Precise Coefficient Estimates for Close-to-Convex Harmonic Univalent Mappings

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The class $SH$ consists of harmonic, univalent, and sense-preserving functions $f$ in the open unit disk $U = \{z : |z| < 1\}$, such that $f = h + \bar{g}$, where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} a_{-n} z^n$. Let $S_{0H}$, $C_{H}$, and $C_{0H}$ denote the subclass of $SH$ with $a_{-1} = 0$, the subclass of $SH$ with $f$ being a close-to-convex mapping, and the intersection of $S_{0H}$ and $C_{H}$, respectively. In this paper, for $f \in C_{0H}$ and $f \in C_{H}$, we prove that the harmonic analogue of the Bieberbach conjecture and the generalization of the Bieberbach conjecture are true.

1. INTRODUCTION

Let $S_{H}$ denote the class of all harmonic, complex valued, orientation-preserving, and univalent mappings $f$ defined in the open unit disk $U = \{z : |z| < 1\}$ normalized at the origin by $f(0) = 0$ and $f'(0) = 1$. Such functions can be written in the form

$$f = h + \bar{g},$$

where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} a_{-n} z^n$ are analytic and $|g'(z)| < |h'(z)|$ in $U$. It follows that $|a_{-1}| < 1$, and hence $(f - a_{-1}\bar{f})/(1 - a_{-1}) < 1$. Let $S_{0H}$, $C_{H}$, and $C_{0H}$ denote the subclass of $S_{H}$ with $a_{-1} = 0$, the subclass of $S_{H}$ with $f$ being a close-to-convex mapping, and the intersection of $S_{0H}$ and $C_{H}$, respectively. In this paper, for $f \in C_{0H}$ and $f \in C_{H}$, we prove that the harmonic analogue of the Bieberbach conjecture and the generalization of the Bieberbach conjecture are true.
\(|a_{-1}|\) belongs to \(S_H\). Thus we may restrict our attention to the subclass

\[
S^0_H = \{ f \in S_H : f_2(0) = 0 \}.
\]

Let \(C_H\) denote the subset of \(S_H\), such that for any \(f \in C_H\), \(f(U)\) is a close-to-convex domain. Let \(C^0_H \cap C_H\), and let \(\overline{C_{S^0}}\) denote the closure of \(C_H\). Clunie and Sheil-Small [1] posed the following conjecture:

I. Harmonic analogue of the Bieberbach conjecture. If \(f \in S^0_H\), then

\[
|a_n| - |a_{-n}| \leq n \quad (n = 2, 3, \ldots)
\]

\[
|a_{-n}| \leq (n - 1)(2n - 1)/6 \quad (n = 2, 3, \ldots).
\]

Later Sheil-Small [2] developed the above conjecture and posed a generalization of the Bieberbach conjecture:

II. If \(f \in S_H\), then

\[
|a_n| < (2n^2 + 1)/3 \quad (|n| = 2, 3, \ldots).\]

It was proved in [1, 2] that conjecture I is true for typically real functions and all functions \(f \in S^0_H\) for which \(f(U)\) is starlike with respect to the origin or \(f(U)\) is convex in one direction. However, it remains open for the close-to-convex class \(C^0_H\). For conjecture II, Clunie and Sheil-Small proved the following facts:

If \(f \in \overline{C_{S^0}}\), then

\[
|a_n| \leq (2n^2 + 1)/3 \quad (|n| = 1, 2, \ldots).
\]

In this paper, we prove that the conjectures I and II are also true for \(f \in C^0_H\) and \(f \in C_H\), respectively.

For convenience, we introduce some notation. Let \(P\) denote the set of analytic functions in \(U\) satisfying \(\text{Re } G > 0\), for \(G \in P\). Let \(H^p\) denote the class of analytic functions \(G\) in \(U\) which have bounded integral mean \(M_p(r, G)\) as \(r \to 1^-\), where

\[
M_p(r, G) = \begin{cases} 
\left( \frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^p \, d\theta \right)^{1/p}, & 0 < p < \infty \\
\max_{0 \leq \theta \leq 2\pi} |G(re^{i\theta})|, & p = \infty.
\end{cases}
\]

A real-valued function \(u(z)\) in \(U\) is said to be harmonic if \(M_p(r, u)\) is bounded as \(r \to 1^-\). The class of harmonic functions is denoted by \(h^p\), \(0 < p \leq \infty\).
2. LEMMAS

