Generic formal fibers of polynomial rings

S. Loepp\textsuperscript{a,*}, Aaron Weinberg\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Williams College, Williamstown, MA 01267, USA
\textsuperscript{b}Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

Received 5 November 1999; received in revised form 4 May 2000
Communicated by C.A. Weibel

Abstract

We construct a noncomplete excellent regular local ring $A$ with maximal ideal $M$ such that the generic formal fiber ring, $\hat{A} \otimes_A K$, (where $\hat{A}$ is the $M$-adic completion of $A$ and $K$ is the quotient field of $A$) is local. In addition, given a small set $L$ of prime ideals of $\hat{A}[[X_1,\ldots,X_n]]$ (where $X_1,\ldots,X_n$ are indeterminates) satisfying some necessary conditions, every element of $L$ is in the generic formal fiber of $A[X_1,\ldots,X_n]/(M,X_1,\ldots,X_n)$. In other words, $Q \cap A[X_1,\ldots,X_n] = (0)$ for every $Q \in L$. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 13J10; 13B25; 13H10

1. Introduction

In this paper, we examine the relationship between a local Noetherian domain $R$ and the ring $R[X_1,\ldots,X_n]/(M_R,X_1,\ldots,X_n)$ (where $M_R$ denotes the maximal ideal of $R$ and $X_1,\ldots,X_n$ are indeterminates) by focusing on the generic formal fibers of these rings. Recall that if $R$ is a local Noetherian domain with maximal ideal $M$ and $M$-adic completion $\hat{R}$, then the generic formal fiber of $R$ is defined to be $\text{Spec}(\hat{R} \otimes_R K)$ where $K$ denotes the quotient field of $R$. Note that the generic formal fiber of $R$ can also be thought of as the inverse image of $(0)$ under the morphism $\text{Spec} \hat{R} \rightarrow \text{Spec} R$ (see [5, p. 47]). In this setting, we will use the notation $\zeta(R)$ to denote the dimension of the ring $(\hat{R} \otimes_R K)$.

In [4], Loepp shows that if $T$ is a complete Noetherian regular local ring of dimension at least two with maximal ideal $M$, $T$ contains the rationals, the cardinality of $T/M$...
is at least the cardinality of the real numbers and $p$ is a nonmaximal prime ideal of $T$, then there exists an excellent regular local ring $A$ such that the completion of $A$ is $T$, and the ring $\hat{A} \otimes_A K$ is local with maximal ideal $p \otimes_A K$ (where $K$ denotes the quotient field of $A$). Note that $\alpha(A)$ in this case is the equal to the height of $p$.

In this paper, we extend this result by not only ensuring $A$ satisfies the above conditions, but also given a “small” set of prime ideals $L$ of $T[[X_1,\ldots,X_n]]$ satisfying certain conditions, that all elements of $L$ are in the generic formal fiber of $A[X_1,\ldots,X_n]_{(M \cap A[X_1,\ldots,X_n])}$. In other words, $Q \cap A[X_1,\ldots,X_n]_{(M \cap A[X_1,\ldots,X_n])} = (0)$ for every $Q$ in $L$. So, we can simultaneously have some control over both the generic formal fiber of $A$ and the generic formal fiber of $A[X_1,\ldots,X_n]_{(M \cap A[X_1,\ldots,X_n])}$.

In particular, the main theorem of the paper is: Let $(T,M)$ be a Noetherian complete regular local ring of dimension at least two containing the rationals, such that the cardinality of the residue field $T/M$ is at least the cardinality of the real numbers. Suppose $p$ is a nonmaximal prime ideal of $T$ and $L$ is a set of prime ideals of $T[[X_1,\ldots,X_n]]$ (where $X_1,\ldots,X_n$ are indeterminates) such that the cardinality of $L$ is strictly less than the cardinality of $T/M$, $Q \cap T \subset p$ for each $Q \in L$, and if $\Pi$ is the prime subring of $T$, then $\Pi[X_1,\ldots,X_n] \cap Q = (0)$ for each $Q \in L$. Then there exists an excellent regular local ring $A$ such that the completion of $A$ is $T$, the generic formal fiber of $A$ is local with maximal ideal $p$ (this means that the ring $T \otimes_A K$ is a local ring with $p \otimes_A K$ its maximal ideal where $K$ is the quotient field of $A$), and $Q \cap A[X_1,\ldots,X_n] = (0)$ for each $Q \in L$. So, given a nonmaximal ideal $p$ of $T$ and a set $L$ of prime ideals of $T[[X_1,\ldots,X_n]]$, we can construct a ring $A$ simultaneously ensuring that $p$ is in the generic formal fiber of $A$ and each element of $L$ is in the generic formal fiber of $A[X_1,\ldots,X_n]_{(M \cap A[X_1,\ldots,X_n])}$. The construction of the ring $A$ is based on Loepp’s construction in [4]. We note here that the conditions $Q \cap T \subset p$ for each $Q \in L$, and if $\Pi$ is the prime subring of $T$, then $\Pi[X_1,\ldots,X_n] \cap Q = (0)$ for each $Q \in L$ are necessary. For if $Q \cap T \not\subset p$ and $p$ is maximal in the generic formal fiber of $A$, then $(Q \cap T) \cap A \neq (0)$ and so $Q \cap A[X_1,\ldots,X_n] \neq (0)$. Clearly, $\Pi[X_1,\ldots,X_n] \cap Q = (0)$ is necessary since $A$ must contain $\Pi$.

It is interesting to note that using this theorem and a result from Matsumura in [6], we can show that $\alpha(A)$ can be “small”, while

$$\alpha(A[X_1,\ldots,X_n]_{(M \cap A[X_1,\ldots,X_n])})$$

is “large”. To see this, let $T = \mathbb{C}[[Y_1,\ldots,Y_r]]$ with $r \geq 2$ and $Y_1,\ldots,Y_r$ indeterminates. In example 2 of [6], Matsumura shows that

$$\alpha(T[X_1,\ldots,X_n]_{(Y_1,\ldots,Y_r,X_1,\ldots,X_n)})$$

with $n \geq 1$ is equal to $n + r - 2$. In other words, there exists a prime ideal $Q$ of $T[[Y_1,\ldots,Y_r,X_1,\ldots,X_n]]$ with $\text{ht}Q = n + r - 2$ and

$$T[X_1,\ldots,X_n]_{(Y_1,\ldots,Y_r,X_1,\ldots,X_n)} \cap Q = (0).$$
Now, apply our main theorem with \( p = (Y_1) \) and \( L = \{Q\} \) to obtain an excellent regular local ring \( A \) such that \( \kappa(A) = 1 \) and

\[
\kappa(A[X_1, \ldots, X_n]_{(M \cap A[X_1, \ldots, X_n])}) \geq n + r - 2.
\]

So, \( \kappa(A) \) is “small” and \( \kappa(A[X_1, \ldots, X_n]_{(M \cap A[X_1, \ldots, X_n])}) \) is “large”.

In this paper, all rings will be commutative with unity. When we say that a ring \( R \) is local, we mean that \( R \) is Noetherian with exactly one maximal ideal. We will use the term quasi-local to denote a ring that has exactly one maximal ideal and is not necessarily Noetherian. When we write \((T, M)\) is a quasi-local ring, we mean that \( T \) is quasi-local with maximal ideal \( M \). We will use \( c \) to denote the cardinality of the real numbers and RLR to denote a regular local ring. Also, \( M_R \) will be used to denote the maximal ideal of a local ring \( R \). We note here that in this paper we will use \( \hat{X} \) to denote the set of indeterminates \( \{X_1, \ldots, X_n\} \).

