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EXTENSION PROBLEMS AND STABLE RANK IN COMMUTATIVE BANACH ALGEBRAS

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We find some estimations of the stable rank of a complex commutative Banach algebra and use them to compute the stable rank of the polydisc algebras.

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topological stable rank	unimodular	reducible
polydisc algebra	n-stable	Bass' stable rank

Introduction

Let A be a complex commutative Banach algebra with identity. It is known that the stable rank of A [2] is related to the topology of its spectrum $X(A)$ (see [4] and [5]). In this paper we explicite this relation and we use it to estimate the stable rank of some uniform algebras. In the first section we improve some results of [4] and [5] and use them to reduce the problem of finding the stable rank of A to study the solutions of some algebraic equations in the Gelfand transforms algebra $\hat{A}$ (Theorem 3). Indeed, by means of a theorem of Arens [1] and its generalizations by Novodvorskii [13] and Taylor [18] the problem is reduced, once more, to a homotopy extension result (Theorem 4). As a corollary we prove that the stable rank of a *regular* algebra A coincides with Vaserstein's number $[d/2]+1$ where d is the topological dimension of $X(A)$. The first section ends with a result relating the reducibility of an unimodular row in A^{n+1} with some separation properties of its Gelfand transform (Theorem 5). In the second section we prove that the stable rank of the polydisc algebra $A(n)$ of $\mathbb{C}^n$ lies between $[n/2]+1$ and n (Theorem 6); surprisingly our lower bound coincides with an estimate obtained first by Vasershtein [19, p. 109, the last remark] and then by Gabel [7] and Vaserstein and Suslin [20, Corollary 19.3]. Their methods, of course, are completely different from ours. We prove also that the algebra $A(X)$ of all continuous complex functions on the compact

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subset X of the complex plane $\mathbb{C}$ which are holomorphic on its interior, has stable rank one. This is a generalization of the result of [10] and [5, Theorem 1.2], where X is the closed unit disc. In Section 3 we deal with the topological stable rank of a Banach algebra A. This notion, introduced by Rieffel [15] (see also [4] and [5]) is easier to handle than Bass' stable rank. We wish to thank the referee for several useful comments.

§ 1

Let A be a ring with identity and n a positive integer. We call *unimodular* the elements of the set $U_n(A)=\{a=(a_1,\ldots,a_n)\in A^n; \sum Aa_i=A\}$. We denote by Latin letters the elements of A^n for $n>1$ and by Greek letters the elements of A. Thus, $(a,\alpha)\in U_{n+1}(A)$ means that $a\in A^n$, $\alpha\in A$ and $\sum Aa_i+A\alpha=A$; (a,α) is *reducible* (in A) if there exists $x\in A^n$ such that $a+x\alpha=(a_1+x_1\alpha_1,\ldots,a_n+x_n\alpha)\in U_n(A)$; A is *n-stable at* $\alpha\in A$ if every $(a,\alpha)\in U_{n+1}(A)$ is reducible. The *stable rank* of A, denoted by $\mathrm{sr}(A)$, is the least n such that A is n-stable at α for every $\alpha\in A$; in other words, $\mathrm{sr}(A)\leq n$ if and only if every $(a,\alpha)\in U_{n+1}(A)$ is reducible. Given a left ideal I of A let $H_n(A;I)=\{a\in A^n$: there exists $\alpha\in I$ with $(a,\alpha)\in U_{n+1}(A)\}$. When I is the left ideal generated by α the symbol $H_n(A;\alpha)$ is used instead of $H_n(A;I)$.

Every ring homomorphism preserving identities $f: A\to B$ induces a mapping $f: U_n(A)\to U_n(B)$. We shall use several times the following fact, proved in [4, Section 3].

Proposition 1.1. *Let A be a commutative ring. Then* $\mathrm{sr}(A)\leq n$ *if and only if the mapping* $U_n(A)\to U_n(A/I)$, *induced by the natural projection* $A\to A/I$, *is onto, for every ideal I of A; moreover, it suffices to consider principal ideals.*

From now on, only commutative Banach algebras are considered. Then, by [4, Theorem 4] it suffices to consider only *closed* ideals in Proposition 1.1.