LEMMA 1 [3]. Let \( t(\theta) \) be a real-valued continuous function on \((-\infty, +\infty)\). If \( t(\theta) \) satisfies \( t(\theta + 2\pi) = t(\theta) + 2\pi \) and \( t(\theta_2) - t(\theta_1) > -\pi \), where \( \theta_1 < \theta_2 < \theta_1 + 2\pi \), then there is a continuous nondecreasing function \( s(\theta) \), such that

\[
s(\theta + 2\pi) = s(\theta) + 2\pi \quad \text{and} \quad |s(\theta) - t(\theta)| \leq \pi/2.
\]

LEMMA 2 [4]. If \( u \in h^p \) for some \( p \), \( 1 < p < \infty \), then its harmonic conjugate \( v \) with \( v(0) = 0 \) also belongs to \( h^p \). Furthermore, for all \( u \in h^p \), there is a constant \( A_p = (p/(p-1))^{1/p} \) depending only on \( p \), such that

\[
M_p(r, u) \leq A_p M_p(r, u), \quad 0 \leq r < 1.
\]

LEMMA 3 [5]. Let \( u(z) \) be real-valued harmonic in \( U \). If \( u(z) \) satisfies

\[
\int_0^{2\pi} |u(re^{i\theta})|^p d\theta = O(1), \quad 0 \leq r < 1, \quad p > 1,
\]

then

\[
u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\varphi - \theta)} f_1(\varphi) d\varphi, \quad z = re^{i\theta},
\]

where \( f_1 \in L^p \), and for almost all \( e^{i\theta} \), \( \lim_{z \to e^{i\theta}} u(z) = f_1(\theta) \) uniformly when \( z \to e^{i\theta} \) from the inside of any fixed Stolz domain in \( U \).

LEMMA 4. If \( f = h + \bar{g} \in C_H \), then there exist real numbers \( \mu, \theta_0 \) and a function \( H(z) \in P \) such that

\[
H(z)\left[ie^{i\theta}(1 - z^2)(e^{-i\mu}h'(e^{i\theta}z) + e^{i\mu}g'(e^{i\theta}z))\right] \in P \quad (|z| < 1).
\]

**Proof of Lemma 4.** Using the approximation method of [1, Theorem 3.7], we may assume that \( f \) extends continuously and smoothly to \( \overline{U} \), with \( f(|z| = 1) \) being a smooth curve whose interior is a close-to-convex domain and

\[
|h'(e^{i\theta})| > |g'(e^{i\theta})|.
\]

Let \( l_1(\theta) = \partial f(e^{i\theta})/\partial \theta = e^{i\theta}ih'(e^{i\theta}) + i\overline{e^{i\theta}g'(e^{i\theta})} \). Then

\[
\arg l_1(2\pi + \theta) = \arg l_1(\theta) + 2\pi,
\]

\[
\arg l_1(\theta_2) - \arg l_1(\theta_1) > -\pi \quad (\theta_1 < \theta_2 < \theta_1 + 2\pi).
\]

By Lemma 1, there is a continuous nondecreasing function \( s(\theta) \) such that

\[
s(\theta + 2\pi) = s(\theta) + 2\pi,
\]

\[
|s(\theta) - \arg l_1(\theta)| \leq \pi/2.
\]
Now setting \( l_2(\theta) = e^{i \theta} \) and \( R(\theta) = l_1(\theta)/l_2(\theta) \), we obtain
\[
\arg R(\theta) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right].
\]
(2.1)

From the proof of [1, Lemma 5.11], we know that there exists a real number \( \theta_0 \), such that
\[
\mathcal{R} \left[ (1 - e^{2i\theta}) e^{-i(\theta + \theta_0)} l_2(\theta + \theta_0) \right] \geq 0 \quad \text{for} \quad \theta \in [0, 2\pi].
\]
(2.2)

For simplicity, we define
\[
e^{-i\theta_0} = e^{-i\mu},
\]
and
\[
Q(\theta) = \begin{cases} 
-\frac{\pi}{2}, & \text{for } \theta \in \{ \theta : 0 \leq \arg l_4(\theta) + \arg R(\theta + \theta_0) \leq \pi \} \\
\frac{\pi}{2}, & \text{for } \theta \in \{ \theta : -\pi \leq \arg l_4(\theta) + \arg R(\theta + \theta_0) < 0 \} \\
0, & \text{for } \theta \in \{0, \pi, 2\pi\},
\end{cases}
\]
(2.5)

where \( E = (0, \pi) \cup (\pi, 2\pi) \). Here, noting (2.1)–(2.4), we have \(-\pi \leq \arg l_4(\theta) + \arg R(\theta + \theta_0) \leq \pi\), for \( \theta \in E \).