2. The construction

The following proposition is Proposition 1 from [3] and its proof can be found there.

**Proposition 1.** If \((R, M \cap R)\) is a quasi-local subring of a complete local ring \((T, M)\), the map \( R \rightarrow T/M^2 \) is onto and \( IT \cap R = I \) for every finitely generated ideal \( I \) of \( R \), then \( R \) is Noetherian and the natural homomorphism \( \hat{R} \rightarrow T \) is an isomorphism.

Our overall goal is to construct an ascending chain of rings, each having specific nice properties. That is, we will construct the rings so that their union will satisfy the conditions of Proposition 1 (ensuring it has the desired completion), have a generic formal fiber that is local with a chosen maximal ideal, and will be excellent.

We begin with a technical lemma. Lemma 2 will allow us to adjoin transcendental elements to rings (by “avoiding” algebraic cosets) which will be useful in future lemmas.

**Lemma 2.** Suppose \((T, M)\) is a local ring with \(|T/M|\) infinite. Let \( C_1 \subset \text{Spec } T \) and \( C_2 \subset \text{Spec } T[[\hat{X}]] \). Suppose \( D_1 \) is a subset of \( T \) and \( D_2 \) is a subset of \( T[[\hat{X}]] \). Let \( w \in T \) such that \( w \not\in P \) for every \( P \in C_1 \) and \( w \not\in Q \) for every \( Q \in C_2 \). If \(|C_1 \times D_1| < |T/M| \) and \(|C_2 \times D_2| < |T/M| \) then we can find a unit \( y \in T \) such that \( wy \not\in \{P + r \mid P \in C_1, \ r \in D_1\} \) and \( wy \not\in \{Q + a \mid Q \in C_2, \ a \in D_2\} \).

**Proof.** First, we define a map \( f_1 : C_1 \times D_1 \rightarrow T \). Suppose \((P, r) \in C_1 \times D_1 \). If \( r + P \not\in (w + P)(T/P) \) then define \( f_1((P, r)) = 0 \). If \( r + P \in (w + P)(T/P) \), then pick an element \( s_1 \in T \) such that \( r + P = (w + P)(s_1 + P) \) and define \( f_1((P, r)) = s_1 \). Let \( S_1 = \text{Image } f_1 \cup \{0\} \). Now, since \(|C_1 \times D_1| < |T/M| \), we have \(|S_1| < |T/M| \). So, we can choose \( t + M \in T/M \) such that \( t + M \neq s_1 + M \) for every \( s_1 \in S_1 \). In fact, since \( T/M \) is infinite, we have \(|T/M|\) choices for such a coset \( t + M \). We define

\[
Z = \{t + M \in T/M \mid t + M \neq s_1 + M \text{ for every } s_1 \in S_1\}.
\]
Note that $|Z|=|T/M|$. Let $\Gamma$ be a set of coset representatives for the elements in $Z$. Then, $|\Gamma|=|T/M|$. Note that since $0\in S_1$, all elements in $\Gamma$ are units in $T$.

Now, define a map $f_2:C_2 \times D_2 \to T[[\hat{X}]]$ as follows. Suppose $(Q,a) \in C_2 \times D_2$. If $a+Q \notin (w+Q)(T[[\hat{X}]]/Q)$, then define $f_2((Q,a))=0$. If $a+Q \in (w+Q)(T[[\hat{X}]]/Q)$, then choose $s_2 \in T[[\hat{X}]]$ such that $a+Q=(w+Q)(s_2+Q)$. Define $S_2=\text{Image } f_2$. Now, let $m$ denote the maximal ideal of $T[[\hat{X}]]$ and define

$$W=\{t+m \in T[[\hat{X}]]/m \mid t \in \Gamma\}.$$

Since $t,t' \in T$ with $t+M \neq t'+M$ implies that $t+m \neq t'+m$, we have $|W|=|\Gamma|=|T/M|$. Note that $|S_2| \leq |C_2 \times D_2| < |T/M|=|W|$, so there exists $y+m \in W$ such that $y+m \neq s_2+m$ for every $s_2 \in S_2$. Observe that since $y+m \in W$, we may assume $y \in \Gamma$.

Hence, $y+M \neq s_1+M$ for every $s_1 \in S_1$.

Now, we claim $wy$ is the desired element. Let $P \in C_1$ and $r \in D_1$. If $r+P \notin (w+P)T/P$, then $r+P \neq (w+P)(y+P)$, so $wy \notin r+P$. On the other hand, if $r+P \in (w+P)T/P$, then $r+P=(w+P)(s_1+P)$ for some $s_1 \in S_1$. So, if $wy \in r+P$ then $wy \in ws_1+P$ and so $w(y-s_1) \in P$. Now, since $P$ is prime and $w \notin P$, we have $y-s_1 \in P \subset M$ and so $y+M=s_1+M$. But, this contradicts the way we chose $y$.

Hence, $wy \notin P+r$ and it follows that

$$wy \notin \bigcup\{P+r \mid P \in C_1, \ r \in D_1\}.$$

Now, let $Q \in C_2$ and $a \in D_2$. If $a+Q \notin (w+Q)(T[[\hat{X}]]/Q)$, then $a+Q \neq (w+Q)(y+Q)$, so $wy \notin a+Q$. On the other hand, if $a+Q \in (w+Q)T[[\hat{X}]]/Q$, then $a+Q=(w+Q)(s_2+Q)$ for some $s_2 \in S_2$. Now, if $wy \in a+Q$, then $wy \in ws_2+Q$, so $w(y-s_2) \in Q$. As $w \notin Q$, we have $y-s_2 \in Q \subset m$ and it follows that $y+m=s_2+m$.

But, this contradicts the way we chose $y$. Hence, $wy \notin a+Q$ and it follows that

$$wy \notin \bigcup\{Q+a \mid Q \in C_2, \ a \in D_2\}. \quad \Box$$

Lemma 3 is a technical lemma used in Lemma 5 and Theorem 13. It is essentially an extension of the Prime Avoidance Theorem. Its proof can be found in [2].

**Lemma 3.** Let $(T,M)$ be a local ring. Let $C \subset \text{Spec } T$, let $I$ be an ideal such that $I \not\subset P$ for every $P \in C$ and let $D$ be a subset of $T$. Suppose $|C \times D| < |T/M|$. Then $I \not\subset \bigcup\{P+r \mid P \in C, \ r \in D\}$.

The definition of an $LQ$ avoiding $p$-subring will allow us to place several necessary conditions on the rings used in the construction.