Denote by $A^{\cdot}$ the group of units of A. A well-known theorem of Michael [11, 14] states that an epimorphism of Banach algebras $f: A\to B$ induces a Serre fibration $A^{\cdot}\to B^{\cdot}$; this result and the fibration properties of the map $\mathrm{GL}_n(A)\to U_n(A)$ which sends the invertible matrix s into its first column se_1 prove the next theorem (see [4]).

Theorem 1.2. *Let $f: A\to B$ be an epimorphism of Banach algebras. Then f induces a Serre fibration $f: U_n(A)\to U_n(B)$. In particular, if $b\in U_n(B)$ belongs to the image V of f, then the connected component of b is contained in V.*

In the following $\pi_0(U)$ denotes the set of connected components of U and $[u]$ denotes the connected component of $u\in U$; of course a map $h: U\to U'$ induces a mapping $\pi_0(h): \pi_0(U)\to\pi_0(U')$. In this paper we apply this functor only to open subsets of Banach spaces, so it concides with the functor 'arc-connected components'. With these notations the second assertion of Theorem 1.2 reads: $b \in V$ if and only if $[b] \in \pi_0(U_n(B))$ belongs to the image of $\pi_0(f)$.

Observe that, for a commutative ring A, $(a, \alpha) \in U_{n+1}(A)$ is reducible if and only if the class $\bar{a}$ of a in $(A/I)^n$ belongs to the image of $U_n(A) \to U_n(A/I)$ where I is the ideal generated by α. Thus, we get:

Theorem 1.3. *Let A be a commutative Banach algebra. Then*:

(i) $(a, \alpha) \in U_{n+1}(A)$ *is reducible if and only if* $[\bar{a}]$ *belongs to the image of* $\pi_0(U_n(A)) \to \pi_0(U_n(A/I))$;

(ii) *A is n-stable at α if and only if the mapping* $\pi_0(U_n(A)) \to \pi_0(U_n(A/I))$ *is onto*;

(iii) $\mathrm{sr}(A) \leq n$ *if and only if the mapping* $\pi_0(U_n(A)) \to \pi_0(U_n(A/I))$ *is onto for every ideal I of the form $\overline{A\alpha}$. In particular, if $U_n(A/I)$ is connected for every such I then* $\mathrm{sr}(A) \leq n$.

The problem of estimating the stable rank of a commutative complex Banach algebra may be transformed into a homotopy extension problem by means of a well-known result of Arens [1] and its generalizations by Novodvorskii [13] and Taylor [18]. Let $X(A)$ denote the spectrum of A, i.e. the set of all non-zero complex homomorphism of A with the weak topology: $h_\alpha \to h$ iff $h_\alpha(a) \to h(a)$ for every a in A; $X(A)$ is a compact Hausdorff space and we consider the supremum norm on the algebra $C(X(A))$ of all complex continuous functions on $X(A)$. The Gelfand transform $\hat{}: A \to C(X(A))$ defined by $\hat{a}(h) = h(a)$, induces a homotopy equivalence $U_n(A) \to U_n(C(X(A)))$ (see [18]]). In particular, this gives a bijection $\pi_0(U_n(A)) \to \pi_0(U_n(C(X(A))) = [X(A), S^{2n-1}]$ (where $[Z, W]$ denotes the set of homotopy classes of continuous maps from Z into W). It is easy to see that $X(A/I)$ is homeomorphic to $\mathrm{hull}(I) = \{M \in X(A): M \supset I\}$.

If I is the closed ideal generated by α, $\mathrm{hull}(I) = \{M \in X(A): \alpha \in M\}$. We denote this space by Z_α; under the usual identification of $X(A)$ with the space of complex homomorphisms of A, Z_α corresponds to the closed set $\{h \in X(A): \hat{\alpha}(h) = h(\alpha) = 0\}$. Finally, the natural map $\pi_0(U_n(A)) \to \pi_0(U_n(A/I))$ corresponds to the restriction mapping $\rho_\alpha \ : [X(A), S^{2n-1}] \to [Z_\alpha, S^{2n-1}]$. Thus, the last result can be rewritten as follows:

