Approximation of the Perturbation Equations of a Quasi-Linear Hyperbolic System in the Neighborhood of a Bicharacteristic

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In 1956 Whitham gave a nonlinear theory for computing the intensity of an acoustic pulse of an arbitrary shape. The theory has been used very successfully in computing the intensity of the sonic bang produced by a supersonic plane. Gubkin (1958) derived an approximate quasi-linear equation for the propagation of a short wave in a compressible medium. These two methods are essentially nonlinear approximations of the perturbation equations of the system of gasdynamic equations in the neighborhood of a bicharacteristic curve (or rays) for weak unsteady disturbances superimposed on a given steady solution. In this paper we have derived an approximate quasi-linear equation which is an approximation of perturbation equations in the neighborhood of a bicharacteristic curve for a weak pulse governed by a general system of first order quasi-linear partial differential equations in \( m + 1 \) independent variables \((t, x_1, ..., x_m)\) and derived Gubkin's result as a particular case when the system of equations consists of the equations of an unsteady motion of a compressible gas. We have also discussed the form of the approximate equation describing the waves propagating upstream in an arbitrary multidimensional transonic flow.

INTRODUCTION

In the theory of hyperbolic systems of partial differential equations in two independent variables \((x, t)\), the characteristic curves play a very important role. This is, of course, due to the compatibility condition along a characteristic curve, which can be easily obtained from the system of equations and in which all dependent variables are differentiated in the direction of the characteristic curve. One of the important properties of a characteristic is that discontinuities in the derivatives of the dependent variables can exist only along a characteristic curve and they can be uniquely determined at any point of a characteristic curve, provided they are known at one point, by solving an ordinary differential equation, called "transport equation" along the
characteristic curve [2]. If a solution of a hyperbolic system is interpreted as a wave propagation, the above result imply that disturbances are carried along the characteristic curves.

In the theory of partial differential equations in \( m + 1 \) independent variables \((x_1, x_2, \ldots, x_m, t) (m > 1)\) a characteristic curve is replaced by an \( m \) dimensional manifold, namely characteristic surface and we can easily derive the corresponding compatibility condition, in which all derivatives of the dependent variables are "interior derivatives." However, there is one important difference when we compare the case of \( m > 1 \) with the case of two independent variables. When \( m > 1 \), there are \( m \) independent interior directions at any point of the characteristic surface, and in the compatibility condition the partial derivatives of different dependent variables will appear in different combinations. Thus, unlike the simpler case of \( m = 1 \), the compatibility condition does not reduce to an ordinary differential equation in a directional derivative. However, at any point of a characteristic surface, there is a privileged direction given by the tangent of the bicharacteristic curve passing through the point. Is it possible to derive a compatibility condition along a bicharacteristic curve, i.e., is it possible to obtain from the system of original equations another equation in which the directional derivatives of the dependent variables are only in the bicharacteristic direction? We do not seem to have a definite answer to this question. Nevertheless, theoretical investigations on the wave propagation in fluid mechanics, particularly in geometrical acoustics, show that such a compatibility condition might exist even for a general hyperbolic system, at least in some approximate sense.

Making use of the results of differential geometry, Coburn (1957) derived an equation containing derivatives only along the bicharacteristic curve from the system of equations representing steady supersonic flow of an inviscid and non-conducting gas. The transport equation along the bicharacteristic curve representing the rate of change of a discontinuity in the normal derivative of a dependent variable has already been derived in the case of general linear [2] and quasilinear [11] systems. The transport equation gives the intensity of a discontinuity in the derivative only at the wave front and, hence, is of very limited use. In this connection Whitham's work on the propagation of weak waves (1956) is outstanding. He discussed this problem for the equations of motion of a gas making use of the result in the linear theory of sound waves and derived an equation giving the rate of change of the amplitude of the wave along the rays. This extension from linear to non-linear theory is essentially a non-linear approximation of the perturbation equations of the system of gasdynamic equations in the neighborhood of a bicharacteristic curve, for weak unsteady disturbances superimposed on a given steady solution. However, it is important to note that Whitham's work on gasdynamic equations does not indicate a definite method for such an
approximation in the case of the general system of partial differential equations. Gubkin (1958), making assumptions of short waves, derived in a very elegant manner an approximate quasilinear equation giving the rate of change in the bicharacteristic direction of the amplitude of a weak disturbance. Unlike the transport equation the approximate treatments of Gubkin and Whitham give the distribution of flow variables not only at the wave front but also to a short distance behind the wave front.