**Definition 4.** Let $(T,M)$ be a complete local ring and $(R,R \cap M)$ a quasi-local subring of $T$. Let $p \neq M$ be a prime ideal of $T$ and $L$ a set of nonmaximal prime ideals of $T[[\hat{X}]]$. Suppose

1. $|R| \leq \sup (\aleph_0, |T/M|)$ with equality implying $T/M$ is countable;
2. $R \cap p = (0)$;
3. \( R \cap P = (0) \) for every \( P \in \text{Ass } T; \)
4. for every \( Q \subseteq L, R[\bar{X}] \cap Q = (0). \)

Then we call \( R \) an \( LQ \) avoiding \( p \)-subring of \( T. \)

With this definition, our goal becomes more distinct. We will try to find an ascending chain of \( LQ \) avoiding \( p \)-subrings of \( T \) whose union, which we will denote by \( A \), is excellent and which satisfy the conditions of Proposition 1. The conditions of the \( LQ \) avoiding \( p \)-subring will ensure, among other things, that \( p \) is in the generic formal fiber of \( A \) and that the elements of \( L \) are in the generic formal fiber of \( A[\bar{X}]_{(M, \bar{X})}. \)

We will use Lemma 5 to eventually find an \( LQ \) avoiding \( p \)-subring \( S \) of \( T \) which will satisfy \( IT \cap S = I \) for every finitely generated ideal of \( S. \) This, of course, is needed to satisfy the conditions of Proposition 1.

**Lemma 5.** Let \((T, M)\) be a complete local ring with \(|T/M| \geq c\) and \( M \notin \text{Ass } T. \) Let \( p \neq M \) be a prime ideal of \( T, \) \( L \) a set of prime ideals of \( T[\bar{X}] \) with \(|L| < |T/M|, \) \( Q \cap T \subseteq p \) for each \( Q \subseteq L \) and \( R \) an infinite \( LQ \) avoiding \( p \)-subring. Suppose \( I \) is a finitely generated ideal of \( R \) and \( b \in IT \cap R. \) Then there exists an \( LQ \) avoiding \( p \)-subring \( S \) of \( T \) with \( R \subseteq S \subseteq T \) and \( b \in IS. \)

**Proof.** Let \( m \) be the number of generators of \( I. \) We will induct on \( m. \)

Let \( m = 1. \) Thus \( I = aR \) for some \( a \in R. \) If \( a = 0 \) then \( S = R \) is the desired subring. Assume \( a \neq 0. \) We have \( b \in IT \cap R = aT \cap R \) so \( b = au \) for some \( u \in T. \) We will show that \( S = R[u]_{R[u] \cap M} \) is the desired subring.

Let \( P \in \{ p \} \cup \text{Ass } T \) and suppose \( f = r_n u^n + \cdots + r_1 u + r_0 \in R[u] \cap P. \) Then

\[
a^nf = r_n (au)^n + \cdots + r_1 a^{n-1}(au) + r_0 a^n
= r_n b^n + \cdots + r_1 a^{n-1}b + r_0 a^n \in P \cap R = (0).
\]

As \( a \in R \) and \( R \) is an \( LQ \) avoiding \( p \)-subring, we know that \( a \) is not a zero-divisor in \( T \) so \( f = 0. \) Now suppose \( g \in R[u][\bar{X}] \cap Q \) where \( Q \subseteq L. \) Then we can find a positive integer \( s \) such that \( a^s g \in R[\bar{X}] \cap Q = (0) \) (as \( au = b \in R. \)) As \( a \) is not a zero divisor in \( T, \) it is not a zero divisor in \( T[\bar{X}] \) and thus \( g = 0. \) Hence, \( R[u, X] \cap Q = (0) \) and so \( S[\bar{X}] \cap Q = (0). \) Hence, \( S[\bar{X}] \cap Q = (0) \) for every \( Q \subseteq L \) and the lemma is true for \( m = 1. \)

Now suppose \( m > 1 \) and assume the lemma holds for \( m - 1. \) Then \( I = (y_1, \ldots, y_m)R. \) Thus, for some \( t_1, \ldots, t_m \in T \) and any \( t \in T \) we have

\[
b = y_1 t_1 + y_2 t_2 + y_3 t_3 + \cdots + y_m t_m
= y_1 t_1 + y_1 y_2 t - y_1 y_2 t + y_2 t_2 + y_3 t_3 + \cdots + y_m t_m
= y_1 (t_1 + y_2 t) + y_2 (t_2 - y_1 t) + y_3 t_3 + \cdots + y_m t_m.
\]

Let \( x_1 = t_1 + y_2 t \) and \( x_2 = t_2 - y_1 t \) where we will choose \( t \) later.

Let \( C_1 = \text{Ass } T \cup \{ p \} \) and suppose \( P \in C_1. \) Note that by hypothesis, \( P \cap R = (0) \) for all \( P \in C_1. \) As \( y_2 \in R \) and \( P \cap R = (0) \) we have \( y_2 \notin P. \) For some \( t, t' \in T \) with \( t + P \neq t' + P, \) assume \( (t_1 + y_2 t) + P = (t_1 + y_2 t') + P. \) Then \( (t_1 + y_2 t) - (t_1 + y_2 t') \in P \)
and so $y_2t - y_2t' \in P$, which implies that $y_2(t - t') \in P$. Since $P$ is prime and $y_2 \not\in P$, we must have $t - t' \in P$. This implies that $t + P = t' + P$, which is a contradiction. Thus $(t_1 + y_2t) + P \neq (t_1 + y_2t') + P$. Thus different choices of $t$ modulo $P$ yield different choices of $x_1$ modulo $P$. (Likewise, if $Q \in L$, $y_2 \not\in Q$, so different choices of $t$ modulo $Q$ in $T[[\hat{X}]]$ yield different choices of $x_1$ modulo $Q$.)

Let $D_{(P)}$ be a full set of coset representatives for those choices of $t$ that make $x_1$ algebraic over $R$ as an element of $T/P$. (That is, there exist $u_0, u_1, \ldots, u_m$ all elements of $R$, such that $u_m(x_1 + P)^n + \cdots + u_0(x_1 + P) = (0 + P)$.) Note that since $R$ is infinite, $|D_{(P)}| \leq |R|$. Define $D_t = \bigcup_{P \in C_t} D_{(P)}$. Let $D_2$ be a full set of coset representatives for those choices of $t$ that make $x_1 + Q$ algebraic over $R[[\hat{X}]]$ as an element of $T[[\hat{X}]]/Q$ for every $Q \in L$. Then, since $R$ is infinite, $|D_2| \leq |R[\hat{X}]/|R|$. Then, since $R$ is infinite, $|D_2| \leq |R[\hat{X}]/|R|$

Note that for every $Q$ in $L$, $M \not\subset Q \cap T$ and for every $P \in C_1$, $M \not\subset P$. Let

$$C = \{Q \cap T \mid Q \in L\} \cup \{P \mid P \in C_1\}.$$  

Then, $|C| < |T/M|$. Now, using Lemma 3 with $D = \{0\}$, we have $M \not\subset \bigcup\{P \mid P \in C\}$. So there exists $w \in M$ with $w \not\in P$ for every $P \in C_1$ and $w \not\in Q$ for every $Q \in L$. Now, use Lemma 2 (with $C_2 = L$) to find a $v \in T$ such that

$$w \not\in \bigcup\{P + r \mid P \in C_1, r \in D_1\} \quad \text{and} \quad w \not\in \bigcup\{Q + a \mid Q \in C_2, a \in D_2\}.$$  

Letting $t = vw$, we have that $x_1 + P$ is transcendental over $R$ as an element of $T/P$ for every $P \in C_1$ and $x_1 + Q$ is transcendental over $R[\hat{X}]$ as an element of $T[[\hat{X}]]/Q$ for every $Q \in L$