Theorem 1.4. *Let A be a complex commutative Banach algebra. Then*:

(i) $(a, \alpha) \in U_{n+1}(A)$ *is reducible if and only if* $[\hat{a}|Z_\alpha]$ *belongs to the image of* ρ_α (*where* $[\hat{c}]$ *means the homotopy class of* $\hat{c}$);

(ii) *A is n-stable at α if and only if ρ_α is onto*;

(iii) $\mathrm{sr}(A) \leq n$ *if and only if ρ_α is onto for every $\alpha \in A$.*

Recall that the hulls of the ideals of A are the closed sets for a topology on $X(A)$, called the hull-kernel topology. A is *regular* if the hull-kernel and the Gelfand topologies coincide.

Corollary 1.5. *Let A be a commutative complex Banach algebra. Suppose that A is regular. Then* $\mathrm{sr}(A)=[d/2]+1$ *(where $[r]$ denotes the greatest integer $\leq r$ and d is the topological dimension of $X(A)$).*

Proof. If A is regular, every closed subset F of $X(A)$ is the hull of some ideal I of A. By Theorems 1.3 and 1.4(iii), $\mathrm{sr}(A)\leq n$ if and only if $\rho_F:[X(A), S^{2n-1}]\to[F, S^{2n-1}]$ is onto for every F; this is exactly one way of saying that $[d/2]+1\leq n$ (see [12, p. 208], for example). □

Remark. When $A=C(X)$ for a compact space X, this gives a classical theorem of Vaserstein [19], which motivated the study of the stable rank in Banach algebras (see also [15]).

Corollary 1.6. *For every complex commutative Banach algebra A it holds* $\mathrm{sr}(A)\leq[d/2]+1$.

Proof. If ρ_F is onto for every closed subset F of $X(A)$, a fortiori it is onto for $F=\mathrm{hull}(I)$ and we apply Theorem 1.4(iii). □

Corollary 1.7. *Let $(a, \alpha)\in U_{n+1}(A)$. Then (a, α) is reducible in A if and only if $(\hat{a}, \hat{\alpha})$ is reducible in $C(X(A))$. Furthermore A is n-stable at α if and only if $C(X(A))$ is n-stable at $\hat{\alpha}$.*

Proof. In fact, $(\hat{a}, \hat{\alpha})$ is reducible in $C(X(A))$ if and only if the homotopy class of $\hat{a}\,|\,Z_\alpha$ is in the image of ρ_α. □

Corollary 1.8. *Let $(a, \alpha)\in U_{n+1}(A)$. Then (a, α) is reducible in A if and only if for every Banach algebra $B\supset A$ such that $X(A)=X(B)$, (a, α) is reducible in B.*

Corollary 1.9. *Let $\alpha\in A$. Then A is n-stable at α for every $n\geq[(d_\alpha+1)/2]+1$, where d_α is the dimension of Z.*

Proof. The condition on n is equivalent to $d\leq 2n-1$, which implies the triviality of $[Z_\alpha, S^{2n-1}]$. It suffices to use Theorem 1.4(ii). □

Corollary 1.10. *The following conditions are equivalent*:

(i) *$U_n(A/I)$ is connected for every closed ideal I;*
(ii) *$[Z_\alpha, S^{2n-1}]$ is trivial for every $\alpha\in A$;*
(iii) $\mathrm{sr}(A)\leq n$ *and $U_n(A)$ is connected;*
(iv) $\mathrm{sr}(A)\leq n$ *and $[X(A), S^{2n-1}]$ is trivial.*

Proof. It is a simple combination of Proposition 1.1 and Theorem 1.4. □

Example 1.11. It is well known that, if A is n-stable at $\alpha \in A$ for *every* $\alpha \in A$ then A is $(n+1)$-stable at α for *every* $\alpha \in A$ [19]. We show now that this assertion fails in general if the word *every* is omitted. Taking $A = C(S^{17})$ and $f \in A$ such that $Z_f = S^9 \subset S^{17}$, we get that A is 4-stable at f, because $[Z_f, S^7] = [S^9, S^7]$ is trivial, but A is not 5-stable at f; because $[X(A), S^9] = [S^{17}, S^9] = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $[Z_f, S^9] = [S^9, S^9] = \mathbb{Z}$ so that $[X(A), S^9] \to [Z_f, S^9]$ can not be onto (see [9, last chapter] for the details).

