, when G is a directed graph (digraph) and H is a subgraph of G (a subgraph of a directed graph means a directed subgraph), a $D(G, H, \lambda)$ *design* is a multiset $\mathcal{D}$ of subgraphs of G each isomorphic to H so that every directed 2-path of G lies in exactly λ subgraphs in $\mathcal{D}$. We call these designs *Dudeney designs* [1].

Let G be a graph [a digraph]. A $D(G, H, \lambda)$ design is *resolvable*[1] if the subgraphs in the design can be partitioned into classes so that every vertex of G appears exactly once in each class. Each such class is called a *parallel class* of the design. A $D(G, H, \lambda)$ design is *edge-resolvable* [*arc-resolvable*] if the subgraphs in the design can be partitioned into classes so that every edge [arc] of G appears exactly once in each class ([2], p. 101).

K_n is the complete graph on n vertices, $K_{n,n}$ is the complete bipartite graph on partite sets with n vertices each, K^*_n is the complete digraph on n vertices, and $K^*_{n,n}$ is the complete bipartite digraph on partite sets with n vertices each.

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[1] We may call it "*vertex-resolvable*".

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K_n^* and $K_{n,n}^*$ are digraphs which are obtained from K_n and $K_{n,n}$, respectively, by substituting two oppositely directed edges (arcs) for each edge. C_k is a cycle on k vertices and $\overrightarrow{C}_k$ is a directed cycle on k vertices.

In this paper, we consider $D(G, H, \lambda)$ designs in which G is K_n, $K_{n,n}$, K_n^* or $K_{n,n}^*$, and H is a cycle or a directed cycle on 4 vertices. In the case of k-cycles where $k \neq 4$, see [3,1].

The following theorems are known.

Theorem A ([4,5]). *Let $n \geq 4$ and $\lambda \geq 1$ be integers.*

(1) *There exists a $D(K_n, C_4, \lambda)$ design if and only if*
 (i) *n is even, or*
 (ii) *$n \equiv 1 \pmod 4$ and λ is even, or*
 (iii) *$n \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 4$.*

(2) *There exists a resolvable $D(K_n, C_4, 1)$ design if and only if $n \equiv 0 \pmod 4$.*

We note that there cannot exist edge-resolvable $D(K_n, C_4, 1)$ designs ([2], p. 106).

Theorem B ([2], p. 110).[2] *Let $n \geq 2$ be an integer.*

(1) *There exists a $D(K_{n,n}, C_4, 1)$ design if and only if n is even.*
(2) *There exists a resolvable $D(K_{n,n}, C_4, 1)$ design if and only if n is even.*
(3) *There exists an edge-resolvable $D(K_{n,n}, C_4, 1)$ design if and only if n is even.*

In this paper, we show the following theorems.

Theorem 1.1. *Let $n \geq 2$ and $\lambda \geq 1$ be integers. There exists a $D(K_{n,n}, C_4, \lambda)$ design if and only if*
 (i) *n is even, or*
 (ii) *n is odd and λ is even.*

Theorem 1.2. *Let $n \geq 2$ and $\lambda \geq 1$ be integers.*

(1) *There exists a $D(K_{n,n}^*, \overrightarrow{C}_4, \lambda)$ design for every n and λ.*
(2) *There exists a resolvable $D(K_{n,n}^*, \overrightarrow{C}_4, 1)$ design if and only if n is even.*
(3) *There exists an arc-resolvable $D(K_{n,n}^*, \overrightarrow{C}_4, 1)$ design if and only if n is even.*

The existence problem of $D(K_n, C_4, \lambda)$ designs has been solved by Theorem A. In this paper, we solved the existence problems of $D(K_{n,n}, C_4, \lambda)$ designs and $D(K_{n,n}^*, \overrightarrow{C}_4, \lambda)$ designs. The remaining problem for $D(G, H, \lambda)$ designs in which G is K_n, $K_{n,n}$, K_n^* or $K_{n,n}^*$, and H is a cycle or a directed cycle on 4 vertices, is the existence problem of $D(K_n^*, \overrightarrow{C}_4, \lambda)$ designs.

2. Proofs of the theorems

Let $n \geq 2$ and $\lambda \geq 1$ be integers.

Lemma 2.1. *If there exists a $D(K_{n,n}, C_4, \lambda)$ design, then*
 (i) *n is even, or*
 (ii) *n is odd and λ is even.*

Proof. We denote by V and W the partite sets of vertices in $K_{n,n}$ with $|V| = |W| = n$. Assume that there exists a $D(K_{n,n}, C_4, \lambda)$ design $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$, where C_i is a 4-cycle of $K_{n,n}$ $(1 \leq i \leq t)$. Note that $\mathcal{C}$ is a multiset of 4-cycles, so it may have a 4-cycle more than once.