Now, we claim that $R[x_1] \cap P = (0)$ for every $P \in C_1$. To see this, suppose $f = r_n x_1^n + \cdots + r_1 x_1 + r_0 \in R[x_1] \cap P$ for some $P \in C_1$. Then, since $x_1 + P$ is transcendental over $R$ as an element of $T/P$, we have that $r_i \in P$ for $i = 0, 1, \ldots, n$. But then, $r_i \in P \cap R = (0)$ and so $f = 0$. Let $Q \in L$ and suppose $g \in R[x_1, \hat{X}] \cap Q$. Then, $g = f_m(\hat{X})x_1^m + \cdots + f_1(\hat{X})x_1 + f_0(\hat{X}) \in Q$ where $f_i(\hat{X}) \in R[\hat{X}]$ for every $i = 1, 2, \ldots, n$. Since $x_1 + Q$ is transcendental over $R[\hat{X}]$, we have $f_i(\hat{X}) \in R[\hat{X}] \cap Q = (0)$ for every $i = 1, 2, \ldots, n$. It follows that $g = 0$. Now, let $R' = R[x_1]/\{0\}$, $R[x_1]$ is an $LQ$ avoiding $p$-subring. Let $J = (y_2, \ldots, y_m)R'$ and $b^* = b - y_1 x_1$. Now, $b \in R \subset R'$ and $x_1 y_1 \in R'$, so $b^* \in R'$. Also, $b^* \in JT \cap R'$, so by our induction assumption, there exists an $LQ$ avoiding $p$-subring $S$ such that $R' \subset S \subset T$, and $b^* \in JS$. So, $b^* = y_2 s_2 + \cdots + y_m s_m$ for $s_i \in S$. Hence, $b = y_1 x_1 + y_2 s_2 + \cdots + y_m s_m \in (y_1, \ldots, y_m)S$. So, $S$ is the desired $LQ$ avoiding $p$-subring.

Given an $LQ$ avoiding $p$-subring $R$ of $T$, Lemmas 6 and 8 will allow us to find an $LQ$ avoiding $p$-subring $S$ of $T$ which contains $R$ and also contains the “factors” of nonzero elements of $R$. This will eventually allow us to make our ring excellent.

**Lemma 6.** Let $(T, M)$ be a complete local ring with $M \not\subset \text{Ass} T$ and $|T/M| \geq c$. Let $p \not\subset M$ be a prime ideal of $T$ and $L$ be a set of prime ideals of $T[[\hat{X}]]$ such that, for every $Q \in L$, $Q \cap T \subset p$ and $|L| < |T/M|$. Suppose $(R, R \cap M)$ is an infinite $LQ$ avoiding $p$-subring and $0 \neq x \in R$ is such that $x = yz$ in $T$ with $y \not\in xT$ and $z \not\in xT$.  

Then there exists an $LQ$ avoiding $p$-subring $S$ of $T$ with $R \subset S \subset T$ and a unit $u \in T$ such that $yu, zu^{-1} \in S$.

**Proof.** As $R$ is an $LQ$ avoiding $p$-subring, its elements are not zero-divisors in $T$. Thus $x$ is not a zero divisor and hence $y$ is not a zero divisor. Note that if $y \in p$ then $yz = x \in p$ which cannot happen as $R \cap p = 0$. As $y \notin p$ we also have $y \notin Q$ for any $Q \in \mathcal{L}$ (otherwise $y \in T \cap Q \subset p$). So if $C_1 = \text{Ass} T \cup \{p\}$, we see $y \notin p$ for every $P \in C_1$ and $y \notin Q$ for every $Q \in \mathcal{L}$.

Let $P \in C_1$. Define $D(P)$ to be a full set of coset representatives for the cosets of $T/P$ that are algebraic over $R$ and define $D_1 = \bigcup_{P \in C_1} D(P)$. Note $|D_1| \leq |R| < |T/M|$.

Let $Q \in L$. Define $D(Q)$ to be a full set of coset representatives for the cosets of $T[[\widehat{X}]]/Q$ that are algebraic over $R[\widehat{X}]$ and define $D_2 = \bigcup_{Q \in L} D(Q)$. Then $|D_2| \leq |R[\widehat{X}]| = |R| < |T/M|$.

Note that $|C_1| < |T/M|$ and $|L| < |T/M|$. In addition $|D_1| < |T/M|$ and $|D_2| < |T/M|$. Thus $|C_1 \times D_1| < |T/M|$ and $|L \times D_2| < |T/M|$. Now, use Lemma 2 to find a unit $u \in T$ such that $yu \notin \bigcup\{P \cap R \mid P \in C_1, r \in D_1\}$ and $yu \notin \bigcup\{Q + a \mid Q \in L, a \in D_2\}$.

Thus $yu + P$ is transcendental over $R$ as an element of $T/P$ for every $P \in C_1$ and $yu + Q$ is transcendental over $R[\widehat{X}]$ as an element of $T[[\widehat{X}]]/Q$ for every $Q \in L$.

Define $S = R[yu, zu^{-1}] / \{R[yu, zu^{-1}] \cap M\}$. Let $f \in R[yu, zu^{-1}] \cap P$ for $P \in C_1$. So

$$ f = r_n(yu)^n + \ldots + r_1(yu) + r_0 + r_{-1}(zu^{-1}) + \ldots + r_{-m}(zu^{-1})^m, $$

$$ (yu)^m f = r_n(yu)^{m+n} + \ldots + r_1(yu)^{m+1} + r_0(yu)^m + r_{-1}(x)(yu)^{m-1} + \ldots + r_{-m}(x)^m \in R[yu] \cap P. $$

We know $yu$ is transcendental over $R$ as an element of $T/P$. As $x \notin P$ and $P$ is prime, we must have $r_i \in P \cap R = 0$ for $i = -m, \ldots, n$. Hence $f = 0$ and thus $S \cap P = 0$.

Now, let $g \in R[yu, zu^{-1}, \widehat{X}] \cap Q$ for some $Q \in L$. So

$$ g = g_n(yu)^n + \ldots + g_1(yu) + g_0 + g_{-1}(zu^{-1}) + \ldots + g_{-m}(zu^{-1})^m, $$

where $g_i \in R[\widehat{X}]$ for every $i = -m, \ldots, n$. As before, $(yu)^mg \in R[yu, \widehat{X}] \cap Q$. As $yu + Q$ is transcendental over $R[\widehat{X}]$, we have $g_x \in R[\widehat{X}] \cap Q = 0$ for $x = 0, 1, \ldots, n$ and $g_{\beta} \in R[\widehat{X}] \cap Q$ for $\beta = -1, \ldots, -m$. Note that as $x \in R \subset T$ and $x \notin Q \cap T \subset p$, we have $x \notin Q$. As $Q$ is prime, $g_{\beta} \in R[\widehat{X}] \cap Q = 0$. Thus $g = 0$. Hence $S[\widehat{X}] \cap Q = 0$ for each $Q \in L$ and $S$ is an $LQ$ avoiding $p$-subring. \(\square\)

**Definition 7.** Let $\Omega$ be a well-ordered set and $x \in \Omega$. We define $\gamma(x) = \sup\{\beta \in \Omega \mid \beta < x\}$.