Example 1.12. Given a compact space X and a Banach algebra A let $C(X, A)$ be the algebra of A-valued continuous functions on X. A result of Corach and Larotonda [6, Theorem 5.12] proves that $\mathrm{sr}(C(X, A)) \leqslant \dim(X) + \mathrm{sr}(A)$. We show now that, if A is the disc algebra, i.e. the algebra of all continuous functions on the unit disc D which are holomorphic on $\mathrm{int}(D)$, then $\mathrm{sr}(C([0,1], A)) = 2$. By the result quoted above and Corollary 1.10 it suffices to exhibite $f \in C([0,1], A)$ such that $[Z_f, S^1]$ is not trivial. An example is $f(t)(z) = z^2 - \sin^2 \pi t$ ($z \in D$, $t \in [0,1]$), for it is easy to see that Z_f is homeomorphic to S^1, as Fig. 1 shows.

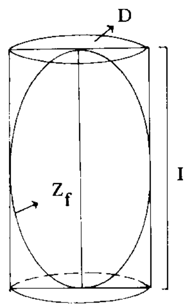

Fig. 1.

The last result of this section gives several equivalent conditions for an unimodular $(a, \alpha) \in U_{n+1}(A)$ to be reducible.

Theorem 1.13. *Let A be a complex commutative Banach algebra. The following conditions on $(a, \alpha) \in U_{n+1}(A)$ are equivalent*:

(1) *(a, α) is reducible*;

(2) *there exist $b \in U_n(A)$ and $\beta \in A$ such that $\sum b_i a_i + \beta\alpha = 1$*;

(3) *there exists $b \in u_n(A)$ such that $\widehat{b \cdot a}(Z_\alpha) \subset \{z \in \mathbb{C}^n : \sum z_i = 1\}$ (where $b \cdot a = (b_1 a_1, \ldots, b_n a_n) \in A^n$)*;

(4) *there exists $b \in U_n(A)$ such that $\widehat{b \cdot a}(Z_\alpha)$ does not separate 0 from ∞ (in $\mathbb{C}^n$)*;

(5) *there exists $c \in U_n(A)$ such that $\hat{a} = \hat{c}$ on Z_α*;

(6) *there exist $f_m \in \mathbb{C}(X(A), \mathbb{C}^n_*) = U_n(C(X(A)))$ such that $\|f_m - \hat{a}\|_{Z_\alpha} \to 0$ (where $\|f\|_{Z_\alpha} = \sup\{\|f(h)\| : h \in Z_\alpha\}$).*

Proof. (1)⇒(2). There exists $x \in A^n$ such that $a + x\alpha \in U_n(A)$, so it suffices to take $b \in U_n(A)$ such that $\sum_{i=1}^n b_i(a_i + x_i\alpha) = 1$ and $\beta = \sum b_i x_i$.

(2)⇒(3). If $\sum b_i a_i + \beta\alpha = 1$, for every $h \in Z_\alpha$ it holds that $1 = \sum h(b_i)h(a_i) + h(\beta)h(\alpha) = \sum h(b_i)h(a_i) = \widehat{b \cdot a}(h)$.

(3)⇒(4). Trivial.

