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Moduli spaces of critical Riemannian metrics in dimension four

Gang Tian*[,1,2], Jeff Viaclovsky[3]

Department of Mathematics 2-175, Massachusetts Institute of Technology, 7 Massachusetts Ave., Cambridge, MA 02139-4307, USA

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Abstract

We obtain a compactness result for various classes of Riemannian metrics in dimension four; in particular our method applies to anti-self-dual metrics, Kähler metrics with constant scalar curvature, and metrics with harmonic curvature. With certain geometric non-collapsing assumptions, the moduli space can be compactified by adding metrics with orbifold-like singularities. Similar results were obtained for Einstein metrics in (J. Amer. Math. Soc. 2(3) (1989) 455, Invent. Math. 97 (2) (1989) 313, Invent. Math. 101(1) (1990) 101), but our analysis differs substantially from the Einstein case in that we do not assume any pointwise Ricci curvature bound.

Keywords: Orbifolds; Anti-self-dual metrics

* Corresponding author. Fax: +1 617 253 4358.

E-mail addresses: tian@math.mit.edu (G. Tian), jeffv@math.mit.edu (J. Viaclovsky).

[1] Current address: Department of Mathematics, Princeton University, Princeton, NJ 08544, USA.

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1. Introduction

Critical points of the total scalar curvature functional (restricted to the space of unit volume metrics)

$$\mathcal{R} : g \mapsto \int_M R_g \, dV_g, \tag{1.1}$$

are exactly the Einstein metrics, and the structure of the moduli space of Einstein metrics has been extensively studied [And89,BKN89,Nak88,Tia90]. In particular, with certain geometric assumptions on non-collapsing, this moduli space can be compactified by adding Einstein metrics with orbifold singularities.

The motivation for this paper is to prove a similar compactness theorem for various classes of metrics in dimension four, where one does not assume a pointwise bound on the Ricci curvature. We will consider the following cases:

(a) half-conformally flat metrics constant scalar curvature,
(b) metrics with harmonic curvature,
(c) Kähler metrics with constant scalar curvature.

Half conformally flat metrics are also known as self-dual or anti-self-dual if $W^- = 0$ or $W^+ = 0$, respectively. These metrics are, in a certain sense, analogous to anti-self-dual connections in Yang–Mills theory (see [FU91,DK90]). The local structure of the moduli space of anti-self-dual metrics, by examining the linearization of the anti-self-dual equations, has been studied, for example, in [AHS78,IT98,KK92]. There has been a considerable amount of research on the existence of anti-self-dual metrics on compact manifolds. In the paper [Poo86], Poon constructed a one-parameter family of anti-self-dual conformal classes on $\overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2$. LeBrun [LeB91a] produced explicit examples on $n\overline{\mathbb{CP}}^2$ for all $n \geqslant 2$. We also mention the work of [Flo91,DF89] for other methods and examples. See also [LeB95] for a nice survey and further references. A very important contribution is Taubes' stable existence theorem for anti-self-dual metrics: for any compact, oriented, smooth four-manifold M, the manifold $M \# n\overline{\mathbb{CP}}^2$ carries an anti-self-dual metric for some n (see [Tau92]). This shows that anti-self-dual metrics exist in abundance, so one would like to understand the moduli space.

In [Bou81], it was proved that a compact four-dimensional Riemannian manifold with harmonic curvature and non-zero signature must be Einstein. Therefore (b) is larger than the class of Einstein metrics only in the case of zero signature. In particular, we have locally conformally flat metrics with constant scalar curvature, which have been studied in [SY88,SY94,Sch91]. For more background about cases (a)–(c) above, see [Bes87]. We also note that case (c) is an important class of extremal Kähler metrics [Cal82,Cal85].

In the sequel, when we say *critical metric* we will mean any of (a)–(c) above. For M compact, we define the Sobolev constant C_S as the best constant C_S so that for all $f \in C^{0,1}(M)$ we have

$$\|f\|_{L^4} \leqslant C_S \|\nabla f\|_{L^2} + Vol^{-1/4} \|f\|_{L^2}, \tag{1.2}$$

where *Vol* is the volume. Note that (1.2) is scale-invariant. For M non-compact, C_S is defined to be the best constant so that

$$\|f\|_{L^4} \leqslant C_S \|\nabla f\|_{L^2} \tag{1.3}$$

for all $f \in C^{0,1}(M)$ with compact support.

We define a *Riemannian orbifold* (M, g) to be a topological space which is a smooth manifold with a smooth Riemannian metric away from finitely many singular points. At a singular point p, M is locally diffeomorphic to a cone $\mathcal{C}$ on S^3/Γ, where $\Gamma \subset SO(4)$ is a finite subgroup acting freely on S^3. Furthermore, at such a singular point, the metric is locally the quotient of a smooth Γ-invariant metric on B^4 under the orbifold group Γ. We note that the notions of smooth orbifold, orbifold diffeomorphism, and orbifold Riemannian metric are well-defined, see [Sat56,Sat57,Thu97,Bor93,TY87] for more background. A Riemannian orbifold (M, g) is a *Kähler orbifold* if g is Kähler, all of the orbifold groups Γ are in $U(2)$, and at each singular point, the metric is locally the quotient of a smooth Kähler metric on a ball in $\mathbf{C}^2$ under the orbifold group.

Consider the disjoint union

$$\tilde{M} = \coprod_{i=1}^{N} M_i, \tag{1.4}$$

where each M_i is a Riemannian orbifold. Then a *Riemannian multi-fold* M is a connected space obtained from $\tilde{M}$ by finitely many identifications of points. Note that points from M_i and M_j, $i \neq j$ can be identified, as well as several points from the same M_i. For example, take M_1 and M_2 to be smooth manifolds, and identify $p_1 \in M_1$ with $p_2 \in M_2$. Another example would be to take just one smooth manifold M_1, and identify $p_1 \in M_1$ with $p_2 \in M_1$. The *singular set* of M is the set of points where M is not a smooth manifold—this will come from the non-trivial orbifold singular points of each M_i, as well as new singular points from the identifications. These latter points look like multiple cone points, thus the terminology *multi-fold*. If there is more than one orbifold in (1.4) ($N > 1$), some M_i is compact, and has only one point which gets identified to the other orbifolds M_j, $i \neq j$ to form M, then we say M *splits off* the compact orbifold M_i. If there is only one cone at a singular point p, then p is called *irreducible*.

A smooth Riemannian manifold (M, g) is called an asymptotically locally Euclidean (ALE) end of order τ if there exists a finite subgroup $\Gamma \subset SO(4)$ acting freely on $\mathbf{R}^4 \setminus B(0, R)$ and a C^∞ diffeomorphism $\Psi : M \to (\mathbf{R}^4 \setminus B(0, R))/\Gamma$ such that under this identification,

$$g_{ij} = \delta_{ij} + O(r^{-\tau}), \tag{1.5}$$

$$\partial^{|k|} g_{ij} = O(r^{-\tau-k}) \tag{1.6}$$

for any partial derivative of order k as $r \to \infty$. We say an end is ALE of order 0 if we can find a coordinate system as above with $g_{ij} = \delta_{ij} + o(1)$, and $\partial^{|k|} g_{ij} = o(r^{-k})$

as $r \to \infty$. A complete, non-compact Riemannian multi-fold (M, g) is called ALE if M can be written as the disjoint union of a compact set and finitely many ALE ends.

We say that a sequence of Riemannian manifolds (M_j, g_j) *converges* to the Riemannian multi-fold (M_∞, g_∞) if the following is satisfied. For $\varepsilon > 0$, consider $M_{\infty,\varepsilon} = M_\infty \setminus S_\varepsilon$, where S_ε is the ε-neighborhood of S, and S is a finite set of points containing all of the singular points of M_∞. Then there exist domains $\Omega_j(\varepsilon) \subset M_j$, and diffeomorphisms $\Phi_{j,\varepsilon} : M_{\infty,\varepsilon} \to \Omega_j(\varepsilon)$, such that $\Phi^*_{j,\varepsilon} g_j$ converges to g_∞ in C^∞ as $j \to \infty$, on compact subsets of $M_{\infty,\varepsilon}$. Furthermore, there exist constants δ, N depending upon ε, such that

$$\max\{Vol(M_j \setminus \Omega_j(\varepsilon)), diam(M_j \setminus \Omega_j(\varepsilon))\} < \delta \tag{1.7}$$

for $j > N$ and $\delta \to 0$ as $\varepsilon \to 0$, where *Vol* and *diam* denote the volume and diameter with respect to the metric g_j, respectively. A sequence of pointed Riemannian manifolds (M_j, g_j, p_j) converges to the pointed Riemannian multi-fold $(M_\infty, g_\infty, p_\infty)$ if for all $R > 0$, $B(p_j, R)$ converges to $B(p_\infty, R)$ as above as pointed spaces.