Next we put
\[
v(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)Q(t)}{1 + r^2 - 2r\cos(\theta - t)} \, dt, \quad z = re^{i\theta} \in U.
\]
then
\[
v(e^{i\theta}) = Q(\theta) \quad \text{a.e. for } \theta \in [0, 2\pi]
\]
(2.6)

and
\[
|v(re^{i\theta})| \leq \pi/2.
\]
(2.7)

Since \( v(re^{i\theta}) \) is real harmonic in \( U \), we have
\[
|v(re^{i\theta})| < \pi/2.
\]
(2.8)

Define
\[
H(z) = e^{\varphi(z)},
\]
(2.8)

where \( \varphi(z) \) is the analytic function in \( U \) and satisfies
\[
\text{Im } \varphi(z) = v(z)
\]
(2.9)
and \( \varphi(0) = iv(0) \). Let \( u(z) = \text{Re} \varphi(z) \). For \( p \geq 2 \), we get from Lemma 2 the following inequality:

\[
\left( \frac{1}{2\pi} \int_0^{2\pi} |u(z)|^p \, d\theta \right)^{1/p} \leq 2 \left( \frac{p}{p-1} \right)^{1/p} \left( \frac{1}{2\pi} \int_0^{2\pi} |v(z)|^p \, d\theta \right)^{1/p} \\
\leq 2^{1+1/p} \left( \frac{1}{2\pi} \int_0^{2\pi} |v(z)|^p \, d\theta \right)^{1/p}.
\]

By (2.7), we have

\[
\int_0^{2\pi} |u(z)|^p \, d\theta \leq 4\pi^{p+1}.
\]

For \( p > 1 \), using Hölder’s inequality, we deduce

\[
\int_0^{2\pi} |H(z)|^p \, d\theta = \int_0^{2\pi} e^{pu(z)} \, d\theta \\
\leq \int_0^{2\pi} p|u(z)| \, d\theta + \sum_{n=2}^{\infty} \frac{1}{n!} \int_0^{2\pi} |pu(z)|^n \, d\theta + 2\pi \\
\leq \int_0^{2\pi} p|u(z)| \, d\theta + \sum_{n=1}^{\infty} \frac{4p^n \pi^{n+1}}{n!} \\
< 4p \pi^3 + 4\pi e^{p\pi}.
\]

From Fatou’s lemma, we get

\[
H(z) \in H^p \quad (p > 1)
\]

and

\[
H(z) \left[ ie^{i\theta_0} (1 - z^2) \left[ e^{-i\mu} h'(e^{i\theta_0} z) + e^{i\mu} g'(e^{i\theta_0} z) \right] \right] \in H^p \quad (p > 1).
\]

Finally set

\[
v_1(z) = \text{Re} \left\{ H(z) ie^{i\theta_0} (1 - z^2) \left[ e^{-i\mu} h'(e^{i\theta_0} z) + e^{i\mu} g'(e^{i\theta_0} z) \right] \right\},
\]

then

\[
v_1(z) \in h^p \quad (p > 1).
\]

Hence applying Lemma 3, we can write \( v_1(z) \) as

\[
v_1(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2) f_2(t)}{1 + r^2 - 2r \cos(\theta - t)} \, dt \quad (z = re^{i\theta}) \quad (2.10)
\]

and

\[
f_2(\theta) = v_1(e^{i\theta}) \quad \text{for a.e. } \theta \in [0, 2\pi]. \quad (2.11)
\]
By virtue of (2.1)–(2.9), we find that for a.e. $\theta \in [0, 2\pi]$,

$$\arg H(e^{i\theta}) = Q(\theta)$$

and

$$H(e^{i\theta})l_3(\theta) = H(e^{i\theta})\frac{l_1(\theta)}{l_3(\theta)}l_4(\theta)$$

$$= H(e^{i\theta})\frac{l_1(\theta + \theta_0)}{l_2(\theta + \theta_0)}l_4(\theta)$$

$$= H(e^{i\theta})R(\theta + \theta_0)l_4(\theta).$$

Therefore

$$-\frac{\pi}{2} \leq \arg [H(e^{i\theta})l_3(\theta)] = Q(\theta) + \arg l_4(\theta)$$

$$+ \arg R(\theta + \theta_0) \leq \frac{\pi}{2}$$  \hspace{1cm} (2.12)

and

$$v_1(e^{i\theta}) = \Re \left[ H(e^{i\theta})i(1 - e^{2i\theta})e^{i(\theta_0 - \mu)}h'(e^{i(\theta_0 + \theta)}) ight.$$  

$$+ H(e^{i\theta})i(1 - e^{2i\theta})e^{i(\theta_0 + \mu)}g'(e^{i(\theta_0 + \theta)}) \bigg]$$