Recently we made some attempt in approximating the perturbation equations of a general system of first order quasi-linear equations in the neighborhood of a bicharacteristic curve, assuming that the sections of the characteristic surface by \( t = \) constants are planes, i.e., in the terminology of wave propagation, by assuming the wave front to be plane \([7, 8]\). In this paper we have extended our previous work to the case when the wave front is of an arbitrary shape. We take here a general hyperbolic system of first order quasilinear equations and consider a perturbation of small amplitude on a given steady solution of the system. The disturbance is bounded by a wave front of an arbitrary shape, whose locus in \((x_a, t)\) space as \( t \) varies is a characteristic surface. We approximate the system of equations governing the perturbation in the neighborhood of the characteristic surface by a single quasilinear partial differential equation which gives the rate of change in the bicharacteristic direction of a quantity giving complete description of the disturbance. As a particular case of our general theory we have derived Gubkin's result for gasdynamic equations. We have also discussed the form of the approximate equations describing the waves propagating upstream in a steady transonic flow. We wish to discuss the main characteristics of transonic pulses in a separate paper, our main aim here is to obtain the derivation of short wave equations for a general system of partial differential equations. We have achieved our goal with the help of a special transformation of dependent variables in terms of the right null vector of the characteristic matrix.

In this context, Ludwig's work (1960) on the hyperbolic system of linear equations also seems to be worth mentioning, but it does not seem to be possible to generalize his method to quasilinear equations.

**Derivation of the Approximate Equation**

Let us consider the system of first order equations,

\[
A_{ij} \frac{\partial u_j}{\partial t} + B_{ij}^{(a)} \frac{\partial u_j}{\partial x_a} + C_i = 0, \quad i = 1, 2, \ldots, n, \tag{2.1}
\]

where \( u_i \) are \( n \) dependent variables, \( x_a, t \) the \( m + 1 \) independent variables
and the coefficients $A_{ij}$, $B_{ij}^{(a)}$, $C_{i}$ are functions of $u_{i}$ and $x_{a}$ only. We have used the convention that a repeated suffix (except the suffixes $L$ and $M$) in a term will imply sum over the spectrum of the suffix. Throughout this paper, we shall take the spectrum of the suffixes $i, j, k$ to be 1, 2, 3,..., $n$ and $\alpha, \beta, \gamma$ to be 1, 2,..., $m$. We assume the system to be hyperbolic with $t$ as a time like variable, i.e., for an arbitrary set of real numbers \{n_{a}\}, there are $n$ real characteristic roots $c_{i}$ (not necessarily distinct) of the equation

$$\det[n_{a}B_{ij}^{(a)} - \lambda A_{ij}] = 0 \quad (2.2)$$

and that there exist $n$ linearly independent left eigenvectors

$$l^{(k)} = (l_{1}^{(k)}, l_{2}^{(k)}, \ldots, l_{n}^{(k)})$$

which imply existence of $n$ linearly independent right eigenvectors

$$r^{(k)} = (r_{1}^{(k)}, r_{2}^{(k)}, \ldots, r_{n}^{(k)})$$

also:

$$I_{i}^{(L)} n_{a}B_{ij}^{(a)} = c_{L} I_{ij}^{(L)} A_{ij} \quad , \quad n_{a}B_{ij}^{(a)} r_{j}^{(L)} = c_{L} A_{ij} r_{j}^{(L)} \quad L = 1, 2, \ldots, n. \quad (2.3)$$

If the equation of the characteristic surface corresponding to the characteristic velocity $c_{k}$ be denoted by

$$\phi_{x_{a}}^{(k)}(x_{a}, t) = 0 \quad (2.4)$$

the derivatives $\phi_{x_{a}}^{(k)}$ and $\phi_{t}^{(k)}$ are proportional to $n_{a}$ and $-c_{k}$. The sections of the characteristic surface by hyperplanes $t =$ constants give the successive positions of a wave front, the vector $(n_{a})$ in $(x_{a})$ space is in the direction of the normal to a wave front and we select $(n_{a})$ such that it represents a unit vector. Then $c_{k}$ represents the velocity of the wavefront in its normal direction $(n_{a})$ and

$$(n_{a}) = \frac{\text{grad } \phi^{(k)}}{|\text{grad } \phi^{(k)}|}, \quad c_{k} = -\frac{\phi_{t}^{(k)}}{|\text{grad } \phi^{(k)}|}. \quad (2.5)$$