We state our main convergence theorem:

Theorem 1.1. *Let (M_i, g_i, p_i) be a sequence of critical metrics g_i on smooth, complete, pointed four-dimensional manifolds M_i satisfying*

$$C_S \leqslant C_1, \tag{1.8}$$

$$\int_{M_i} |Rm_{g_i}|^2 \, dV_{g_i} \leqslant \Lambda, \tag{1.9}$$

$$Vol(g_i) > \lambda > 0, \tag{1.10}$$

$$b_1(M_i) < B_1, \tag{1.11}$$

where C_1, Λ, λ are constants, and $b_1(M_i)$ denotes the first Betti number. Then there exists a subsequence $\{j\} \subset \{i\}$, a pointed, connected, critical Riemannian multi-fold $(M_\infty, g_\infty, p_\infty)$, and a finite singular set $S \subset M_\infty$ such that

(1) *(M_j, g_j, p_j) converges to $(M_\infty, g_\infty, p_\infty)$.*
(2) *The limit space $(M_\infty, g_\infty, p_\infty)$ does not split off any compact orbifold.*
(3) *If M_∞ is non-compact, then $(M_\infty, g_\infty, p_\infty)$ is ALE of order τ for any $\tau < 2$.*
(4) *If $b_1(M_i) = 0$, then $(M_\infty, g_\infty, p_\infty)$ is a Riemannian orbifold.*
(5) *In the Kähler case (c), $(M_\infty, g_\infty, p_\infty)$ is a Kähler orbifold.*

Remark. We note that the definition of convergence given here implies, in particular, Gromov–Hausdorff convergence. Moreover, we will show in Section 7 that the convergence is even stronger, in the sense that suitable rescalings of the metrics near the singular points converge to ALE multi-folds. The singular set S is the singular set of convergence, it necessarily contains the multi-fold singular set of M_∞, but it is possible for some points in S to be smooth points of M_∞. This is in contrast to the Einstein case, where the Bishop–Gromov volume comparison theorem implies that convergence is smooth at any smooth point in M_∞.

Remark. A positive lower bound on the Yamabe invariant $Y(M_i, [g_i])$ will imply the Sobolev constant bound, and in certain geometric situations, this bound will be automatically satisfied. The bound in (1.9) will also follow automatically in certain geometric situations. We will discuss these points in Section 3 below. Furthermore, the main elements of our proof only require a *local* Sobolev constant bound, see Theorem 7.3 below.

In conjunction with Theorem 1.1, we have the volume comparison theorem:

Theorem 1.2. *Let (M, g) be a critical metric on a smooth, complete four-dimensional manifold M satisfying*

$$C_S \leqslant C_1, \tag{1.12}$$

$$\int_M |Rm_g|^2 \, dV_g \leqslant \Lambda, \tag{1.13}$$

$$b_1(M) < B_1, \tag{1.14}$$

where $C_1, \Lambda,$ and B_1 are constants. Then there exists a constant V_1, depending only upon $C_1, \Lambda,$ and B_1, such that $Vol(B(p, r)) \leqslant V_1 \cdot r^4$, for all $p \in M$ and $r > 0$.

Finally, we restate Theorem 1.1 in the compact case:

Theorem 1.3. *Let (M_i, g_i) be a sequence of critical metrics on smooth, closed four-dimensional manifolds M_i satisfying*

$$C_S \leqslant C_1, \tag{1.15}$$

$$\int_{M_i} |Rm_{g_i}|^2 \, dV_{g_i} \leqslant \Lambda, \tag{1.16}$$

$$Vol(M_i, g_i) = 1, \tag{1.17}$$

$$b_1(M_i) < B_1, \tag{1.18}$$

where C_1, Λ, B_1 are constants. Then the conclusion of Theorem 1.1 holds. That is, the limit space (M_∞, g_∞) is a compact, connected, critical Riemannian multi-fold which *does not split off any compact orbifold. In the Kähler case* (c), *or if* $b_1(M_i) = 0$, *then* M_∞ *is an orbifold.*

Remark. All of our results hold in the more general Bach-flat case (see Section 2), with the exception that at a singular point, we can only show the metric g is locally the quotient by the orbifold group of a C^0 metric on a standard ball, smooth away from the origin, and in the ALE case, the metric is ALE of order 0.

2. Critical metrics

In this section, we discuss the systems of equations satisfied in cases (a), (b), and (c), and justify the terminology *critical metric*.

2.1. Half-conformally flat metrics and metrics with harmonic curvature

These systems were discussed in [TV], so we just briefly review them here.

The Euler–Lagrange equations of the functional

$$\mathcal{W} : g \mapsto \int_M |W_g|^2 \, dV_g, \tag{2.1}$$

in dimension four, are

$$B_{ij} = \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R^{kl} W_{ikjl} = 0. \tag{2.2}$$

where W_{ijkl} and R_{kl} are the components of the Weyl and Ricci tensors, respectively (see [Bes87,Der83]). Since the Bach tensor arises in the Euler–Lagrange equations of a Riemannian functional, it is symmetric, and since the functional (2.1) is conformally invariant, it follows that the Bach-flat equation (2.2) is conformally invariant. The Bach tensor arises as the Yang–Mills equation for a twistor connection [Mer84], see also [BM87,LeB91b] for other occurrences of the Bach tensor.

We note that (see [ACG03])

$$B_{ij} = 2\nabla^k \nabla^l W^+_{ikjl} + R^{kl} W^+_{ikjl} = 2\nabla^k \nabla^l W^-_{ikjl} + R^{kl} W^-_{ikjl}, \tag{2.3}$$

so that both self-dual and anti-self-dual metrics are Bach-flat.

Using the Bianchi identities, a computation shows that we may rewrite the Bach-flat equation (in dimension four) as

$$(\Delta Ric)_{ij} = 2(R_{il} g_{jk} - R_{ikjl} - W_{ikjl})(R_{kl} - (R/6) g_{kl}). \tag{2.4}$$

Introducing a convenient shorthand, we write this as

$$\Delta Ric = Rm * Ric. \tag{2.5}$$

The condition for harmonic curvature is that

$$\delta Rm = -R_{ijkl;i} = 0. \tag{2.6}$$

This condition was studied in [Bou81,Der85,Bes87], and is the Riemannian analogue of a Yang–Mills connection. An equivalent condition for harmonic curvature that $\delta W = 0$ and $R = constant$. In particular, locally conformally flat metrics with constant scalar curvature have harmonic curvature. A computation shows (2.5) is also satisfied, but in this case we moreover have an equation on the full curvature tensor. We compute

$$\begin{aligned}(\Delta Rm)_{ijkl} &= R_{ijkl;m;m}\\ &= (-R_{ijlm;k} - R_{ijmk;l})_{;m}\\ &= -R_{ijlm;mk} - R_{ijmk;ml} + Q(Rm)_{ijkl} = Q(Rm)_{ijkl}.\end{aligned}$$

where $Q(Rm)$ denotes a quadratic expression in the curvature tensor. In the shorthand, we write this as

$$\Delta Rm = Rm * Rm. \tag{2.7}$$

2.2. Kähler metrics with constant scalar curvature

We assume that (M, g) is a Kähler manifold with Kähler metric g. In [Cal82] it was shown that if dR is a holomorphic vector field, then g is critical for the L^2 norm of the scalar curvature, restricted to a Kähler class. In particular Kähler and constant scalar curvature implies extremal.