Let a, b be any vertices in V $(a \neq b)$. Let $\mathcal{C}_{a,b}$ be a multiset of all 4-cycles (a, x, b, y) $(x, y \in W)$ in $\mathcal{C}$. Put $\mathcal{P}_{a,b} = \{2 - \text{path } (a, x, b) \mid x \in W\}$, then we have $|\mathcal{P}_{a,b}| = n$. Since each 4-cycle (a, x, b, y) in $\mathcal{C}_{a,b}$ has two 2-paths in $\mathcal{P}_{a,b}$, we have $|\mathcal{C}_{a,b}| = \lambda n/2$. Therefore we have $\lambda n \equiv 0 \pmod 2$ which follows Lemma 2.1. □

[2] It is easy to see that if there exists a $D(K_{n,n}, C_4, 1)$ design, then n is even (see Lemma 2.1). Together with Theorem 3.1 of [2] (p. 110), Theorem B is obtained.

Lemma 2.2. *When n is odd, there exists a* $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ *design.*

Proof. We denote by V and W the partite sets of vertices in $K_{n,n}$ with $|V| = |W| = n$. Assume that n is odd. Let $\mathcal{H} = \{H_i \mid 1 \le i \le (n-1)/2\}$ and $\mathcal{G} = \{G_i \mid 1 \le i \le (n-1)/2\}$ be Hamilton decompositions of K_V and K_W, respectively. (K_V and K_W are the complete graphs with vertex sets V and W, respectively.) We put arbitrary directions to Hamilton cycles $H_i \in \mathcal{H}$ and $G_i \in \mathcal{G}$ $(1 \le i \le (n-1)/2)$, then denote those by $\overrightarrow{H}_i$ and $\overrightarrow{G}_i$. For any i $(1 \le i \le (n-1)/2)$, we define a set of directed 4-cycles $\overrightarrow{H}_i \times \overrightarrow{G}_i$ as

$$\overrightarrow{H}_i \times \overrightarrow{G}_i = \{(a, x, b, y) \mid (a, b) \in \overrightarrow{H}_i, (x, y) \in \overrightarrow{G}_i\}$$

and put

$$\mathcal{C} = \bigcup_{i=1}^{(n-1)/2} \overrightarrow{H}_i \times \overrightarrow{G}_i.$$

Note that we have $|\mathcal{C}| = n^2(n-1)/2$.

For any directed 2-path (u, v, w) of $K^*_{n,n}$ with $u, w \in V$, $v \in W$, there exists a directed Hamilton cycle $\overrightarrow{H}_k$ such that an arc (u, w) or (w, u) lies on $\overrightarrow{H}_k$, and there are vertices $s, t \in W$ such that arcs (s, v), (v, t) lie on $\overrightarrow{G}_k$. So, if $(u, w) \in \overrightarrow{H}_k$, then we have $(u, v, w, t) \in \overrightarrow{H}_k \times \overrightarrow{G}_k$, and if $(w, u) \in \overrightarrow{H}_k$, then we have $(w, s, u, v) \in \overrightarrow{H}_k \times \overrightarrow{G}_k$. In both cases, the directed 2-path (u, v, w) belongs to $\overrightarrow{H}_k \times \overrightarrow{G}_k$. Similarly, for any directed 2-path (u, v, w) of $K^*_{n,n}$ with $u, w \in W$, $v \in V$ belongs to $\mathcal{C}$.

Since the number of directed 2-paths of $K^*_{n,n}$ is $2n^2(n-1)$, we see that every directed 2-path of $K^*_{n,n}$ lies on exactly one directed 4-cycle in $\mathcal{C}$. Hence $\mathcal{C}$ is a $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ design. □

Lemma 2.3. *When n is odd, there exists a* $D(K_{n,n}, C_4, 2)$ *design.*

Proof. When n is odd, there is a $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ design from Lemma 2.2. Replace each directed 4-cycle in the design by an undirected 4-cycle, then we obtain a $D(K_{n,n}, C_4, 2)$ design. □

Lemma 2.4. *When n is even, then there exist a* $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ *design, a resolvable* $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ *design, and an arc-resolvable* $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ *design.*

Proof. When n is even, there are a $D(K_{n,n}, C_4, 1)$ design, a resolvable $D(K_{n,n}, C_4, 1)$ design, and an edge-resolvable $D(K_{n,n}, C_4, 1)$ design from Theorem B. Replace each 4-cycle by two oppositely directed 4-cycles, then we obtain a $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ design, a resolvable $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ design, and an arc-resolvable $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ design. □

Lemma 2.5. (1) *If there exists a resolvable* $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ *design, then n is even.*
(2) *If there exists an arc-resolvable* $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ *design, then n is even.*

Proof. If there is a resolvable $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ design, then the number of vertices in $K^*_{n,n}$, $2n$, is a multiple of 4, so n is even. If there is an arc-resolvable $D(K^*_{n,n}, \overrightarrow{C}_4, 1)$ design, then the number of arcs in $K^*_{n,n}$, $2n^2$, is a multiple of 4, so n is even. □

To prove Theorems 1.1 and 1.2, we note that a $D(G, H, k\mu)$ design can be obtained by taking k copies of a $D(G, H, \mu)$ design. From Theorem B(1), Lemmas 2.1 and 2.3, we obtain Theorem 1.1. From Lemmas 2.2 and 2.4, we obtain Theorem 1.2(1). From Lemmas 2.4 and 2.5, we obtain Theorem 1.2(2) and (3). This completes the proofs of Theorems 1.1 and 1.2.

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