Let us introduce a new set of $m + 1$ independent variables $(x_{a}', \phi^{(L)})$, where

$$\phi^{(L)} = \phi^{(L)}(x_{a}, t), \quad x_{a}' = x_{a}, \quad (2.6)$$

$L$ being a fixed number taken from the set (1, 2,..., $n$) and the system (2.1) reduces to

$$(A_{ij} \phi_{x_{a}'}^{(t)} + B_{ij}^{(a)} \phi_{x_{a}}^{(t)}) \frac{\partial u_{i}}{\partial \phi_{L}} + B_{ij}^{(a)} \frac{\partial u_{i}}{\partial x_{a}} + C_{i} = 0. \quad (2.7)$$
When $\phi^{(L)} = 0$ is one of the characteristic surfaces and $l^{(L)}$ the corresponding left eigenvector as given by (2.3), we can multiply (2.7) by $l^{(L)}_i$ and get

$$l^{(L)}_i B^a_i j \frac{\partial u_j}{\partial x^a} + l^{(L)}_i C_i = 0, \quad (2.8)$$

which is the compatibility condition along the characteristic surface. It is important to note that $\frac{\partial}{\partial x^a}$ is an interior derivative in the characteristic surface $\phi^L = 0$ and is given by

$$\frac{\partial}{\partial x^a} = \frac{\partial}{\partial x^a} + \frac{n}{\delta L} \frac{\partial}{\partial \sigma}. \quad (2.8)$$

If the equation of a bicharacteristic curve lying in the characteristic surface $\phi^{(L)}(x, t) = 0$ be written as

$$x_a = x_a(\sigma), \quad t = t(\sigma), \quad (2.9)$$

from the lemma on bicharacteristics [2] we can suitably choose $\sigma$ such that

$$\frac{dx_a}{d\sigma} = l^{(L)}_i B^a_i j r_j^{(L)}, \quad \frac{dt}{d\sigma} = l^{(L)}_i A_{ij} r_j^{(L)}. \quad (2.10)$$

The directional derivative in the direction of the bicharacteristic curve is given by

$$\frac{d}{d\sigma} = \frac{dt}{d\sigma} \frac{\partial}{\partial t} + \frac{dx_a}{d\sigma} \frac{\partial}{\partial x^a} = (l^{(L)}_i B^a_i j r_j^{(L)}) \frac{\partial}{\partial x^a}. \quad (2.11)$$

Let us consider a known steady solution $u_i = u_{i0}(x_a)$ of the system of equations (2.1) such that

$$B^{a}_{ij0} \frac{\partial u_{i0}}{\partial x^a} + C_{i0} = 0, \quad (2.12)$$

where the suffix 0 on a variable represents its value in the steady solution, i.e.,

$$B^{a}_{ij0} = B^{a}_{ij0}(x, u_{c0}). \quad (2.13)$$

We consider a perturbation

$$v_i = u_i - u_{i0} \quad (2.14)$$

on the given steady state such that the amplitude of $v_i$ is of the order of a small quantity $\delta$. From (2.8) we get

$$l^{(L)}_i B^a_i j \frac{\partial v_j}{\partial x^a} + F^{(L)} = 0 \quad (2.15)$$
where

\[ F^{(l)} = l_i^{(l)} C_i + l_i^{(l)} B_{ij}^{(a)} \frac{\partial u_{j0}}{\partial x_a}. \]  

(2.16)

Expanding the functions \( C_i \) and \( B_{ij}^{(a)} \) about the steady solution \( u_{i0} \) and using (2.12) we get

\[ C_i + B_{ij}^{(a)} \frac{\partial u_{j0}}{\partial x_a} = (F_{\text{ev}})_0 v_k + O(\delta^2) \]

(2.17)

where

\[ (F_{\text{ev}})_0 = \left( \frac{\partial C_i}{\partial u_k} \right)_0 + \left( \frac{\partial B_{ij}^{(a)}}{\partial u_k} \right)_0 \frac{\partial u_{j0}}{\partial x_a}. \]