$$= \Re \left[ H(e^{i\theta})i(1 - e^{2i\theta})e^{i(\theta_0 - \mu)}h'(e^{i(\theta_0 + \theta)}) ight.$$  

$$+ \frac{H(e^{i\theta})i(1 - e^{2i\theta})e^{i(\theta_0 + \mu)}g'(e^{i(\theta_0 + \theta)})}{\theta_0} \bigg]$$

$$= \Re \left\{ H(e^{i\theta})e^{-i\mu}[i(1 - e^{2i\theta})e^{i\theta_0}h'(e^{i(\theta_0 + \theta)}) ight.$$  

$$- (e^{i\theta} - e^{-i\theta})ie^{i(\theta_0 + \mu)}g'(e^{i(\theta_0 + \theta)})] \bigg\}$$

$$= \Re \left\{ H(e^{i\theta})e^{-i(\mu + \theta)}[i(1 - e^{2i\theta})e^{i(\theta_0 + \theta)}h'(e^{i(\theta_0 + \theta)}) ight.$$  

$$- (e^{2i\theta} - 1)ie^{i(\theta_0 + \mu)}g'(e^{i(\theta_0 + \theta)})] \bigg\}$$

$$= \Re \left\{ H(e^{i\theta})e^{-i(\mu + \theta)}[i(1 - e^{2i\theta})e^{i(\theta_0 + \theta)}h'(e^{i(\theta_0 + \theta)}) ight.$$  

$$+ ie^{i(\theta + \theta_0)}g'(e^{i(\theta_0 + \theta)})] \bigg\}$$

$$= \Re \left\{ [H(e^{i\theta})l_3(\theta)] \right\}$$

$$= \Re \left\{ H(e^{i\theta})l_4(\theta)R(\theta + \theta_0) \right\} > 0 \hspace{1cm} \text{for a.e. } \theta \in [0, 2\pi].$$  \hspace{1cm} (2.13)

By using (2.10)–(2.13), we have $v_1(z) \geq 0$. 
Applying an argument similar to that in the proof of (2.7), we can prove
\[ v_1(z) > 0 \quad \text{for } z \in U. \]

Consequently,
\[ H(z) \{ i e^{i \theta_0} (1 - z^2) [ e^{-i \mu} h'(e^{i \theta_0} z) + e^{i \mu} g'(e^{i \theta_0} z) ] \} \in P. \] (2.14)

From (2.5)–(2.9), we get \( H(z) \in P \). Hence the assertion of the lemma holds.

3. PROOFS ON CONJECTURES I AND II FOR \( f \in C^0_H \) AND \( f \in \widetilde{C}_H \)

For conjecture I, we prove the following theorem:

**Theorem 1.** If \( f(z) = z + \sum_{n=2}^\infty a_n z^n + \sum_{n=2}^\infty a_{-n} z^{-n} \in C^0_H \), then
\[ ||a_n| - |a_{-n}| || \leq n \quad (n = 2, 3, \ldots) \] (3.1)
\[ |a_{-n}| \leq \frac{(n - 1)(2n - 1)}{6} \quad (n = 2, 3, \ldots) \] (3.2)

and
\[ |a_n| \leq \frac{(n + 1)(2n + 1)}{6} \quad (n = 2, 3, \ldots). \] (3.3)

**Proof.** From (2.14), we have
\[ p_1(z) = H(z) \{ i e^{i \theta_0} (1 - z^2) [ e^{-i \mu} h'(e^{i \theta_0} z) + e^{i \mu} g'(e^{i \theta_0} z) ] \} \in P. \]
Let \( p_2(z) = 1/H(z) \). We therefore have
\[ i e^{i \theta_0} [ e^{-i \mu} h'(e^{i \theta_0} z) + e^{i \mu} g'(e^{i \theta_0} z) ] = \frac{p_1(z) p_2(z)}{1 - z^2}. \] (3.4)