The bisectional curvature tensor is given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{s\bar{t}} \frac{\partial g_{s\bar{j}}}{\partial \bar{z}_l} \frac{\partial g_{i\bar{t}}}{\partial z_k},$$

in local coordinates $(z_1, \ldots, z_n)$ of M. Contracting with the inverse of $\{g_{i\bar{j}}\}$, we obtain for the Ricci and scalar curvature

$$\begin{aligned}R_{i\bar{j}} &= -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}(\log \det(g_{k\bar{l}})),\\ R &= -2\Delta_g \log \det(g_{k\bar{l}}) = -2\frac{\partial^2}{\partial z_k \partial \bar{z}_k}(\log \det(g_{i\bar{j}})).\end{aligned}$$

In particular these imply the following Bianchi identities

$$R_{i\bar{j}k\bar{l};m} = R_{i\bar{j}m\bar{l};k},$$
$$R_{i\bar{j};\bar{k}} = R_{i\bar{k};\bar{j}},$$
$$R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}}.$$

It follows then that in local unitary frames

$$\begin{aligned}\Delta_g(Ric)_{i\bar{j}} &= R_{i\bar{j};k;\bar{k}}\\ &= R_{k\bar{j};i;\bar{k}}\\ &= R_{k\bar{j};\bar{k};i} + R_{i\bar{s}}R_{s\bar{j}} - R_{k\bar{s}}R_{i\bar{j}s\bar{k}}\\ &= \frac{1}{2}R_{;i;\bar{j}} + R_{i\bar{s}}R_{s\bar{j}} - R_{k\bar{s}}R_{i\bar{j}s\bar{k}}.\end{aligned}$$

Therefore if the scalar curvature is constant, we have

$$\begin{aligned}\Delta_g(Ric)_{i\bar{j}} &= R_{i\bar{s}}R_{s\bar{j}} - R_{k\bar{s}}R_{i\bar{j}s\bar{k}}\\ &= (R_{i\bar{s}}g_{k\bar{j}} - R_{k\bar{s}}R_{i\bar{j}s\bar{k}})R_{k\bar{s}},\end{aligned} \tag{2.8}$$

so in this case, we again have the equation

$$\Delta Ric = Rm * Ric. \tag{2.9}$$

3. Geometric bounds

In this section, we will discuss some special cases for which various assumption in Theorem 1.1 will be automatically satisfied. We recall that the Yamabe invariant in dimension four is defined by

$$Y(M,[g_0]) = \inf_{g\in[g_0]} Vol(g)^{-1/2}\int_M R_g\, dV_g.$$

We define the Sobolev constant as the best constant such that for all $\phi \in C_c^{0,1}(M)$,

$$\|\phi\|_{L^4} \leqslant C_S\|\nabla\phi\|_{L^2} + Vol^{-1/4}\|\phi\|_{L^2}.$$

Proposition 3.1. *If g is a Yamabe minimizer, and $Y(M,[g]) > 0$, then $C_S(M,g) \leqslant \sqrt{6}\, Y(M,[g])^{-1/2}$.*

Proof. From the definition of the Yamabe invariant, for any $u \in L^2_1(M)$,

$$\int_M \left(6|\nabla_g u|_g^2 + R_g u^2\right) dV_g \geqslant Y(M) \left\{\int_M u^4\, dV_g\right\}^{1/2},$$

where we use g as the background metric. Since g has constant scalar curvature, this implies

$$\frac{6}{Y(M)} \int_M |\nabla u|^2 + \frac{R_g Vol(g)^{1/2}}{Y(M)} Vol(g)^{-1/2} \int_M u^2 \geqslant \left\{\int_M u^4\right\}^{1/2}.$$

Since g is Yamabe, we have $R_g Vol(g)^{1/2} = Y(M)$, so we obtain

$$\|u\|_{L^4} \leqslant \sqrt{6} Y(M)^{-1/2} \|\nabla u\|_{L^2} + Vol(g)^{-1/4} \|f\|_{L^2}. \qquad \Box$$

In dimension four, the Gauss–Bonnet and signature formulas are (see [Bes87])

$$8\pi^2 \chi(M) = \frac{1}{6}\int_M R^2 - \frac{1}{2}\int_M |Ric|^2 + \int_M |W|^2, \tag{3.1}$$

$$12\pi^2 \tau(M) = \int_M |W^+|^2 - \int_M |W^-|^2. \tag{3.2}$$

In the anti-self-dual case, $W^+ \equiv 0$, we have

$$8\pi^2 \chi(M) = \frac{1}{6}\int_M R^2 - \frac{1}{2}\int_M |Ric|^2 + \int_M |W^-|^2, \tag{3.3}$$

$$12\pi^2 \tau(M) = -\int_M |W^-|^2. \tag{3.4}$$

Since the anti-self-dual equation is conformally invariant, we can make a conformal change to a Yamabe minimizer (see [Aub82,Sch84,LP87]), and add these equations together to obtain

$$8\pi^2 \chi(M) + 12\pi^2 \tau(M) = \frac{R^2}{6} Vol(M) - \frac{1}{2}\int_M |Ric|^2. \tag{3.5}$$

If $R > 0$, and $2\chi(M) + 3\tau(M) > 0$ then we obtain the estimate

$$Y(M,[g]) = R Vol(M)^{1/2} \geqslant 2\sqrt{6}\pi(2\chi + 3\tau) > c > 0. \tag{3.6}$$

Proposition 3.2. *Let* (M, g) *be Yamabe with* $R > 0$*, and anti-self-dual. Then* $\|Rm_g\|_{L^2} < C$*, where* C *depends only on* $\chi(M), \tau(M)$*. Furthermore, if* $2\chi(M) + 3\tau(M) > 0$*, then the Sobolev constant is uniformly bounded from above,*

$$2\pi(C_S)^2 \leqslant \sqrt{6}\,(2\chi(M) + 3\tau(M))^{-1}\,. \tag{3.7}$$

Proof. For the first statement, (3.5) gives a bound on $\|Ric\|_{L^2}$, since $Y(M, [g]) \leqslant Y(S^n)$ (see [LP87]). Eq. (3.4) gives a bound on $\|W\|_{L^2}$. The second statement follows from (3.6). □

We next consider the Kähler case. Let $c_1(M)$ denote the first Chern class of M. It is known that for complex surfaces,

$$c_1^2(M) = 2\chi(M) + 3\tau(M)$$

and therefore on a complex surface,

$$Q'(M, [g]) \equiv c_1^2(M) - \frac{1}{3}\frac{(c_1(M) \cdot \omega_g)^2}{\omega_g^2} \tag{3.8}$$

is a conformal invariant. It follows that when $Q'(M, [g]) > 0$,

$$Y(M, [g]) \geqslant 3\pi^2 \sqrt{Q'(M, [g])}. \tag{3.9}$$

This implies

Proposition 3.3. *For* (M, g) *Kähler satisfying*

$$3c_1^2(M) > (c_1(M) \cdot [\omega_g])^2, \tag{3.10}$$

the Sobolev constant is uniformly bounded from above.

3.1. On the Sobolev inequality

All of the results in this paper are still valid if the weaker Sobolev inequality is assumed

$$\|\phi\|_{L^4} \leqslant C_S \left(\|\nabla\phi\|_{L^2} + Vol^{-1/4}\|\phi\|_{L^2} \right) \tag{3.11}$$

with the exceptions being in (2) in Theorem 1.1, M_∞ might split off a compact orbifold, and in (4) of Theorem 1.1, even if $b_1(M_i) = 0$, the limit may be reducible, see the proof of Proposition 7.2 below. Furthermore, as remarked in the introduction, the main elements of our proof only require a *local* Sobolev constant bound, see Theorem 7.3 below.

If we have a conformal class with positive Yamabe invariant, we have shown above that the Sobolev constant of the Yamabe minimizer is bounded. However, if we instead choose a non-minimizing constant scalar curvature metric, we will have a Sobolev inequality of type (3.11).

4. Local regularity

In all the above cases, the equation take the form

$$(\Delta Ric)_{ij} = A_{ikjl} R_{kl}, \tag{4.1}$$

where A_{ikjl} is some linear expression in the curvature tensor. Using a convenient shorthand, we write this as

$$\Delta Ric = Rm * Ric. \tag{4.2}$$

Using the Bianchi identities, any Riemannian metric satisfies

$$\Delta Rm = L(\nabla^2 Ric) + Rm * Rm, \tag{4.3}$$

where $L(\nabla^2 Ric)$ denotes a linear expression in second derivatives of the Ricci tensor.