(2.18)

From (2.16) we get

\[ F^{(l)} = l_i^{(l)} (F_{\text{ev}})_0 v_k + O(\delta^2). \]

(2.19)

Since the set \( \{r^{(1)}, r^{(2)}, \ldots, r^{(n)}\} \) of right eigenvectors is linearly independent we can replace the set \( \{v_i\} \) by a new set of dependent variables \( w_1, w_2, \ldots, w_n \) through the transformation

\[ v_j = r_j^{(k)} \omega_k. \]

(2.20)

and the equations (2.15) become

\[ l_i^{(l)} B_{ij}^{(a)} r_j^{(k)} \frac{\partial \omega_k}{\partial x_a'} + R_H^{(a)} \frac{\partial \omega_j}{\partial x_a'} + F^{(l)} = 0, \quad L = 1, 2, \ldots, n \]

(2.21)

where

\[ F^{(l)} = l_i^{(l)} (F_{\text{ev}})_0 r_j^{(k)} \omega_j + O(\delta^2), \]

(2.22)

and each of \( \omega_1, \omega_2, \ldots, \omega_n \) are at most of the order of \( \delta \). Using (2.11), we write (2.21) in the form

\[ \frac{d\omega_L}{d\sigma_L} + \sum_{k \neq L} l_i^{(l)} B_{ij}^{(a)} r_j^{(k)} \frac{\partial \omega_k}{\partial x_a'} + l_i^{(l)} B_{ij}^{(a)} \omega_k \frac{\partial r_j^{(k)}}{\partial x_a'} + F^{(l)} = 0, \quad L = 1, 2, \ldots, n. \]

(2.23)

If we create an arbitrary disturbance on a given steady solution \( u_{i0}(x_a) \), in general, the disturbance will break into \( n \) modes propagating with the characteristic velocities \( c_1, c_2, \ldots, c_n \). The locus of the wave front of the part of the disturbance moving with the velocity \( c_k \) will be a member of the family of characteristic surfaces \( \phi^{(k)} = \text{constants} \). However, we consider here only
those disturbances which consist of only a single mode, i.e., in which the disturbance stays in the neighborhood of the characteristic surface \( \phi^{(M)}(x_\alpha, t) = 0 \) where \( M \) is a fixed integer from 1, 2, ..., \( n \). We shall now make short wave assumption, i.e., that the disturbance is localized in the neighborhood of the wave front so that it is non-zero only over a distance of the order of \( \epsilon \) from the wave front where \( \epsilon \) is small compared to the radius of curvature \( R \) of the wave front and compared to the characteristic length \( H \) in the steady state over which \( u_{i_0} \) varies significantly. We further assume, for simplicity, that the characteristic \( \phi^{(M)} = 0 \) is simple (we can easily extend the theory when \( \phi^{(M)} = 0 \) is a multiple characteristic) [1]. We shall show now that each \( \omega_L (L \neq M) \) is small compared to \( \omega_M \).

We consider (2.23) for \( L \neq M \). Then the curve \( \sigma_L \) remains in the disturbed region only over a distance of the order of \( \epsilon \) and integrating (2.23) along \( \sigma_L \) we get

\[
\omega_L = O \left( \frac{\epsilon \delta}{R} \right) + O \left( \frac{\epsilon \delta}{H} \right) + O(\delta \epsilon), \quad L \neq M. \tag{2.24}
\]

Substituting (2.24) in (2.23) for \( L = M \) and neglecting all terms of orders \( \epsilon \delta/R, \epsilon \delta/H \) and \( \delta \epsilon \) we get

\[
\frac{d\omega_M}{d\sigma_M} + \left( j_{i_0}^{(M)} B_{ij0}^{(a)} \frac{\partial \phi^{(M)}}{\partial x_\alpha} \right) \omega_M + \left( j_{i_0}^{(M)} F_{i_0 j_0} r^{(M)} \right) \omega_M = 0,
\]

there is no sum over \( M \), \tag{2.25}

and the relation (2.20), to the same approximation, becomes

\[
v_j = r_{j_0}^{(M)} \omega_M. \tag{2.26}
\]