**Lemma 8.** Let $(T,M)$ be a complete local ring with $M \notin \text{Ass} T$ and $|T/M| > c$. Let $p \neq M$ be a prime ideal of $T$ and $L$ a set of prime ideals of $T[[\widehat{X}]]$ such that, for each $Q \in L$, $Q \cap T \subset p$ and $|L| < |T/M|$. Suppose $R$ is an infinite $LQ$ avoiding $p$-subring. Then there exists an $LQ$ avoiding $p$-subring $S$ with $R \subset S \subset T$ such that for every nonzero $x \in R$ such that $x = yz \in T$ with $y \notin xT$ and $z \notin xT$, there exists a unit $t \in T$ with $yt, zt^{-1} \in S$. 
Proof. If $R$ contains no such element, then $S = R$ works, so assume $R$ contains such an $x$. Define $\Omega = \{x \in R \mid x = yz \text{ in } T \text{ with } y \notin xT \text{ and } z \notin xT\}$ \cup \{0\}.

Well-order $\Omega$ so that 0 is its initial element and so that every element of $\Omega$ has fewer than $|\Omega|$ predecessors. Define $R_0 = R$. We define an ascending chain of subrings recursively as follows:

Let $x \in \Omega$ and assume $R_\beta$ has been defined for all $\beta < x$.

If $\gamma(x) < x$, define $R_\gamma$ to be the $LQ$ avoiding $p$-subring obtained from Lemma 6 so that $\gamma(x) = yz$ with $y \notin \gamma(x)T$, $z \notin \gamma(x)T$ and $ytzt^{-1} \in R_\gamma$ for some unit $t \in T$.

If $\gamma(x) = x$ then $R_\gamma = \bigcup_{\beta < x} R_\beta$. We will show that in this case, $R_\gamma[\tilde{X}] \cap Q = (0)$ for each $Q \in L$. For some $Q \in L$ suppose $f \in R_\gamma[\tilde{X}] \cap Q$. Since $R_\gamma = \bigcup_{\beta < x} R_\beta$, there exists a $\delta < x$ such that $f \in R_\delta[\tilde{X}] \cap Q$. Since $R_\delta$ is an $LQ$ avoiding $p$-subring, we have $f = 0$.

Let $S = \bigcup_{x \in \Omega} R_x$. By transfinite induction, every $R_x$ is an $LQ$ avoiding $p$-subring. So $S$ is an $LQ$ avoiding $p$-subring.

Suppose $x \in R$ with $x = yz$, $y \notin xT$ and $z \notin xT$. Then $x = \gamma(x)$ for some $x$ with $\gamma(x) < x$. So $ytzt^{-1} \in R_x \subset S$ for some unit $t$ in $T$. \(\square\)

Lemma 9 will allow us to find an $LQ$ avoiding $p$-subring $S$ of $T$ which contains $R$ and also contains an element of a prime ideal $q$ of $T$, where $q \not\subset p$. This will eventually allow us to make $p$ the maximal ideal of the generic formal fiber, as every prime ideal $q \not\subset p$ will not intersect down to the zero ideal (and so will not be in the generic formal fiber).

Lemma 9. Let $(T, M)$ be a complete local ring with $|T/M| \geq c$, $p$ a nonmaximal prime ideal of $T$ containing the associated prime ideals of $T$ and $q$ a prime ideal of $T$ not contained in $p$. Let $L$ be a set of prime ideals of $T[[\tilde{X}]]$ with $|L| < |T/M|$ such that for each $Q \in L$, $Q \cap T \subset p$ and suppose $R$ is an $LQ$ avoiding $p$-subring of $T$. Then there exists an infinite $LQ$ avoiding $p$-subring $S$ of $T$ such that $R \subset S \subset T$ and $q \cap S \neq (0)$.

Proof. Let $C_1 = \text{Ass } T \cup \{p\}$. Suppose $P$ is a prime ideal of $T$ and define $D_{(P)}$ to be a full set of coset representatives of elements of $T/P$ that are algebraic over $R$. Let $D_1 = \bigcup_{P \in C_1} D_{(P)}$. Suppose $Q$ is an element of $L$ and define $D_{(Q)}$ to be a full set of coset representatives of elements of $T[[\tilde{X}]]/Q$ that are algebraic over $R[\tilde{X}]$. Let $D_2 = \bigcup_{Q \in L} D_{(Q)}$.

Note that $|T/M| \geq c$. If $|R| = |T/M|$ then by definition of an $LQ$ avoiding $p$-subring $|T/M|$ is countable. As $|T/M| \geq c$, this is a contradiction, so $|R| < |T/M|$. Now if $R$ is infinite, then $|D_2| \leq |R[\tilde{X}]| = |R| < |T/M|$ and if $R$ is finite then $|D_2| \leq |R[\tilde{X}]| = \aleph_0 < |T/M|$.

Note that for every $Q$ in $L$, $q \not\subset Q \cap T$ and for every $P \in C_1$, $q \not\subset P$. Let

$$C = \{Q \cap T \mid Q \in L\} \cup \{P \mid P \in C_1\}.$$ 

Then, $|C| < |T/M|$. Now, using Lemma 3 with $D = \{0\}$, we have $q \not\subset \bigcup\{P \mid P \in C\}$. 


Thus there is a $w \in q$ such that $w \not\in Q$ and $w \not\in P$ for every $P \in C_1$ and every $Q \in L$. Note that $|C_1| < |T/M|$ and $|L| < |T/M|$. In addition $|D_1| < |T/M|$ and $|D_2| < |T/M|$. Thus $|C_1 \times D_1| < |T/M|$ and $|L \times D_2| < |T/M|$. Now use Lemma 2 to find a unit $u \in T$ such that $wu \not\in \{P + r | P \in C_1, r \in D_1\}$ and $wu \not\in \{Q + a | Q \in L, a \in D_2\}$. Let $v = wu$ and note that $v \in q$.

Let $S = R[v, (R[v]) \cap M]$. Note that by the way we chose $v$, it is transcendental over $R$ and so $S$ is infinite. We see that $S$ is the desired $LQ$ avoiding $p$-subring. (Verifying the other conditions is routine.) □

Lemma 10 allows us to construct an $LQ$ avoiding $p$-subring $S$ of $T$ which, in addition to containing an element of $q$ where $q \in \text{Spec} T$ such that $q \not\subset p$, will contain a coset representative of an element of $T/M^2$. This will eventually allow us to make the map $A \to T/M^2$ surjective.

**Lemma 10.** Let $(T, M)$ be a complete local ring with $|T/M| \geq c$, $p$ a nonmaximal prime ideal of $T$ containing the associated prime ideals of $T$ and $q$ a prime ideal of $T$ not contained in $p$. Let $L$ be a set of prime ideals of $T[[X]]$ such that $|L| < |T/M|$ and for each $Q \in L$, $Q \cap T \subset p$ and suppose $R$ is an $LQ$ avoiding $p$-subring of $T$. Let $u \in T$. Then there exists an infinite $LQ$ avoiding $p$-subring $S$ of $T$ such that $R \subset S \subset T$, $S \cap q \neq (0)$ and there is a $b \in S$ with $u - b \in M^2$.

**Proof.** First use Lemma 9 to find an infinite $LQ$ avoiding $p$-subring $R'$ such that $R' \cap q \neq (0)$. Let $C_1 = \text{Ass} T \cup \{p\}$ and suppose $P \in C_1$. Define $D_i(p)$ to be a full set of coset representatives of elements $t + P \in T/P$ that make $(u + t) + P$ algebraic over $R$ as an element of $T/P$. Let $D_1 = \bigcup_{P \in C_1} D_i(p)$.