Even though second derivatives of the Ricci occur in (4.3), overall the principal symbol of the system (4.2) and (4.3) in triangular form. The Eqs. (4.2) and (4.3), when viewed as an elliptic system, together with the bound on the Sobolev constant, yield the following ε-regularity theorem:

Theorem 4.1 (*Tian and Viaclovsky [TV, Theorem 3.1]*). *Assume that* (4.2) *is satisfied, choose* $r < diam(M)/2$, *and let* $B(p,r)$ *be a geodesic ball around the point* p, *and* $k \geqslant 0$. *Then there exist constants* ε_0, C_k (*depending upon* C_S) *so that if*

$$\|Rm\|_{L^2(B(p,r))} = \left\{ \int_{B(p,r)} |Rm|^2 \, dV_g \right\}^{1/2} \leqslant \varepsilon_0,$$

then

$$\sup_{B(p,r/2)} |\nabla^k Rm| \leqslant \frac{C_k}{r^{2+k}} \left\{ \int_{B(p,r)} |Rm|^2 \, dV_g \right\}^{1/2} \leqslant \frac{C_k \varepsilon_0}{r^{2+k}}.$$

Remark. We state the following slight variation of the above. Let $C_S(r)$ denote the Sobolev constant for compactly supported functions in $B(p,r)$, that is,

$$\|f\|_{L^4(B(p,r))} \leqslant C_S(r)\|\nabla f\|_{L^2(B(p,r))} \tag{4.4}$$

for all $f \in C_c^{0,1}(B(p,r))$. Then there exists a universal constant ε_0 such that if

$$\left\{C_S(r)^4 \cdot \int_{B(p,r)} |Rm|^2 \, dV_g\right\}^{1/2} \leqslant \varepsilon_0, \tag{4.5}$$

then

$$\sup_{B(p,r/2)} |Rm| \leqslant \frac{C}{r^2}\left\{C_S(r)^4 \cdot \int_{B(p,r)} |Rm|^2 \, dV_g\right\}^{1/2} \leqslant \frac{C\varepsilon_0}{r^2},$$

where C is a universal constant, the proof being the same as that of Theorem 4.1. It is also interesting to bound $C_S(r)$ in terms of the volume of $B(p,r)$. For the manifolds being considered in this paper, it may be possible that $C_S(r) \cdot Vol(B(p,r))^{1/4} < Cr$, for some uniform constant C.

Theorem 4.1 may be applied to non-compact orbifolds to give a rate of curvature decay at infinity. Assume that (M,g) is a complete, non-compact orbifold with finitely many singular points, with a critical metric, bounded Sobolev constant (for functions with compact support), and finite L^2 norm of curvature. Fix a basepoint p, and let $r(x) = d(p,x)$. Given $\varepsilon < \varepsilon_0$ from Theorem 4.1, there exists an R large so that there are no singular points on $D(R/2)$ and

$$\int_{D(R)} |Rm|^2 \, dV_g < \varepsilon < \varepsilon_0,$$

where $D(R) = M \setminus B(R)$. Choose any $x \in M$ with $d(x,p) = r(x) > 2R$, then $B(x,r) \subset D(R)$. From Theorem 4.1, we have

$$\sup_{B(x,r/2)} |\nabla^k Rm| \leqslant \frac{C_k}{r^{2+k}}\left\{\int_{B(x,r)} |Rm|^2 \, dV_g\right\}^{1/2} \leqslant \frac{C_k\varepsilon}{r^{2+k}},$$

which implies

$$|\nabla^k Rm|(x) \leqslant \frac{C_k\varepsilon}{r^{2+k}}.$$

As we take R larger, we may choose ε smaller, and we see that M has better-than-quadratic curvature decay, along with decay of derivatives of curvature.

5. Volume growth

One of the crucial aspects of this problem is to obtain control on volume growth of metric balls from above. We let (M, d) be a length space with distance function d, and basepoint $p \in M$.

Definition 5.1. We say a component $A_0(r_1, r_2)$ of an annulus $A(r_1, r_2) = \{q \in M \mid r_1 < d(p, q) < r_2\}$ is *bad* if $S(r_1) \cap \overline{A_0(r_1, r_2)}$ has more than 1 component, where $S(r_1)$ is the sphere of radius r_1 centered at p.

As we remarked after [TV, Theorem 4.1], the proof of our volume growth theorem requires only that there are finitely many disjoint bad annuli, therefore we have

Theorem 5.2 (*Tian and Viaclovsky [TV, Theorem 4.1]*). *Let (M, g) be a complete, noncompact, four-dimensional Riemannian orbifold* (*with finitely many singular points*) *with base point p. Assume that there exists a constant $C_1 > 0$ so that*

$$Vol(B(q, s)) \geqslant C_1 s^4 \tag{5.1}$$

for any $q \in M$, and all $s \geqslant 0$. Assume furthermore that as $r \to \infty$,

$$\sup_{S(r)} |Rm_g| = o(r^{-2}), \tag{5.2}$$

where $S(r)$ denotes the sphere of radius r centered at p. If (M, g) contains only finitely many disjoint bad annuli, then (M, g) has finitely many ends, and there exists a constant C_2 so that

$$Vol(B(p, r)) \leqslant C_2 r^4, \tag{5.3}$$

Furthermore, each end is ALE of order 0.

Proof. Since there are no orbifold singular points outside of a compact set, the proof of [TV, Theorem 4.1] is valid in this case. To see this, from [Bor93, Proposition 15] any minimizing geodesic segment cannot pass through the singular set unless it begins or ends on the singular set, and the set $M \setminus S$ is geodesically convex. Therefore, all of the standard tools from Riemannian geometry used in the proof of [TV, Theorem 4.1] apply in this setting. □

By taking instead sequences of dyadic annuli $A(s^{-j-1}, s^{-j})$, $1 < s$, around a singular point, the proof of [TV, Theorem 4.1] can also be applied directly to components of isolated singularities:

Theorem 5.3. *Let (X, d, x) be a complete, locally compact length space, with basepoint x. Let $B(x,1) \setminus \{x\}$ be a C^∞ connected four-dimensional manifold with a metric g of class (a), (b), or (c) satisfying*

$$\int_{B(x,1)} |Rm_g|^2 \, dV_g < \infty, \tag{5.4}$$

$$\|u\|_{L^4(B(x,1)\setminus\{x\})} \leqslant C_{\mathrm{S}} \|\nabla u\|_{L^2(B(x,1)\setminus\{x\})}, \quad u \in C^{0,1}(B(x,1) \setminus \{x\}), \tag{5.5}$$

$$b_1(X) < \infty, \tag{5.6}$$

where C_{S}, V_1 are positive constants. Then there exists a constant $C_1 > 0$ so that $Vol(B(x,r)) \leqslant C_1 r^4$. The basepoint x is an orbifold point, and the metric g extends to $B(x,1)$ as a C^0-orbifold metric. That is, for some small $\delta > 0$, the universal cover of $B(x,\delta) \setminus \{x\}$ is diffeomorphic to a punctured ball $B^4 \setminus \{0\}$ in $\mathbf{R}^4$, and the lift of g, after diffeomorphism, extends to a C^0 metric $\tilde{g}$ on B^4, which is smooth away from the origin.

Remark. This is valid for components of $B(x,\delta) \setminus \{x\}$, we will prove below that there are finitely many components for the limit space arising in Theorem 1.1. To show x is a C^0-orbifold point, one uses a tangent cone analysis as in [TV, Theorem 4.1]. Furthermore, in Theorem 6.4 below, we will show g is a *smooth* orbifold metric.

6. Asymptotic curvature decay and removal of singularities with bounded energy

We first discuss curvature decay results from [TV, Section 6], and using the same technique, we prove a singularity removal result.