We may be tempted to approximate the expression (2.21) for the bicharacteristic derivative by replacing \( t_{i_0}^{(M)} B_{ij}^{(a)} c_{i_0}^{(M)} \) by \( t_{i_0}^{(M)} B_{ij0}^{(a)} r_{j_0}^{(M)} \), so that the resulting equation is linear. However, this will lead to inaccuracy if we follow the wave for a sufficiently long time when the accumulating non-linear effects will become important [12]. Linearising (2.23) will lead to an inaccurate description of the wave even over small distance if the bicharacteristic velocity components \( \chi_\alpha \):

\[
\chi_\alpha = \frac{j_{i_0}^{(M)} B_{ij}^{(a)} c_{i_0}^{(M)}}{j_{i_0}^{(M)} A_{ij} c_{i_0}^{(M)}} \tag{2.27}
\]

vanish at some critical point in the steady state [5, 8]. Therefore, for an uniformly valid non-linear theory of wave propagation, we must retain the first order terms of the order of \( \delta \) in the approximation of \( t_{i_0}^{(M)} B_{ij}^{(a)} c_{i_0}^{(M)} \) in (2.11).
Equation (2.25) is our final equation giving an approximation of the perturbation equations of an arbitrary system of quasilinear hyperbolic equations (2.1) or the equivalent system (2.23) in the neighborhood of a bicharacteristic curve.

We have not said anything about the unit normal \( (n_a) \) to the wave front which appears in the equation (2.25). If the wave is continuous, the wave front which separates the disturbed region from the undisturbed region can be obtained by solving the system of bicharacteristic equations

\[
\frac{dx_a}{ds_M} = \frac{\partial Q}{\partial \phi_{x_a}}, \quad \frac{dt}{ds_M} = \frac{\partial Q}{\partial \phi_t},
\]

and

\[
\frac{d\phi_{x_a}}{ds_M} = -\frac{\partial Q}{\partial x_a}, \quad \frac{d\phi_t}{ds_M} = -\frac{\partial Q}{\partial t},
\]

where

\[Q = \det[A_{ij}(\phi_t) + B_{ij}(\phi_\alpha)]\]

and \( s_M \) is a function of \( \sigma_M \). The equations (2.22) and (2.29) can be transformed into another system of ordinary differential equations for \( x_a \) and \( n_a \) by the relation \( n_a = \phi_{x_a}/(\phi_{x_a^2} \phi_{\sigma_a})^{1/2} \). Thus, given the initial wave front and hence the normal \( n_a \) at \( t = 0 \) we can determine \( n_a \) as functions of \( x_a \) (see the example in the next section).

If the wave is not continuous but headed by a weak shock, the situation is complicated. However, for the short wave, we assume here that we shall determine \( n_a \) from the same equations (2.28) and (2.29) but with the initial wave front as the shock front.

Solutions of the short wave problems is now reduced to the solution of a single quasi-linear equation (2.25) for a single unknown \( \omega_1 \). The method of solution of this equation and its use will be discussed in the next section with an example, namely, the equations of unsteady motion of a compressible fluid.

**Derivation of Gubkin’s Equation and Approximation for Unsteady Transonic Pulses**

The equations of motion of a nonviscous, nonconducting polytropic gas can be represented by Eq. (2.1) where

\[ n = 5, \quad m = 3, \quad u_a = q_a, \quad u_k = \rho, \quad u_5 = \rho, \]

\[ A_{ij} = \delta_{ij}, \quad C_i = 0, \]

\[ \text{409/50 /3-3} \]
and there exist three linearly independent eigenvectors corresponding to $n_0 q_\alpha$.

Gubkin has followed the waves corresponding to the characteristic velocity $c_5$, however, in order to avoid a duplication of the algebra in the case of our result on transonic flows we consider here upstream propagating waves and, therefore, we take $M = 1$. The left and right eigenvectors are given by:

$$
l^{(1)}_\alpha = n_\alpha, \quad l^{(1)}_4 = -\frac{1}{\rho a}, \quad l^{(1)}_5 = 0,\]

$$
r^{(1)}_\alpha = n_\alpha, \quad r^{(1)}_4 = -\rho a, \quad r^{(1)}_5 = -\frac{\rho}{a}.
$$