Suppose $Q \in L$. Define $D_i(Q)$ to be a full set of coset representatives of elements $a + Q \in T[[X]]/Q$ that make $(u + a) + Q$ algebraic over $R[[X]]$ as an element of $T[[X]]/Q$. Let $D_2 = \bigcup_{Q \in L} D_i(Q)$.

Note that for every $Q \in L$, $M^2 \not\subset Q \cap T$ and for every $P \in C_1$, $M^2 \not\subset P$. Let

$$C = \{Q \cap T | Q \in L\} \cup \{P | P \in C_1\}$$

Then, $|C| < |T/M|$. Now, using Lemma 3 with $D = \{0\}$, we have $M^2 \not\subset \bigcup \{P | P \in C\}$.

Thus there exists a $w \in M^2$ such that $w \not\in Q$ for every $Q \in L$ and $w \not\in P$ for every $P \in C_1$. Now use Lemma 2 (with $C_2 = \{L\}$) to find a $v \in M^2$ such that $v \not\in \{P + r | P \in C_1, r \in D_1\}$ and $v \not\in \{Q + a | Q \in L, a \in D_2\}$.

Let $S = R'[v + u, (R'[v + u]) \cap M]$. We see that if $b = v + u$ then we have $u - b \in M^2$. Hence $S$ is the desired $LQ$ avoiding $p$-subring. □

Lemma 11 is a technical lemma used in Lemma 12 and Theorem 13. It is needed to show that unions of certain chains of $LQ$ avoiding $p$-subrings remain $LQ$ avoiding $p$-subrings.

**Lemma 11.** Let $(T, M)$ be a complete local ring with $p$ a nonmaximal prime ideal of $T$ and $L$ a set of prime ideals of $T$. Let $R_0$ be an $LQ$ avoiding $p$-subring of $T$.
Let $\Omega$ be a well-ordered set with least element 0, with $\Omega$ either countable or $|\{\beta \in \Omega \mid \beta < \alpha\}| < |T/M|$ for every $\alpha \in \Omega$. Suppose $\{R_\alpha \mid \alpha \in \Omega\}$ is an ascending chain of subrings of $T$ such that if $\gamma(\alpha) = \alpha$ then $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$, while if $\gamma(\alpha) < \alpha$, then $R_\alpha$ is an LQ avoiding $p$-subring. Then $S = \bigcup_{\alpha \in \Omega} R_\alpha$ has a cardinality bounded above by $\sup\{\aleph_0, \max\{R_\alpha\}, |\Omega|\}$ and satisfies the properties of an LQ avoiding $p$-subring except for possibly the cardinality condition.

**Proof.** We first show that $R_\alpha$ is an LQ avoiding $p$-subring for every $\alpha \in \Omega$. We proceed by transfinite induction, the base case holding by hypothesis. Assume $\alpha \in \Omega$ and that $R_\beta$ is an LQ avoiding $p$-subring for every $\beta < \alpha$. We know that if $\gamma(\alpha) \neq \alpha$ that $R_\alpha$ is an LQ avoiding $p$-subring. So assume $\gamma(\alpha) = \alpha$. We will show that in this case $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$ is an LQ avoiding $p$-subring. We have $|R_\alpha| \leq \sum_{\beta < \alpha} |R_\beta| \leq |\{\beta \in \Omega \mid \beta < \alpha\}| \cdot \sup_{\beta < \alpha} |R_\beta|$. We know that each $R_\beta$ is an LQ avoiding $p$-subring and thus $\sup_{\beta < \alpha} |R_\beta| \leq \sup\{\aleph_0, |T/M|\}$. As defined, we have $|\{\beta \in \Omega \mid \beta < \alpha\}| < |T/M|$. Thus we have $|\{\beta \in \Omega \mid \beta < \alpha\}| \cdot \sup_{\beta < \alpha} |R_\beta| \leq \sup\{\aleph_0, |T/M|\}$. As each $R_\beta \cap p = (0)$ for all $\beta < \alpha$, clearly we have $R_\alpha \cap p \neq (0)$. Similarly, $R_\alpha \cap P = (0)$ for every $P \in \text{Ass} T$. Similarly we have $R_\alpha[\hat{x}] \cap Q = (0)$ for every $Q \in L$.

Now look at $S = \bigcup_{\alpha \in \Omega} R_\alpha$. Note that while each $R_\alpha$ is an LQ avoiding $p$-subring, the cardinality of their union may be larger than their individual cardinalities. We see $|S| \leq \sup\{\aleph_0, \max\{R_\alpha\}, |\Omega|\}$ and $S$ may not satisfy the cardinality condition.

As we have $R_\alpha \cap p = (0)$ for each $R_\alpha$, clearly we have $S \cap p = (0)$. Similarly we have $S \cap P = (0)$ for every $P \in \text{Ass} T$.

For every prime ideal $Q \in L$ we have $R_\alpha[\hat{x}] \cap Q = (0)$. By the same reasoning as above, we have $S[\hat{x}] \cap Q = (0)$.

Thus $S$ satisfies all conditions of an LQ avoiding $p$-subring except for possibly the cardinality condition. \qed

Lemma 12 pulls together the results from Lemmas 5, 8 and 10. It will allow us to find an LQ avoiding $p$-subring $S$ of $T$ containing $R$ having all the properties in the respective lemmas.

**Lemma 12.** Let $(T,M)$ be a complete local ring with $|T/M| \geq c$. Let $p$ be a non-maximal prime ideal of $T$ containing the associated prime ideals of $T$ and $q$ a prime ideal of $T$ not contained in $p$. Suppose $L$ is a set of prime ideals of $T[[\hat{x}]]$ with $|L| < |T/M|$ such that $Q \cap T \subset p$ for each $Q \in L$. Let $\bar{t} \in T/M^2$. Suppose $R$ is an LQ avoiding $p$-subring of $T$. Then there exists an LQ avoiding $p$-subring $S$ of $T$ such that
1. $R \subset S \subset T$.
2. $S \cap q \neq (0)$.
3. $\bar{t} \in \text{Image}(S \to T/M^2)$.
4. For every finitely generated ideal $I$ of $S$, we have $IT \cap S = I$.
5. For every $x \in S$ with $x = yz$, $y \notin xT$ and $z \notin xT$, we have $yt, zt^{-1} \in S$ for some unit $t \in T$. 


Proof. First use Lemma 10 to find an infinite $LQ$ avoiding $p$-subring $R_0$ such that $R \subset R_0 \subset T$, $\bar{t} \in \text{Image}(R_0 \to T/M^2)$ and $R_0 \cap q \neq (0)$. We will construct $S$ to contain $R_0$ so conditions 1, 2, and 3 will follow automatically.

Let $\Omega = \{(I,b) \mid I \text{ a finitely generated ideal of } R_0 \text{ and } b \in IT \cap R_0 \}$. Since $I$ can be $R_0$ we have $|R_0| \leq |\Omega|$. Since the number of finite subsets of $R_0$ is $|R_0|$ we have $|\Omega| \leq |R_0|$. Thus $|\Omega| = |R_0|$. Well-order $\Omega$ so that it does not have a maximal element and let $0$ denote its initial element.