(4)⇒(2). Let $r = \sup\{\|z\|: z \in b \cdot z(Z_\alpha)\} + 1$ and choose $\varphi: [1/2, 1] \to \mathbb{C}^n - \widehat{b \cdot a}(Z_\alpha)$, such that $\varphi(1/2) = (r, 0, \ldots, 0)$ and $\varphi(1) = 0$. Define $\gamma_1: [0, 1] \to \mathbb{C}$ by $\gamma_1(t) = 2t$ for $t \in [0, 1/2]$; $\gamma_1(t) = 1$ for $t \in [1/2, 1]$, and $\gamma_2: [0, 1] \to \mathbb{C}^n$ by $\gamma_2(t) = (r, 0, \ldots, 0)$ for $t \in [0, 1/2]$, $\gamma_2(t) = \varphi(t)$ for $t \in [1/2, 1]$. We must prove that $(\widehat{b \cdot a}, \alpha)$ is reducible in A; by Theorem 1.4(i) it suffices to show that $\widehat{b \cdot a} \,|\, Z_\alpha$ is null-homotopic in $C(Z_\alpha, \mathbb{C}^n_*)$. For this, define $F(h, t) = \gamma_1(t)\widehat{b \cdot a}(h) - \gamma_2(t)$; it is clear that F is a homotopy between $\widehat{b \cdot a}$ and the constant $(r, 0, \ldots, 0)$, so, it suffices to show that $F(h, t) \in \mathbb{C}^n_*$ for every $t \in [0, 1]$: if $0 \leq t \leq 1/2$, $F(h, t) = 2th(b \cdot a) - (r, 0, \ldots, 0)$ and $r = \|(r, 0, \ldots, 0)\| > |h(b \cdot a)\|$ for every $h \in Z_\alpha$; if $1/2 \leq t \leq 1$ $F(h, t) = h(b \cdot a) - \varphi(t)$ and $\varphi(t) \notin \widehat{b \cdot a}(Z_\alpha)$.

(2)⇒(1). Observe that $\beta = \sum b_i c_i$ for some $c_i \in A$, so $\sum b_i(a_i + c_i\alpha) = 1$.

(1)⇒(5). Taking $x \in A^n$ such that $c = a + X\alpha \in U_n(A)$, it is clear $\hat{a} = \hat{c}$ on Z_α.

(5)⇒(6). It suffices to take $f_m = \hat{c}$ $(m \in \mathbb{N})$.

(6)⇒(1) We shall prove that $(\hat{a}, \hat{\alpha})$ is reducible in $C(X(A))$ and use Corollary 1.7. By (6) $F_t = \hat{a} + t(f_m - \hat{a})$ defines a path in $H_n(C(X(A)); \hat{\alpha})$: in fact, $F_t(h) \neq 0$ in $\mathbb{C}^n$ if $h \in Z_\alpha$ and m is such that $\|f_m - \hat{a}\|_{Z_\alpha} < \min\{\|h(a)\|; h \in Z_\alpha\}$. Thus $\hat{a} = F_0$ belongs to the connected component of the unimodular f_m in $H_n(C(X(A)); \hat{\alpha})$ and this means that $[\hat{a}]$ is in the image of $U_n(C(X(A)) \to U_n(C(X(A))/I)$ where I = closed ideal generated by $\hat{\alpha}$; thus, $(\hat{a}, \hat{\alpha})$ is reducible in $C(X(A))$ by Theorem 1.3(i). □

§ 2

This short section contains some applications of the results just obtained, to the case of some interesting function algebras. In a previous paper [5] (see also [10]), it has been proved that the disc algebra has stable rank 1. We find an estimation for the stable rank of the polydisc algebra $A(n)$ of $\mathbb{C}^n$.

Theorem 2.1. *Let $A(n)$ be the polydisc algebra of $\mathbb{C}^n$, i.e. the algebra of all continuous functions on $D^n = \{z \in \mathbb{C}^n: |z_i| \leq 1, i = 1, \ldots, n\}$ which are holomorphic on its interior. Then $[n/2] + 1 \leq \mathrm{sr}(A(n)) \leq n$.*

Proof. We prove first that $\mathrm{sr}(A(n)) \leq n$. For this, it suffices to show that $[Z_g, S^{2n-1}]$ is trivial for every $g \in A(n)$. For $g = 0$ it is obvious because $X(A(n)) = D^n$, which is contractible. (The fact that $X(A(n))$ is (homeomorphic to) D^n is an easy 'corona theorem'; see Rudin [16, Theorem 11.7]; for the hard 'corona problem' see Carleson [3].) For $g \neq 0$ it is known that the dimension d_g of Z_g is at most $2n - 2$ (it is a combination of a theorem of Cartan and a result of dimension theory which asserts that a k-dimensional subset of S^{2n-1} with $k<2n-1$ has no interior). So $[Z_g, S^{2n-1}]$ is trivial.