From (2.14) and (2.26) we get

$$
v_{\alpha} = q_{\alpha} - q_{s0} = n_\alpha \omega_1,
$$

$$
v_{4} = \rho - \rho_0 = -\rho_0 a_0 \omega_1 \quad \text{and} \quad v_{5} = \rho - \rho_0 = -\frac{\rho_0}{a_0} \omega_1$$

and

$$
\frac{d}{d\sigma_1} = 2(q_{\alpha} - an_\alpha) \frac{\partial}{\partial x_\alpha}.
$$

We can also show that

$$
l^{(1)}_{i0} B^{(1)}_{i0} \frac{\partial r^{(1)}_{i0}}{\partial x_\alpha} = -a_0 \frac{\partial n_\alpha}{\partial x_\alpha} + \frac{1}{\rho_0 a_0} (q_{s0} - a_0 n_\alpha) \frac{\partial (\rho_0 a_0)}{\partial x_\alpha},
$$

(3.6)
and
\[ f_{x_0}^t (F_{x_0})_0 t_{x_0}^t = \gamma \frac{\partial q_{x_0}}{\partial x_0} + n_\alpha n_\beta \frac{\partial q_{x_0}}{\partial x_\alpha} \cdot \quad (3.9) \]

Since
\[ \frac{\partial}{\partial x_\alpha'} = \frac{\partial}{\partial x_\alpha} + n_\alpha \frac{\partial}{\partial t} \cdot \]

from (3.7) we get
\[ \frac{1}{2} \frac{d}{d\tau} - \frac{\partial}{\partial t} + (q_\alpha - an_\alpha) \frac{\partial}{\partial x_\alpha} \cdot \quad (3.10) \]

With the help of (3.8), (3.9) and (3.10), Eq. (2.25) for \( M = 1 \) becomes
\[ \frac{\partial \omega_1}{\partial t} + (q_\alpha - an_\alpha) \frac{\partial \omega_1}{\partial x_\alpha} = (K - a_0 \Omega) \omega_1 \cdot \quad (3.11) \]

where
\[ K = -\frac{\gamma}{2} \frac{\partial q_{x_0}}{\partial x_\alpha} - \frac{1}{2} n_\alpha n_\beta \frac{\partial q_{x_0}}{\partial x_\alpha} - \frac{1}{2} (q_{x_0} - u_0 a_0) \frac{1}{\rho_0 a_0} \frac{\partial}{\partial x_\alpha} (\rho_0 a_0) \quad (3.12) \]

and
\[ \Omega = -\frac{1}{2} \frac{\partial n_\alpha}{\partial x_\alpha} \quad (3.13) \]

is the mean curvature of the wave front.

Taking
\[ \Omega = \phi_t + \phi_{x_\alpha} q_{x_0} - a_0(\phi_{x_\alpha} \phi_{x_\alpha})^{1/2}, \]

the bicharacteristic equations (2.28) and (2.29) give us
\[ \frac{dx_\alpha}{dt} = q_{x_0} - n_\alpha a_0 \quad (3.14) \]

and
\[ \frac{dn_\alpha}{dt} = (n_\alpha n_\gamma - \delta_{\alpha\gamma}) \left( n_\beta \frac{\partial q_{x_0}}{\partial x_\beta} - \frac{\partial a_0}{\partial x_\gamma} \right) \cdot \quad (3.15) \]

Consider a solution \( x_\alpha = x_\alpha(a, b, t), n_\alpha = n_\alpha(a, b, t) \) of the equations (3.14) and (3.15) with initial conditions at \( t = 0 \):
\[ n_\alpha = g_\alpha(a, b) \quad (3.16) \]

and
\[ x_\alpha = h_\alpha(a, b) \quad (3.17) \]
where \((g_1, g_2, g_3)\) is the unit normal to the initial position of the wave front given by (3.17). Solving \((a, b, t)\) in terms of \(x_a\) from \(x_a = x_a(a, b, t)\) and substituting in \(n_a(a, b, t)\) we can determine \(n_a\) as functions of \(x_a\) i.e. \(n_a(\varphi, n_a')\) for \(\varphi = 0\).

To obtain Gubkin's equation we should follow the wave front moving with the velocity \(c_5\) and make the substitution

\[
\omega_1 = \frac{A}{\rho_0 a_0}
\]

(3.18)

in the equation corresponding to (3.11). Thus we have derived the approximate theory of Gubkin (1958) for short waves.