Now we will define a family of $LQ$ avoiding $p$-subrings. Begin with $R_0$. If $\gamma(x) = (I,b) \neq x$ then choose $R'_x$ to be the $LQ$ avoiding $p$-subring extension of $R_0^{(x)}$ obtained from Lemma 5 so that $b \in IR'_x$. Define $R_x$ to be the $LQ$ avoiding $p$-subring extension of $R'_x$ from Lemma 8 such that for every $x \in R'_x$ with $x = yz$ in $T$ and $y \notin xT$ and $z \notin xT$ we have $yt,zt^{-1} \in R_x$ for some unit $t \in T$. Note that since $R_0^{(x)} \subset R'_x \subset R_x$ we have that $b \in IR_x$ and for every $x \in R_0^{(x)}$ with $x = yz$ in $T$ and $y \notin xT$ and $z \notin xT$ we have $yt,zt^{-1} \in R_x$ for some unit $t \in T$.

If $\gamma(x) = x$ choose $R_x = \bigcup_{\beta < x} R_\beta$. Let $R_1 = \bigcup R_x$ where $x \in \Omega$. By Lemma 11, we see that $R_1$ is an $LQ$ avoiding $p$-subring. Now suppose $x \in R_1$ such that $x = yz \in T$ and $y \notin xT$ and $z \notin xT$. Then $x \in R_0^{(x)}$ for some $x$ with $\gamma(x) \neq x$. It follows that $yt,zt^{-1} \in R_1$ for some unit $t \in T$. If $I$ is any finitely generated ideal of $R_0$ and $b \in IT \cap R_0$ then $(I,b) = \gamma(x)$ for some $x \in \Omega$. So $b \in IR_x \subset IR_1$. Thus $IT \cap R_0 \subset IR_1$.

We repeat the process to obtain $LQ$ avoiding $p$-subring extension $R_2$ of $R_1$ such that $x \in R_2$ with $x = yz \in T$ and $y \notin xT$, $z \notin xT$ implies that $yt,zt^{-1} \in R_2$ for some unit $t \in T$ and $IT \cap R_1 \subset IR_2$ for every finitely generated ideal $I$ of $R_1$.

Continue this process to obtain an ascending chain $R_0 \subset R_1 \subset \cdots$ such that $IT \cap R_n \subset R_{n+1}$ for every finitely generated ideal $I$ of $R_n$. Then $S = \bigcup_{i=1}^\infty R_i$ is an $LQ$ avoiding $p$-subring.

If $I$ is a finitely generated ideal of $S$, then some $R_n$ contains a generating set for $I$. Let $\{y_1, \ldots, y_k\}$ be such a generating set. If $b \in IT \cap S$ then $b \in R_m$ for some $m \geq n$. So $b \in \langle y_1, \ldots, y_k \rangle T \cap R_m$, so $b \in \langle y_1, \ldots, y_k \rangle R_{m+1} \subset I$. Thus $IT \cap S = I$ and condition 4 holds.

Suppose $x \in S$ with $x = yz$ in $T$ and $y \notin xT$, $z \notin xT$. Then $x \in R_i$ for some $i$, so $yt,zt^{-1} \in R_i \subset S$ for some unit $t \in T$. □

In Theorem 13, we take the results of Lemma 12 and construct the local domain $A$ whose completion is $T$. We do so by defining an ascending chain of $LQ$ avoiding $p$-subrings, all of which have the properties from Lemma 12, and whose union will be $A$.

**Theorem 13.** Let $(T,M)$ be a complete local ring with $|T/M| \geq c$ and such that no integer is a zero divisor. Let $p$ be a nonmaximal prime ideal of $T$ containing the associated primes. Suppose $L$ is a set of prime ideals of $T[[X]]$ with $|L| < |T/M|$ such that $Q \cap T \subset p$ for each $Q \in L$. Let $\Pi$ denote the prime subring of $T$ and assume that $p \cap \Pi = (0)$ and $\Pi[[X]] \cap Q = (0)$ for each $Q \in L$. Then there exists a
local domain $A$ such that

1. $\hat{A} = T$.
2. If $x \in A$ with $x = yz$ in $T$ and $y \notin xT$, $z \notin xT$, then $yt, zt^{-1} \in A$ for some unit $t \in T$.
3. The generic formal fiber of $A$ is local with maximal ideal $p$.
4. $Q \cap A[\bar{x}] = (0)$ for each $Q \in L$.

**Proof.** Note that if $p = (0)$, then $A = T$ works. So assume $p \neq (0)$.

Assume Spec $T \neq \{p, M\}$ (we will deal with the case Spec $T = \{p, M\}$ later). We define a set $\Omega_1 = \{q \in \text{Spec } T | q \not\subseteq p\}$, well-ordered so that each element has fewer than $|\Omega_1|$ predecessors. As $T$ is Noetherian, each $q$ is finitely generated. It follows that $|\Omega_1| \leq |T|$. By hypothesis we have $|T/M| \geq c$ and thus $|T| = |T/M|$. So $|\Omega_1| \leq |T/M|$.

Now suppose $|\Omega_1| < |T/M|$. Let

$$C = \{P \in \text{Spec } T | ht(P) = 1 \text{ and } P \not\subset p\} \cup \{p\}$$

and let $D = \{0\}$. Note that since $C - \{p\} \subset \Omega_1$ and $C$ is infinite,

$$|C \times D| = |C| = |C - \{p\}| \leq |\Omega_1| < |T/M|.$$ 

By Lemma 3, $M \not\subset \bigcup\{P \in \text{Spec } T | ht(P) = 1 \text{ and } P \not\subset p\} \cup \{p\}$. As we actually have $M = \bigcup\{P \in \text{Spec } T | ht(P) = 1 \text{ and } P \not\subset p\} \cup \{p\}$, this is a contradiction and $|\Omega_1| = |T/M|$.

Pick an index set $\beta$ for $\Omega_1$. Define $T$ to be the prime subring of $T$ and let $R_0 = P(\beta \cap M)$ (thus $R_0$ is isomorphic to $Q$, $\mathbb{Z}$, or $\mathbb{Z}_r$ for some prime integer $r$). Clearly $|T| \leq |T/M|$. By hypothesis $P(\bar{x}) \cap Q = (0)$ for each $Q \in L$, $P \cap p = (0)$ and $P \cap p = (0)$ for $P \in \text{Ass } T$. As these properties extend to the localization of $T$, we see that $R_0$ is an $LQ$ avoiding $p$-subring.

Let $\Omega_2 = T/M^2$, well-ordered so that each element of $\Omega_2$ has fewer than $|\Omega_2|$ predecessors. Since $|\Omega_1| = |\Omega_2|$ and we are ordering both sets so that each element of $\Omega_2$ has fewer than $|\Omega_1|$ predecessors, we can use $\beta$ for both the index set of $\Omega_1$ and $\Omega_2$.

If Spec $T \neq \{p, M\}$ let $\Omega = \{(q_a, \bar{t}_a) | q_a \in \Omega_1, \bar{t}_a \in \Omega_2 \text{ where } a \in \beta\}$ and if Spec $T = \{p, M\}$, let $\Omega = \{(M, \bar{t}_a) | \bar{t}_a \in \Omega_2 \text{ where } a \in \beta\}$, well-ordered in the obvious way using $\beta$ as the index set. Note that if Spec $T \neq \{p, M\}$, $\Omega$ is the diagonal of $\Omega_1 \times \Omega_2$. Let $0$ designate the first element of $\Omega$.