Where \( |p_1(0)| p_2(0) = 1 \), \( p_1(z) p_2(z) = p_1(z)/|p_1(0)| \cdot p_2(z)/|p_2(0)| \).
Since \( (1 + z) / (1 - z) \) and \( 1 / (1 - z^2) \) have positive coefficients, by (3.4) we obtain
\[ e^{-i \mu} h'(e^{i \theta_0} z) + e^{i \mu} g'(e^{i \theta_0} z) \ll \left( \frac{1 + z}{1 - z} \right)^2 \frac{1}{1 - z^2} = \frac{1 + z}{(1 - z)^3}, \]
where the symbol \( \ll \) means the moduli of the coefficients of the function on the left are bounded by the corresponding coefficients of the function on the right. Hence,
\[ |e^{-i \mu} a_n + e^{i \mu} a_{-n}| \leq n \quad \text{for } n = 1, 2, \ldots. \]
The result of (3.1) holds.
On the other hand, we have

\[ g'(z) = w(z)h'(z) \quad (z \in U), \]

where \(|w(z)| \leq |z|\). Again by (3.4), we get

\[ h'(e^{i\theta_0}z) = -\frac{ie^{i(\mu-\theta_0)}p_1(z)p_2(z)}{(1 - z^2)(1 + e^{2i\mu}w(e^{i\theta_0}z))} \ll \frac{1 + z}{(1 - z)^3} \frac{1}{1 - z} = \frac{1 + z}{(1 - z)^4} \]

and

\[ g'(e^{i\theta_0}z) = -\frac{ie^{i(\mu-\theta_0)}p_1(z)p_2(z)w(e^{i\theta_0}z)}{(1 - z^2)(1 + e^{2i\mu}w(e^{i\theta_0}z))} \ll \frac{1 + z}{(1 - z)^3} \frac{z}{1 - z} = \frac{z(1 + z)}{(1 - z)^4}. \]

These inequalities give (3.2) and (3.3).

**Remark.** The estimates (3.1)–(3.3) are sharp, for we may choose a function

\[ k_0(z) = \text{Re} \left( \frac{z + (1/3)z^3}{(1 - z)^3} \right) + i \text{Im} \left( \frac{z}{(1 - z)^3} \right) \in C_H^0, \]

which is extremal (see [1]).

For conjecture II, we improve Clunie and Sheil-Small’s result and prove the following theorem:

**Theorem 2.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} a_{-n} z^n \in C_H, \) then

\[ ||a_n|| - |a_{-n}| \leq (1 + |a_{-1}|)n \quad (n = 2, 3, \ldots) \tag{3.5} \]

\[ |a_n| \leq \frac{(n+1)(2n+1)}{6} + \frac{|a_{-1}|}{6}(n - 1)(2n - 1) < \frac{2n^2 + 1}{3} \quad (n = 1, 2, 3, \ldots) \tag{3.6} \]

\[ |a_{-n}| \leq \frac{(n-1)(2n-1)}{6} + \frac{|a_{-1}|}{6}(n + 1)(2n + 1) < \frac{2n^2 + 1}{3} \quad (n = 1, 2, 3, \ldots). \tag{3.7} \]

**Proof.** As in the proof of Theorem 1, by (2.14) we have a similar result,

\[ \frac{e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)}{e^{-i\mu} + a_{-1}e^{i\mu}} \ll \left( \frac{1 + z}{1 - z} \right)^2 \frac{1}{1 - z^2} = \frac{1 + z}{(1 - z)^3}. \]

This yields

\[ ||a_n|| - |a_{-n}| < (1 + |a_{-1}|)n \quad (n = 2, 3, \ldots). \]
Since the class $C_H$ is affine and linear invariant, for any $f \in C_H$, we conclude

$$f_0 = \frac{f - a_{-1} \hat{f}}{1 - |a_{-1}|^2} \in C_H^0.$$ 

Therefore we can write $f = f_0 + a_{-1} \hat{f}_0$. Now (3.6) and (3.7) follow from (3.2) and (3.3), respectively. In particular, we have

$$|a_n| - |a_{-n}| < 2n \quad (n = 1, 2, \ldots)$$

and

$$|a_n| < \frac{2n^2 + 1}{3} \quad (|n| \geq 2).$$

This completes the proof.

**Remark.** Obviously, Theorem 2 implies Theorem 1, so that we may rewrite the generalization of the Bieberbach conjecture as follows:

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} a_{-n} z^{-n} \in S_H$, then

$$|a_n| - |a_{-n}| \leq (1 + |a_{-1}|)n \quad (n = 1, 2, \ldots)$$

$$|a_n| \leq \frac{(n + 1)(2n + 1)}{6} + \frac{|a_{-1}|}{6} (n - 1)(2n - 1) \quad (n = 2, 3, \ldots)$$

$$|a_{-n}| \leq \frac{(n - 1)(2n - 1)}{6} + \frac{|a_{-1}|}{6} (n + 1)(2n + 1) \quad (n = 2, 3, \ldots).$$

In particular,

$$|a_n| - |a_{-n}| < 2n \quad (n = 2, 3, \ldots)$$

and

$$|a_n| < \frac{2n^2 + 1}{3} \quad (|n| \geq 2).$$

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