Theorem 6.1. *Let (M, g) be a complete, non-compact four-dimensional irreducible Riemannian orbifold with g of class (a), (b), or (c), and finitely many singular points. Assume that*

$$\int_M |Rm_g|^2 \, dV_g < \infty, \ \ C_{\mathrm{S}} < \infty, \ \ and \ \ b_1(M) < \infty. \tag{6.1}$$

Then (M, g) has finitely many ends, and each end is ALE of order τ for any $\tau < 2$.

Remark. In case (M, g) is a manifold, from [Car99, Theorem 1], we have a bound on the number of ends depending only upon the Sobolev constant and the L^2-norm of curvature (moreover, all of the L^2-Betti numbers are bounded). In the Kähler case, an argument as in [LT92] shows that there can be at most 1 non-parabolic end, we remark that the analysis there is valid for irreducible orbifolds with finitely many singular points. Since any ALE end is non-parabolic, this implies there only one end. The argument in [LT92, Theorem 4.1] is roughly, to construct a non-constant bounded harmonic function with finite Dirichlet integral if there is more than 1 non-parabolic end. This function must then be pluriharmonic, and under the curvature decay conditions, it must therefore be constant.

Proof. Theorem 6.1 was proved in [TV, Theorem 1.3], the proof there is also valid for orbifolds. We briefly outline the details.

Lemma 6.2. *If (M, g) satisfies (a), (b), or (c), then*

$$\big|\nabla|E|\big|^2 \leqslant \frac{2}{3}|\nabla E|^2, \tag{6.2}$$

at any point where $|E| \neq 0$, where E denotes the traceless Ricci tensor.

This is due to Tom Branson, the proof follows from his general theory of Kato constants developed in [Bra00], see [TV, Lemma 5.1] for the details of this case, the proof being valid also in all cases (a), (b), and (c). We remark that the same constant follows from the methods in [CGH00]. The case considered in Lemma 6.2 is exactly the case $r = s = 2$ in the last line of the table on [CGH00, p. 253], giving immediately the best constant $2/3$.

Using this improved Kato constant, we now have the equation

$$\Delta|E|^{1/2} \geqslant -C|E|^{1/2}|Rm|. \tag{6.3}$$

Using a Moser iteration argument from [BKN89], and since the scalar curvature is constant, this allows one to improve the Ricci curvature decay to $|Ric| = O(r^{-2-\delta})$ for any $\delta < 2$, where $r(x) = d(p, x)$ is the distance to a basepoint p. Next, using a Yang–Mills argument (inspired by the proof of Uhlenbeck for Yang–Mills connections [Uhl82], also [Tia90, Section 4]) the following was proved in [TV, Lemma 6.5]

Lemma 6.3. *Let $D(r) = M \setminus B(p, r)$. For $\delta < 2$, and r sufficiently large, we have*

$$\sup_{D(2r)} \|Rm_g\|_g \leqslant \frac{C}{r^{2+\delta}}. \tag{6.4}$$

The result then follows by the construction of coordinates at infinity in [BKN89]. $\square$

Next we discuss a removable singularity result, this is an analogue of [BKN89, Theorem 5.1], [Tia90, Lemma 4.5]. This theorem is crucial to obtain smoothness of the limit orbifold.

Theorem 6.4. *Let (X, d, x) be a complete; locally compact length space, with basepoint x. Let $B(x,1)\setminus\{x\}$ be connected C^∞ four-dimensional manifold with a metric g of class (a), (b), or (c) satisfying*

$$\int_{B(x,1)} |Rm_g|^2 \, dV_g < \infty, \tag{6.5}$$

$$\|u\|_{L^4(B(x,1)\setminus\{x\})} \leqslant C_s \|\nabla u\|_{L^2(B(x,1)\setminus\{x\})}, \;\; u \in C^{0,1}(B(x,1)\setminus\{x\}), \tag{6.6}$$

$$Vol(B(x,r)) \leqslant V_1 r^4, \;\; r > 0, \tag{6.7}$$

where C_s, V_1 are positive constants. Then the metric g extends to $B(x,1)$ as a smooth orbifold metric. That is, for some small $\delta > 0$, universal cover of $B(x,\delta)\setminus\{x\}$ is diffeomorphic to a punctured ball $B^4\setminus\{0\}$ in $\mathbf{R}^4$, and the lift of g, after diffeomorphism, extends to a smooth critical metric $\tilde{g}$ on B^4.

Proof. The argument in [TV, Lemma 6.5] for ALE spaces examined the behavior at infinity, we now imitate the argument using balls around a singular point. From Theorem 5.3 above, we know the singularity is orbifold of order 0, and the tangent cone at a singularity is a cone on a spherical space form S^3/Γ, We lift by the action of the orbifold group to obtain a critical metric in $B(0,\delta)\setminus\{0\}$ with bounded energy, bounded Sobolev constant, and $Vol(B(0,s)) < Cs^4$. From the Kato inequalities in cases (a), (b), and (c), we obtain the estimate $|Ric| = O(r^{-2+\delta})$, where r now denotes distance to the origin, for any $\delta < 2$. The argument from [TV, Lemma 6.5] shows that $|Rm| = O(r^{-2+\delta})$. As in [Tia90, Lemma 4.4], we can then find a self-diffeomorphism ψ of $B(0,\delta)$ so that $\nabla(\psi^* g) = O(r^{-1+\delta})$, and $\psi^* g = O(r^\delta)$. Choosing δ close to 2, the metric $\psi^* g$ has a $C^{1,\alpha}$ extension across the origin. From the results of [DK81], this is sufficient regularity to find a harmonic coordinate system around the origin. We view the equation as coupled to the equation for g in harmonic coordinates:

$$\Delta Ric = Rm * Ric, \tag{6.8}$$

$$\Delta g = Ric + Q(g, \partial g). \tag{6.9}$$

From (6.8), as in [BKN89, Lemma 5.8], it is not hard to conclude that $Ric \in L^p$ for any $p < \infty$ (this is because from assumption we have a Sobolev constant bound, and we also have an upper volume growth bound). Since g is $C^{1,\alpha}$, from elliptic regularity, (6.9) implies that $g \in W^{2,p}$, and therefore $Rm \in L^p$ for any p. Equation (6.8) then implies $Ric \in W^{2,p}$, and (6.9) gives $g \in W^{3,p}$. Bootstrapping in this manner, we find that $g \in C^\infty$. $\square$

7. Convergence

In this section, we complete the proofs of Theorems 1.1, 1.2, and 1.3. We first describe the construction of the limit space, we will be brief since this step is quite standard (see for example [Aku94, Section 4], [And89, Section 5], [Nak88, Section 4], [Tia90, Section 3]). From the Sobolev constant bound (1.8) and lower volume bound (1.10), we obtain a lower growth estimate on volumes of geodesic balls. That is, there exists a constant $v > 0$ so that $Vol(B(x,s)) \geqslant vs^4$, for all $x \in M_j$ and $s \leqslant s_0$, for some $s_0 > 0$ [Heb96, Lemma 3.2]. For $R > 0$ large, let $M_{j,R} = M_j \cap B(p_j, R)$, and for $r > 0$ small, we take a maximally r-separated set of $M_{j,R}$, that is, a collection of points $p_{i,j} \in M_{j,R}$ so that $B(p_{i,j}, r) \cap B(p_{i',j}, r) = \emptyset$ for $i \neq i'$, and the collection $B(p_{i,j}, 2r)$ covers $M_{j,R}$. From the assumed bound (1.9) on the L^2-norm of curvature, only a uniformly finite number of the balls $B(p_{i,j}, r)$ may satisfy $\int_{B(p_{i,j},r)} |Rm_{g_j}|^2 \, dV_j \geqslant \varepsilon_0$, where ε_0 is the constant from Theorem 4.1. By passing to a subsequence, we may assume that the number of these points is constant. Let us denote this collection of points by S_j, let $S_j(r)$ denoted the r-neighborhood of S_j, and let $\Omega_j(r) = M_{j,R} \setminus S_j(r)$. From Theorem 4.1, the curvature and all covariant derivatives are uniformly bounded on compact subsets of $\Omega_j(r)$. Furthermore, the lower volume growth estimate implies an injectivity radius estimate (see [CGT82]), so we may apply a version of the Cheeger-Gromov convergence theorem (see [And89,Tia90]) to find a subsequence such that $(\Omega_j(r), g_j)$ converges smoothly to $(\Omega_\infty(r), g_\infty)$ as $j \to \infty$ on compact subsets. That is, there exist diffeomorphisms $\Phi_{j,r} : \Omega_\infty(r) \to \Omega_j(r)$ such that $\Phi_{j,r}^* g_j$ converges to g_∞ in C^∞ on compact subsets of $\Omega_\infty(r)$. By choosing a sequence $r_i \to 0$, and by taking diagonal subsequences, we obtain limit spaces with natural inclusions $\Omega_\infty(r_i) \subset \Omega_\infty(r_{i+1})$. Letting $i \to \infty$, we obtain a limit space $(M_{\infty,R}, g_\infty)$. This is done for each R large, and taking a sequence $R_i \to \infty$, we obtain a pointed limit space $(M_\infty, g_\infty, p_\infty)$.