We now recursively define a family of $LQ$ avoiding $p$-subrings as follows:

$R_0$ is already defined. Let $\lambda \in \Omega$ and assume $R_{\beta}$ has been defined for every $\beta < \lambda$.

Let $\lambda \in \Omega$. If $\gamma(\lambda) < \lambda$ use Lemma 12 to define $R_{\lambda}$ to be an $LQ$ avoiding $p$-subring with $R_{\gamma(\lambda)} \subset R_{\lambda}$ and so that for $\gamma(\lambda) = (q, \bar{t})$ we have $\bar{t} \in \text{Im}(R_{\lambda} \rightarrow T/M^2)$, $q \cap R_{\lambda} \neq (0)$. In addition, if $x \in R_{\lambda}$ with $x = yz$ in $T$ with $y \notin xT$ and $z \notin xT$ then $yt, zt^{-1} \in R_{\lambda}$ for some unit $t \in T$, and we have $IT \cap R_{\lambda} = I$ for every finitely generated ideal $I$ of $R_{\lambda}$.

If $\gamma(\lambda) = \lambda$, define $R_{\lambda} = \bigcup_{\mu < \lambda} R_{\mu}$. Note that for every $R_{\lambda}$, if $x \in R_{\lambda}$ with $x = yz$ and $y \notin xT$, $z \notin xT$, then $yt, zt^{-1} \in R_{\lambda}$ for some unit $t \in T$. Note that by Lemma 11 $R_{\lambda}$ has the properties of an $LQ$ avoiding $p$-subring except for possibly the cardinality condition.
We claim that \( A = \bigcup_{\lambda \in \Omega} R_\lambda \) is the desired domain. First use Proposition 1 to show that \( A \) is Noetherian and \( A = T \). By construction, \( A \rightarrow T/M^2 \) is surjective. Now let \( I \) be a finitely generated ideal of \( A \) with generating set \( \{ y_1, \ldots, y_k \} \) and suppose \( b \in IT \cap A \). Then \( \{ b, y_1, \ldots, y_k \} \subset R_\lambda \) for some \( \lambda \in \Omega \). Since we have \( IT \cap R_\lambda = I \) for every finitely generated ideal \( I \) of \( R_\lambda \), we see \( (y_1, \ldots, y_k)R_\lambda = (y_1, \ldots, y_k)T \cap R_\lambda \). Thus we have \( b \in (y_1, \ldots, y_k)R_\lambda \subset IA \). Hence \( IT \cap A = I \).

It follows by Proposition 1 that \( A \) is Noetherian and \( \hat{A} = T \). Note that by construction, \( A \cap p = (0) \), \( A \cap Q = (0) \) for each \( Q \in L \) and for every \( q \notin p \), we have \( A \cap q \neq (0) \). It follows that the generic formal fiber of \( A \) is local with \( p \) the maximal ideal. Also note that the construction gives us that \( A \) enjoys property 2 of the theorem.

Finally, the construction will draw to a close. Theorem 14 takes the result from Theorem 13 and shows that in certain cases, the ring we have constructed is indeed excellent. Note that the proof is the same as the proof of Theorem 14 in [4], but it is short and so we include it here.

**Theorem 14.** Let \( (T, M) \) be a complete regular local ring of dimension at least two containing the rationals with \( \frac{|T|}{|M|} \geq c \). Let \( p \neq M \) be a prime ideal of \( T \) and \( L \) a set of prime ideals of \( T[\bar{X}] \) such that \( |L| < \frac{|T|}{|M|} \), \( Q \cap T \subset p \) for each \( Q \in L \) and \( \Pi[\bar{X}] \cap Q = (0) \) for each \( Q \in L \) where \( \Pi \) is the prime subring of \( T \). Then there exists an excellent RLR \( A \) such that \( \hat{A} = T \), the generic formal fiber of \( A \) is local with \( p \) its maximal ideal and \( Q \cap A[\bar{X}] = (0) \) for each \( Q \in L \) (each \( Q \) is in the generic formal fiber of \( A[\bar{X}]_{(M_0, \bar{X})} \)).

**Proof.** We apply Theorem 13 to obtain a local ring \( A \) such that the completion of \( A \) is \( T \), the generic formal fiber of \( A \) is local with \( p \) the maximal ideal, and if \( x \in A \) with \( x = yz \) in \( T \) where \( y \notin xT \) and \( z \notin xT \) then \( yt, zt^{-1} \in A \) for some unit \( t \in T \). Since \( T \) is a RLR, so is \( A \). Note that this implies both are unique factorization domains.

We must now show that \( A \) is excellent.

We claim that if \( P \) is a prime ideal of \( A \) then the ring \( T \otimes_A k(P) \) is a RLR (in fact, except for the case \( P = (0) \), it is a field) where \( k(P) \) is the field \( A_P/PAP \). Note that \( T \otimes_A k((0)) \cong T_T \) so as \( T \) is regular, the claim holds for \( P = (0) \).

Suppose \( P \neq (0) \) Let \( q \) be a prime ideal of \( T \) such that \( q \cap A = P \). We will show that \( q = PT \). As \( q \notin p \), let \( u \in q - p \in T \). Factor \( u \) into its prime elements: \( u = u_1 \cdots u_m \).

As \( q \) is prime, we know that at least one of these factors is in \( q \). Furthermore, none of these factors can be in \( p \) (for then we would have \( u \notin p \)). Without loss of generality, let \( u_1 \in q - p \). Thus \( (u_1) \cap A = (x)A \) with \( x \neq 0 \) a prime element of \( A \). We see that \( x \) must also be prime in \( T \); if it were not we could write \( x = yz \in T \) where \( y \notin xT \) and \( z \notin xT \). However, as constructed we would then have \( yt, zt^{-1} \in A \) and thus \( ytz^{-1} = yt = x \in A \) and \( x \) would not be prime in \( A \) which is a contradiction. Thus \( x \) is prime in \( T \) and \( xT = (u_1) \).

As \( u_1 \in q \) we must have \( x \in q \). As we also have \( x \in A, x \in q \cap A = P \) so \( u_1 \in PT \). As \( PT \) is an ideal, we have \( u_1 \cdot (u_2 \cdots u_m) = u \in PT \). For every \( u \in q \) where \( u \notin p \) we
thus have \( u \in PT \). Thus \( q \subseteq PT \cup p \). By the Prime Avoidance Theorem, this implies \( q \subseteq PT \). As \( q \cap A = P \) we have \( PT \subseteq q \) and thus \( q = PT \).

Note this implies that for \( P \neq (0) \), there is a one-to-one correspondence of prime ideals of \( T \) not in the generic formal fiber of \( A \) and prime ideals \( P \neq (0) \) of \( A \) given by \( P \rightarrow PT \). As \( T \) contains the rationals, the ring \( T \otimes_A k(P) \) is a field of characteristic zero.

Now we use Theorem (19.6.4) and Corollaire (19.6.5) from [1] to see that \( A \) is a \( G \)-ring. It is clear (by Theorem 31.6 in [8]) that \( A \) is universally catenary. Hence, \( A \) is excellent.

Acknowledgements

The first author would like to thank the National Science Foundation for their support of this research (DMS-9773069). Part of this research was conducted at Michigan State University and the first author is grateful for their hospitality.

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