We prove the other inequality by showing $g \in A(n)$, $n>1$, such that $[Z_g, S^{2[n/2]-1}]$ is not trivial. Let $g(z)=z_1 z_2 \cdots z_n - 1$. It is clear that Z_g is homeomorphic to the $(n-1)$-dimensional torus T^{n-1}, so $[Z_g, S^{2[n/2]-1}]=[T^{n-1}, S^{n-1}]$ for n even and $=[T^{n-1}, S^{n-2}]$ for n odd. Both sets are no trivial [7, last chapter.] □

Corollary 2.2. $\mathrm{sr}(A(1))=1$ *and* $\mathrm{sr}(A(2))=2$.

In [5, Theorem 3.11] we proved that, for every compact subset X of the complex plane $\mathbb{C}$, the algebra $P(X)$ (resp. $R(X)$) of uniform limits of polynomial functions (resp. rational functions without poles on X) has stable rank 1. The next result shows that the same statement holds for $A(X)$, the algebra of continuous complex functions on X which are holomorphic on $\mathrm{int}(X)$. This proves a conjecture of [5].

Theorem 2.3. *Let X be a compact subset of $\mathbb{C}$. Then* $\mathrm{sr}(A(X))=1$.

Proof. Given $(f, g) \in U_2(A(X))$ it suffices to prove that (f, g) is reducible in $C(X)$, by Corollary 1.7. Observe first that $Z_g \cap \overline{\mathrm{int}(X)}$ is zero-dimensional; by Corollary 1.9 $C(\overline{\mathrm{int}(X)})$ is 1-stable at $g|\overline{\mathrm{int}(X)}$ so there exists an extension $F: \overline{\mathrm{int}(X)} \to \mathbb{C}_* = \mathbb{C}-0$ of $f|Z_g \cap \overline{\mathrm{int}(X)}$. It is clear that the function $H: \overline{\mathrm{int}(X)} \cup Z_g \to \mathbb{C}$ defined by $F(x)$ for $x \in \overline{\mathrm{int}(X)}$ and by $f(x)$ for $x \in Z_g$, is continuous. The dimension of $X - (\overline{\mathrm{int}(X)} \cup Z_g)$ is at most 1 [12, p. 98], so $H \in C(\overline{\mathrm{int}(X)} \cup Z_g, \mathbb{C}_*)$ has an extension $K \in C(X, \mathbb{C}_*)$ [12, p. 54]. □

Remark. This theorem and Corollary 1.8 give another proof of the equalities $\mathrm{sr}(P(X))=\mathrm{sr}(R(X))=1$. Observe that, if $\tilde{X}$ is the polynomial hull of X, $P(X)=P(\tilde{X}) \subset A(\tilde{X})$.

§ 3

In [4, 5, 15] it is proved that for every Banach algebra A such that $U_n(A)$ is dense in A^n it holds that $\mathrm{sr}(A) \leq n$. Rieffel defines the *topological stable rank* of A, $\mathrm{tsr}(A)$, as the least n such that $U_n(A)$ is dense in A. Herman and Vaserstein prove that $\mathrm{sr}(A)=\mathrm{tsr}(A)$ for every C^*-algebra [8]. We show that the polydisc algebra of $\mathbb{C}^n$ has topological stable rank $n+1$. This result and Theorem 2.1 show a whole family of Banach algebras whose topological stable ranks are different from their (Bass) stable ranks.

Theorem 3.1. *Let $A(n)$ be the polydisc algebra of $\mathbb{C}^n$. Then* $\mathrm{tsr}(A(n))=n+1$.