We will now show how the main part of Theorem 1.1 follows assuming Theorem 1.2, and then we will complete the proof of Theorem 1.2 below. In fact, we only require the volume growth estimate from Theorem 1.2 to hold only for $r \leqslant r_0$, where r_0 is some fixed scale. That is, let us assume that

$$Vol(B_{g_i}(p,r)) \leqslant Vr^4 \tag{7.1}$$

for all $p \in M_i$, and all $r \leqslant r_0$. The volume growth estimate (7.1) implies that we may add finitely many points to M_∞ to obtain a complete length space; this follows since $\#|S_j|$ is uniformly bounded (see [And89, Section 5], [Tia90, Section 3]) for more details). For notational simplicity, we will continue to denote the completion by M_∞.

The estimate (7.1), together with a global lower volume bound, imply a lower diameter bound $\mathrm{diam}(M_i, g_i) > \lambda > 0$, which implies that $M_\infty \neq S$. From (7.1), it follows also that we have local volume convergence, and (M_j, g_j, p_j) converges to $(M_\infty, g_\infty, p_\infty)$ in the Gromov–Hausdorff distance, moreover, the convergence is of length spaces.

To analyze the singular points of M_∞, for $p \in S$ we look at $B(p,\delta) \setminus \{p\}$ for δ small. The volume growth estimate (7.1) implies the number of components of $B(p,\delta) \setminus \{p\}$

is finite (see [Tia90, Lemma 3.4]). Restricting to each component, Theorem 6.4 implies that the singularities are metric orbifold singularities, that is, the metric is locally a quotient of a *smooth* metric on each cone. Consequently, M_∞ is a Riemannian multifold. Using what we have proved so far about limits (i.e., under the assumption (7.1)), we next prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 4.1, if g is critical, and

$$\|Rm\|_{L^2(B(p,2))} = \left\{ \int_{B(p,2)} |Rm|^2 \, dV_g \right\}^{1/2} \leqslant \varepsilon_0,$$

then

$$\sup_{B(p,1)} |Rm| \leqslant \frac{1}{4} C \varepsilon_0.$$

By Bishop's volume comparison theorem, we must have $Vol(B(p,1)) \leqslant A_0$, where A_0 depends only on the Sobolev constant.

We also note the following fact, for any metric,

$$\lim_{r \to 0} Vol(B(p,r)) r^{-4} = \omega_4,$$

where ω_4 is the volume ratio of the Euclidean metric on $\mathbf{R}^4$. Clearly, $A_0 \geqslant \omega_4$.

For any metric (M, g), define the maximal volume ratio as

$$MV(g) = \max_{x \in M, r \in \mathbf{R}^+} \frac{Vol(B(x,r))}{r^4}. \tag{7.2}$$

If the theorem is not true, then there exists a sequence of critical manifolds (M_i, g_i), with $MV(g_i) \to \infty$, that is, there exist points $x_i \in M_i$, and $t_i \in \mathbf{R}^+$ so that

$$Vol(B(x_i, t_i)) \cdot t_i^{-4} \to \infty, \tag{7.3}$$

as $i \to \infty$. We choose a subsequence (which for simplicity we continue to denote by the index i) and radii r_i so that

$$2 \cdot A_0 = Vol(B(x_i, r_i)) \cdot r_i^{-4} = \max_{r \leqslant r_i} \; Vol(B(x_i, r)) \cdot r^{-4}, \tag{7.4}$$

We furthermore assume that x_i is chosen so that r_i is minimal, that is, the smallest radius for which there exists some $p \in M_i$ such that $Vol(B_{g_i}(p, r)) \leqslant 2A_0 r^4$, for all $r \leqslant r_i$.

First let us assume that r_i has a subsequence converging to zero. For this subsequence (which we continue to index by i), we consider the rescaled metric $\tilde{g}_i = r_i^{-2} g_i$, so that

$B_{g_i}(x_i, r_i) = B_{\tilde{g}_i}(x_i, 1)$. From the choice of x_i and r_i, the metrics $\tilde{g}_i$ have bounded volume ratio, in all balls of unit size.

From the argument above, some subsequence converges on compact subsets to a complete length space $(M_\infty, g_\infty, p_\infty)$ with finitely many point singularities. The limit could either be compact or non-compact. In either case, the arguments above imply that the limit is a Riemannian multi-fold.

Claim 7.1. *The limit $(M_\infty, g_\infty, p_\infty)$ contains at most B_1 disjoint bad annuli.*

Proof. We know that $(M_i, \tilde{g}_i, x_i)$ converges to $(M_\infty, g_\infty, p_\infty)$ as pointed spaces. Assume that $(M_\infty, g_\infty, p_\infty)$ contains $B_1 + 1$ disjoint bad annuli $A_l, 1 \leqslant l \leqslant B_1 + 1$. Then there exists a radius R so that $\cup A_l \subset B(p_\infty, R)$. Since the convergence is of pointed spaces, given any $\varepsilon > 0$, there exist pointed, continuous ε-almost isometries $\Phi_{i,\varepsilon} : B_{\tilde{g}_i}(x_i, 2R) \to B_{g_\infty}(p_\infty, 2R + \varepsilon)$ for i sufficiently large (see [BBI01]). For ε sufficiently small, it is easy to see that for each l, $\Phi_{i,\varepsilon}^{-1}(A_l)$ will be ε-close to a bad annulus in $(M_i, \tilde{g}_i, x_i)$. Applying the Mayer–Vietoris argument in [TV, Lemma 4.7] to this collection, we conclude that the number must be bounded by B_1, a contradiction. $\square$

If M_∞ is noncompact, the remarks at the end of Section 4 shows that assumption (5.2) is satisfied. Also, from [TV, Lemma 6.1], the Sobolev constant bound implies a lower volume growth bound (this is valid for orbifolds), so (5.1) is satisfied. Theorem 5.2 then implies that M_∞ has only finitely many ends, and that there exists a constant $A_1 \geqslant 2A_0$ so that

$$Vol(B_{g_\infty}(p_\infty, r)) \leqslant A_1 r^4, \quad \text{for all} \quad r > 0. \tag{7.5}$$

If M_∞ is compact, then clearly the estimate (7.5) is valid for some constant $A_1 \geqslant 2A_0$, since the limit has finite diameter and volume, and the estimate holds for $r \leqslant 1$.

The inequality

$$\int_{B_{g_i}(x_i, r_i)} |Rm_i|^2 \, dV_i \geqslant \varepsilon_0, \tag{7.6}$$

must hold; otherwise, as remarked above, we would have $Vol(B_{g_i}(x_i, r_i)) \leqslant A_0 r_i^4$, which violates (7.4).

If the r_i are bounded away from zero then there exists a radius t so that

$$Vol(B_{g_i}(p, r)) \leqslant 2A_0 r^4, \quad \text{for all} \quad r \leqslant t, \, p \in M_i. \tag{7.7}$$

We repeat the argument from the first case, but without any rescaling. Since the maximal volume ratio is bounded on small scales, we can extract an orbifold limit. The limit can either be compact or non-compact, but the inequality (7.5) will also be satisfied for some A_1, Following the same argument, we find a sequence of balls satisfying (7.6).