Expanding the bicharacteristic velocity components \(q_a - an_a\) about the steady solution and retaining only the first order terms, we get

\[
q_a - n_a a = q_{a0} - n_a a_0 + \frac{\gamma + 1}{2} n_a \omega_1.
\]

(3.19)

Thus, we get the final equation in a single unknown \(\omega_1\),

\[
\frac{\partial \omega_1}{\partial t} + \left\{(q_{a0} - n_a a_0) + \frac{\gamma + 1}{2} n_a \omega_1\right\} \frac{\partial \omega_1}{\partial x_a} = (K - \Omega a_0) \omega_1.
\]

(3.20)

The method of solution of this equation is simple. Let us assume a given distribution of \(\omega_1\) at \(t = 0\):

\[
\omega_1(x_a, 0) = f(x_a)
\]

(3.21)

in a domain bounded by the initial wave front (3.17). Consider a solution \(x_a = x_a(\xi, \eta, \zeta, t), \omega_1 = \omega_1(\xi, \eta, \zeta, t)\) of the characteristic equations

\[
\frac{dx_a}{dt} = \left(q_{a0} - n_a a_0 + \frac{\gamma + 1}{2} n_a \omega_1\right)
\]

(3.22)

and

\[
\frac{d\omega_1}{dt} = (K - \Omega a_0) \omega_1,
\]

(3.23)

with conditions:

\[
\begin{align*}
x_1 &= \xi, & x_2 &= \eta, & x_3 &= \zeta \\
\omega_1 &= f(\xi, \eta, \zeta) & \text{at} & & t = 0.
\end{align*}
\]

(3.24)

Solving \(\xi, \eta, \zeta\) from \(x_a = x_a(\xi, \eta, \zeta, t)\) in terms of \(x_a\) and \(t\) and substituting in
\( \omega_1(\xi, \eta, \zeta, t) \) we get the solution \( \omega_1(x_a, t) \) giving the distribution of flow variables at any instant behind the wave front in the short wave. Weak shocks should be fitted wherever the solution is not single valued and their motion should be determined from the result that for weak shocks the normal velocity of a shock wave is equal to the arithmetic mean of the wave front velocities (i.e., values of \( q_a n_a - a \)) just ahead and just behind the shock front.

Recently some interest has been revived in the propagation of weak pulses in a transonic flow [8, 10] in order to resolve the famous transonic controversy. It was initially proposed that due to trapping of upstream propagating waves (only these waves are likely to be trapped) [8] at different points of a sonic surface, a continuous mixed supersonic and subsonic flow is unstable. However, Spee used the theory of geometrical acoustics to calculate the wave front and showed that due to the variation of the flow variables in the directions normal to the stream lines, a wave front perpendicular to the stream lines turns and thus escapes trapping in the transonic region. Still, when we look at the photographs of a transonic flow [10] we find the presence of a large number of weak shocks perpendicular to the streamlines and moving slowly in the transonic region. This is simple to explain. In most of the practical examples of transonic flows we deal with thin aerofoils or quasi-onedimensional nozzles with small curvature at the throat and in such cases the derivatives \( \partial q_{30}/\partial x_\gamma \) and \( \partial q_\alpha/\partial x_\gamma \) are small quantities and thus, from (3.15), the time spent in turning of a upstream propagating wave is large compared to the time taken by the other waves in traversing the transonic region. Thus, the upstream propagating waves perpendicular to the streamlines for which each of the bicharacteristic velocity components vanish at a sonic point, remain in the transonic region for a longer time compared to other waves and this explains the existence of almost trapped waves with weak shocks in a transonic flow. In our previous paper [8] we tried to discuss the propagation of these trapped waves by assuming the wave front to be a plane perpendicular to a stream line passing through a sonic point \( (x^*) \). However, based on this assumption, we could not take into account the important multidimensional turning effect. On the assumption that the characteristic length \( H \) in the steady flow in large compared to the radius of curvature \( R \) of the wave front and that the angle between the normal to the wave front and the streamlines is small, we can easily discuss the turning effect of the waves from a model which we can easily deduce from the equations (3.14), (3.15) and (3.20). Finally we can combine this result qualitatively with the results of our previous paper to get full picture of transonic pulses.

Since the main aim of this paper was the derivation of short wave equations for waves governed by a general system of partial differential equations, we shall discuss the transonic pulses in a separate paper.
REFERENCES