Proof. Let z_i, $i=1,2,\dots,n$, be the coordinate functions. We shall prove that $z=(z_1,\dots,z_n)$ is not approximable by elements of $U_n(A(n))$. Suppose on the contrary that there exists $f\in U_n(A)$ such that $\|f-z\|\leqslant\varepsilon<1$. Then $F_t:\partial D^n\to\mathbb{C}^n_*$ defined by $F_t(z)=z-t(f(z)-z)$ is a homotopy from z to f (observe that $F_t(z)\in\mathbb{C}^n_*$ because if $z\in\partial D^n$ there is a z_k with $|z_k|=1$ and $z_k+t(f_k(z)-z_k)=0$ would imply that $1=|z_k|=t|f_k(z)-z_k|\leqslant\varepsilon<1$). The unimodularity of f shows that $G_t(z)=f(tz)$ defines a homotopy $\partial D^n\times[0,1]\to\mathbb{C}^n_*$ between f and a constant. Thus, $z:\partial D^n\to\mathbb{C}^n_*$ would be null-homotopic, contradiction. The contradiction proves that $\operatorname{tsr}(A)\geqslant n+1$. The other inequality holds for every n-generated commutative Banach algebra but the proof is quite involved.

In the case under consideration, we shall prove that $H_n(A,g)$ is dense in A^n for every $g\in A$. Observe, in general, that $U_{n+1}(A)=\bigcup_{g\in A}H_n(A;g)\times\{g\}$. Given $f=(f_1,\dots,f_n)\in A^n$ and $\varepsilon>0$ there exist polynomials $p_1,\dots,p_n$ such that $\|p-f\|<\varepsilon$ where $p=(p_1,\dots,p_n)$. The smoothness of p implies that $p(Z_g)$ has no interior, so a sequence $\lambda^{(m)}\in\mathbb{C}^n-p(Z_g)$ may be chosen such that $\lambda^{(m)}\to 0$. Then $p-\lambda^{(m)}\in H_n(A;g)$ and $p-\lambda^{(m)}\to p$. □

References

[1] R. Arens, The group of invertible elements of a commutative Banach algebra, Studia Math. Zeszyt (Ser. Specjalna) 1 (1963) 21–23.

[2] H. Bass, K-theory and stable algebra, in: Publications Mathématiques 22 (Institut des Hautes Etudes Scientifiques, Paris, 1964) 489–544.

[3] L. Carleson, Interpolations by bounded analytic functions and the corona problem, Ann. Math. 76 (1962) 542–559.

[4] G. Corach and A.R. Larotonda, Stable range in Banach algebras, J. Pure Appl. Algebra 32 (1984) 289–300.

[5] G. Corach and F.D. Suárez, Stable range in holomorphic function algebras, Illinois J. Math., to appear.

[6] G. Corach and A.R. Larotonda, A stabilization theorem for Banach algebras, J. Algebra, to appear.

[7] M.R. Gabel, Lower bounds of the stable range of polynomial rings, Pacific J. Math. 61 (1975) 117–120.

[8] R. Herman and L.N. Vaserstein, The stable range of C^*-algebras, Invent. Math. 77 (1984) 553–555.

[9] S.T. Hu, Homotopy Theory (Academic Press, New York, 1959).

[10] P. Jones, D. Marshall and T. Wolff, Stable rank of the disc algebra, Proc. Amer. Math. Soc., to appear.

[11] E. Michael, Convex structures and continuous selections, Canad. J. Math. 11 (1959) 571.

[12] J. Nagata, Modern Dimension Theory (North-Holland, Amsterdam, 1965).

[13] M.E. Novodvorski, Certain homotopical invariants of spaces of maximal ideals, Mat. Zametki 1 (1967) 487–494.

[14] R.S Palais, Homotopy theory of infinite dimensional manifolds, Topology 5 (1966) 1–16.

[15] M. Rieffel, Dimension and stable rank in the K-theory of C^*-algebras, Proc. London Math. Soc. (3) 46 (1983) 301–333.

[16] W. Rudin, Functional Analysis (McGraw-Hill, New York, 1973).

[17] R.G. Swan, Topological examples of projective modules, Trans. Amer. Math. Soc. 230 (1977) 201–234.

[18] J.L. Taylor, Topological invariants of the maximal ideal space of a Banach algebra, Adv. Math. 19 (1976) 149–206.

[19] L.N. Vaserstein, Stable rank of rings and dimensionality of topological spaces, Functional Anal. Appl. 5 (1971) 102–110.

[20] L.N. Vaserstein and A.A. Suslin, Serre's problem on projective modules over polynomial rings and algebraic K-theory, Math. USSR Izv. 10 (1976) 937–1001.