We next return to the (sub)sequence (M_i, g_i) and extract another subsequence so that

$$200 \cdot A_1 = Vol(B(x_i', r_i')) \cdot (r_i')^{-4} = \max_{r \leqslant r_i'} Vol(B(x_i', r)) \cdot r^{-4}. \tag{7.8}$$

Again, we assume that x_i' is chosen so that r_i' is minimal, that is, the smallest radius for which there exists some $p \in M_i$ such that $Vol(B_{g_i}(p, r)) \leqslant 200 A_1 r^4$, for all $r \leqslant r_i$. Clearly, $r_i < r_i'$.

Arguing as above, if $r_i' \to 0$ as $i \to \infty$, then we repeat the rescaled limit construction, but now with scaled metric $\tilde{g}_i = (r_i')^{-2} g_i$, and basepoint x_i'. We find a limiting orbifold $(M_\infty', g_\infty', p_\infty')$, and a constant $A_2 \geqslant 200 A_1$ so that

$$Vol(B_{g_\infty'}(p_\infty', r)) \leqslant A_2 r^4 \text{ for all } r > 0.$$

For the same reason as above, we must have

$$\int_{B_{g_j}(x_j', r_j')} |Rm_j|^2 \, dV_j \geqslant \varepsilon_0.$$

If r_i' is bounded below, we argue similarly, but without any rescaling.

We claim that for i sufficiently large, the balls $B(x_i, r_i)$ (from the first subsequence) and $B(x_i', r_i')$ (from the second) must be disjoint because of the choice in (7.8). To see this, if $B(x_i, r_i) \cap B(x_i', r_i') \neq \emptyset$, then $B(x_i', r_i') \subset B(x_i, 3r_i')$. Then (7.5) and (7.8) yield

$$200 A_1 (r_i')^4 = Vol(B(x_i', r_i')) \leqslant Vol(B(x_i, 3r_i')) \leqslant 2A_1 (3r_i')^4 = 162 A_1 (r_i')^4,$$

which is a contradiction (note the last inequality is true for i sufficiently large since (7.5) holds for the limit).

We repeat the above procedure. The process must terminate in finitely many steps from the bound $\|Rm_i\|_{L^2} < \Lambda$. This contradicts (7.3), which finishes the proof. $\square$

The convergence statement in Theorem 1.1 now follows from Theorem 1.2, since (7.1) is satisfied. Statement (3) follows from Theorem 6.1, since the multi-fold is the union of irreducible orbifolds. Note also that the volume bound in Theorem 1.2 gives a uniform bound for the number of irreducible pieces, and the number of ends of the limit multi-fold.

To finish the proof of Theorem 1.1 we need to verify statements (2), (4), and (5). The next proposition gives a direct argument to bound the number of components of $B(p, \delta) \setminus \{p\}$ for δ small in terms of the Sobolev constant and first Betti number.

Proposition 7.2. *For $p \in M_\infty$, and δ sufficiently small, the number of components of $B(p, \delta) \setminus \{p\}$ can be estimated in terms of the first Betti number and the Sobolev constant* (*defined as in* (1.2 *or* 1.3)). *If $b_1(M_i) = 0$, then p is irreducible. Furthermore, M_∞ does not split off any compact orbifold.*

If the weaker Sobolev inequality (3.11) *is assumed, then the number of components of* $B(p,\delta)\setminus\{p\}$ *can be still be estimated in terms of the Sobolev constant and the first Betti number* (*but in this case it is possible that if* $b_1(M_i)=0$, *a singular point could be reducible, and it is also possible that* M_∞ *could split off a compact orbifold*).

Proof. Let p be a non-irreducible singular point. We have shown around p, M_∞ is a finite union of orbifold cones, with the basepoints identified. For each orbifold, since the convergence is smooth away from the singular points, we look before the limit, and this gives us a portion of a cone on S^3/Γ in the original manifold, very small, which we call $N_i \subset M_i$ and $N_i = (a_i, 2a_i)\times(S^3/\Gamma)$, which is close, in any C^m-topology, to an annulus $A(a_i,2a_i)$ in a cone on a spherical space form $\mathcal{C}(S^3/\Gamma)$, and where $a_i \to 0$ as $i\to\infty$.

If $\{a_i\}\times S^3/\Gamma$ bounds a region in M_i, equivalently, if N_i separates M_i into two components, then this decomposes M_i into a disjoint union $A_i \cup N_i \cup B_i$. Since the point p is non-irreducible and the convergence is smooth away from the singular points, we must have $Vol(A_i)$ and $Vol(B_i)$ uniformly bounded away from zero. Without loss of generality, assume $Vol(A_i)\leqslant Vol(B_i)$.

We take a function f_i which is 1 on A_i, 0 on B_i, since the neck N_i is C^m-close to the annulus $A(a_i,2a_i)$ in a flat cone, we may take $|\nabla f| = 1/a_i$ on the neck portion N_i. We compute

$$\|f_i\|_{L^4} = \left\{\int_{A_i} 1\, dV_{g_i} + \int_{N_i} f_i\, dV_{g_i}\right\}^{1/4} \sim Vol(A_i)^{1/4}. \tag{7.9}$$

Next,

$$\|\nabla f_i\|_{L^2}^2 = \int_{N_i} \frac{1}{a_i}\, dV_{g_i} \sim \frac{1}{a_i} C((2a_i)^4 - (a_i)^4) = C a_i^3, \tag{7.10}$$

since N_i is C^m-close to $A(a_i,2a_i)$. Using the Sobolev inequality (1.2), we obtain

$$Vol(A_i)^{1/4} \leqslant C_S C' a_i^{3/2} + Vol(M_i)^{-1/4} Vol(A_i)^{1/2}, \tag{7.11}$$

Rearranging terms,

$$Vol(A_i)^{1/4}(1 - Vol(M_i)^{-1/4} Vol(A_i)^{1/4}) \leqslant C_S C' a_i^{3/2}. \tag{7.12}$$

We have $Vol(M_i)\geqslant 2Vol(A_i)$, therefore

$$Vol(A_i)^{1/4}(1 - 2^{-1/4}) \leqslant C_S C' a_i^{3/2}. \tag{7.13}$$

Since $Vol(A_i)$ is uniformly bounded away from zero, this is a contradiction for i large. Therefore none of the necks N_i around a non-irreducible singular point can bound regions in M_i. Using the intersection pairing, any of these embedded space forms will give a generator of $b_1(M_i)$. At most two of these may give rise to the same generator, so from the assumed bound on $b_1(M_i)$, there may only be finitely many, and if $b_1(M_i) = 0$, the singular point p must be irreducible.

Note that in case of the Sobolev inequality (1.3), a similar argument works, and a similar argument shows that M_∞ does not split off any compact orbifold.

In the case (3.11) is satisfied, let p be non-irreducible singular point. Again, we have shown around p, M_∞ is a finite union of orbifold cones, with the basepoints identified. For each orbifold group Γ_j at p, since the convergence is smooth away from the singular points, we look before the limit, and this gives us a portions of cones on S^3/Γ_j in the original manifold, $N_{i,j} \subset M_i$, very small, $N_{i,j} = (a_i, 2a_i) \times (S^3/\Gamma_j)$, which is close, in any C^m-topology to an annulus $A(a_i, 2a_i)$ in a cone on a spherical space form $\mathcal{C}(S^3/\Gamma_j)$, and where $a_i \to 0$ as $i \to \infty$.

Take any collection of $Q > 16C_S^4$ irreducible orbifolds at p. Then we claim at least one of the necks $N_{i,j}$ cannot bound a region in M_i, i.e., $N_{i,j}$ cannot separate M_i into 2 components. If all of the $N_{i,j}$ bound, then this decomposes M_i into a disjoint union $A_i \cup (\cup_j N_{i,j}) \cup B_i$, where A_i is taken to be on the side of the neck where convergence is smooth, B_i is the rest of M_i. Since we have a finite collection, and convergence is smooth on A_i, so $Vol(A_i)$ is uniformly bounded away from zero. Now A_i is the union of Q regions, therefore, one of the regions, which we call $R_{i,j}$, must satisfy $Vol(R_{i,j}) < \frac{1}{Q} Vol(A_i)$.

We take a function f_i which is 1 on the region $R_{i,j}$, since the neck $N_{i,j}$ bounding $R_{i,j}$ is C^∞-close to the annulus $A(a_i, 2a_i)$ in a flat cone, we may take $|\nabla f| = 1/a_i$ on the neck portion N_i, with $f_i = 0$ otherwise.

As in (7.12) above, but using the Sobolev inequality (3.11), we obtain

$$Vol(R_{i,j})^{1/4}(1 - C_S Vol(M_i)^{-1/4} Vol(R_{i,j})^{1/4}) \leqslant C_S C' a_i^{3/2}. \tag{7.14}$$

We have $Vol(M_i) \geqslant Q Vol(R_{i,j})$, therefore

$$Vol(R_{i,j})^{1/4}(1 - C_s Q^{-1/4}) \leqslant C_S C' a_i^{3/2}, \tag{7.15}$$

from the choice of Q, we obtain

$$\frac{1}{2} Vol(R_{i,j})^{1/4} \leqslant C_S C' a_i^{3/2}. \tag{7.16}$$

Since $Vol(R_{i,j})$ is uniformly bounded away from zero, this is a contradiction for i large. Therefore, for any collection of $Q > 16C_S^4$ irreducible orbifolds at p, one of the neck $N_{i,j}$ cannot bound a regions in M_i. Using the intersection pairing, the corresponding embedded space form $S^3/\Gamma_{i,j}$ will give a generator of $b_1(M_i)$.

If there are $k * Q$ orbifolds at p, then we find k generators $b_1(M_i)$. At most 2 of these may give rise to the same generator, so from the assumed bound on $b_1(M_i)$, there may only be finitely many. $\square$

We remark that we may characterize the singular set as follows: with ε_0 as in Theorem 4.1, we have

$$S = \{x \in M_\infty \,|\, \liminf_{j\to\infty} \int_{B(x_j,r)} |Rm_{g_j}|^2 \, dvol_{g_j} \geqslant \varepsilon_0$$
$$\text{for any sequence } \{x_j\} \text{ with } \lim_{j\to\infty} x_j = x, \text{ and all } r > 0\}. \tag{7.17}$$

We next give a description of the convergence at the singular points, by rescaling the sequence at a singular point $x \in S$. Several bubbles may arise in the degeneration, so we have to rescale properly, and possibly at several different scales. This was done in [Ban90a] for the Einstein case, and with a few minor changes, the proof works in our case. We outline the details here. For $0 < r_1 < r_2$, we let $D(r_1, r_2)$ denote $B(p, r_2) \setminus B(p, r_1)$. Given a singular point $x \in S$, we take a sequence $x_i \in (M, g_i)$ such that $\lim_{i\to\infty} x_i = x$ and $B(x_i, \delta)$ converges to $B(x, \delta)$ for all $\delta > 0$. We choose a radius r_∞ sufficiently small and the sequence x_i to satisfy

$$\sup_{B(x_i, r_\infty)} |Rm_{g_i}|^2 = |R_{g_i}|^2(x_i) \to \infty \quad \text{as} \quad j \to \infty, \tag{7.18}$$

and

$$\int_{B(x, r_\infty)} |R_{g_\infty}|^2 \, dV_{g_\infty} \leqslant \varepsilon_0/2. \tag{7.19}$$

We next choose $r_0(j)$ so that

$$\int_{D(r_0, r_\infty)} |R_{g_j}|^2 \, dV_{g_j} = \varepsilon_0, \tag{7.20}$$

where ε_0 is as in Theorem 4.1, and again $D(r_o, r_\infty) = B(x_i, r_\infty) \setminus B(x_i, r_0)$. An important note, which differs from the Einstein case, the annulus $D(r_0, r_\infty)$ may have several components.

Since the curvature is concentrating at x, $r_o(j) \to 0$ as $j \to \infty$. From Theorem 1.1, the rescaled sequence $(M, r_o(j)^{-2} g_i, x_i)$ has a subsequence which converges to a complete, non-compact multi-fold with finitely many singular points, which we denote by M_{i_1}, $1 \leqslant i_1 \leqslant \#\{S\}$. Since

$$\int_{D(1,\infty)} |Rm|^2 \, dV_g \leqslant \varepsilon_0, \tag{7.21}$$

there are no singular points outside of $B(x, 1)$.

On the non-compact ends, from Theorem 6.1, the metric is ALE of order τ for any $\tau < 2$. As in [Ban90a, Proposition 4], we conclude that the neck region (for large i) will be arbitrarily close to a portion of a flat cone $\mathbf{R}^4/\Gamma$, possibly several cones if M_{i_1} has several ends. So the convergence at a singular point x_{i_1} is that the ALE multi-fold M_{i_1} is bubbling off, or scaled down to a point in the limit.

To further analyze the degeneration at the singular points, we look at the multi-fold M_{i_1} with singular set S_{i_1}. If S_{i_1} is empty, then we stop. We do the same process as above for each singular point of M_{i_1}, and obtain ALE multi-folds M_{i_1,i_2}, $1 \leqslant i_2 \leqslant \#\{S_{i_1}\}$. If M_{i_1,i_2} has singularities, then we repeat the procedure. This process must terminate in finite steps, since in this construction, each singularity takes at least ε_0 of curvature. As pointed out in [Ban90b], there could be some overlap if any singular point lies on the boundary of $B(1)$ at some stage in the above construction. But there can only be finitely many, and then there must also be a singular point in the interior of $B(1)$, so we still take away at least ε_0 of curvature at each step.

In the Kähler case, one can use the methods of [LT92] to show that boundary of sufficiently small balls around the singular points of M_∞ are connected. If a singular point $p \in M_\infty$ is non-irreducible, then using the above bubbling analysis, at some step one must find an irreducible Kähler ALE orbifold with more than one end. From the remark following Theorem 6.1, this is not possible, therefore in the Kähler case (c), the limit is irreducible. This completes the proof of Theorem 1.1.

7.1. Local Sobolev inequality

As we have noted throughout the paper, many of our results hold with a weaker assumption on the Sobolev constant. We have the following notion of local Sobolev constant. We define $C_S(r)$ to be the best constant such that

$$\|f\|_{L^4} \leqslant C_S(r)\|\nabla f\|_{L^2} \tag{7.22}$$

for all $f \in C^{0,1}$ with compact support in $B(p,r)$, and for all $p \in M$.

The following is the analogue of Theorem 1.3 with a local Sobolev constant bound (the proof is identical):

Theorem 7.3. *Let (M_i, g_i) be a sequence of critical metrics g_i on smooth, four-dimensional manifolds M_i satisfying*

$$C_S(r_0) \leqslant C_1 \quad (\textit{for some fixed } r_0 > 0), \tag{7.23}$$

$$\int_{M_i} |Rm_{g_i}|^2 \, dV_{g_i} \leqslant \Lambda, \tag{7.24}$$

$$Vol(g_i) = 1, \tag{7.25}$$

$$b_1(M_i) < B_1, \tag{7.26}$$

where C_1, Λ, λ *are constants, and* $b_1(M_i)$ *denotes the first Betti number, Then there exists a subsequence* $\{j\} \subset \{i\}$, *a compact, connected, critical Riemannian multifold* (M_∞, g_∞), *and a finite singular set* $S \subset M_\infty$ *such that* (M_j, g_j) *converges to* (M_∞, g_∞).

8. Further remarks

We conclude by listing here some interesting problems.

(1) We considered above the case of constant scalar curvature Kähler metrics. We conjecture that these results extend to the more general extremal Kähler case in dimension four [Cal82,Cal85].

(2) It is an interesting problem to generalize our results to higher dimensions. We conjecture that the following is true for the higher dimensional extremal Kähler case. Assuming fixed Kähler class, first and second Chern classes, the limit space has at most a codimension four singular set, and the singular set is a holomorphic subvariety. Even in the case of Bach-flat or harmonic curvature in higher dimensions, under the bound $\|Rm\|_{L^2} < \Lambda$, the limit space should have a most a codimension four singular set, with top strata modeled on orbifold singularities. This was proved for Einstein metrics in [CC97,CC00a,CC00b,CCT02].

(3) It would be very interesting to remove the Sobolev constant assumption and understand the collapsing case.

(4) In the general Bach-flat case in dimension four, one should be able to show that the orbifold singularities are *smooth* metric singularities, and that in the ALE case, one can obtain a positive order of decay.

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