ARITHMETICAL EXTENSIONS WITH PRESCRIBED CARDINALITY

BY

MICHAEL O. RABIN

(Communicated by Prof. A. HEYTING at the meeting of May 30, 1959)

In their paper [6, p. 95] TARSKI and VAUGHT discuss the following question. Does every relational system R with an arbitrary number of relations have a proper arithmetically equivalent or arithmetical extension of the same cardinality. In case the number of relations of R does not exceed the cardinality of R a proper arithmetical extension of the same cardinality always exists. This was shown to be, essentialy, a consequence of the completeness theorem. But the situation in the general case was not known.

It turns out that there are even countable relational systems which have no proper countable arithmetical or arithmetically equivalent extensions (Theorem 6). In fact we have an almost complete picture of the situation as follows. If the cardinal $\mathfrak{a}$ satisfies $\mathfrak{a}^{\aleph_0}=\mathfrak{a}$ then every realtional system of cardinality $\mathfrak{a}$ has a proper arithmetical extension of cardinality $\mathfrak{a}$ (Theorem 5 and Corollary). If $\mathfrak{a}$ is smaller than the first weakly inaccessible cardinal and $\mathfrak{a}^{\aleph_0}>\mathfrak{a}$ then there exists a relational system of cardinality $\mathfrak{a}$ having no proper arithmetically equivalent extension (Theorem 8; when $\mathfrak{a}>\aleph_0$ the generalized continuum hypothesis is used in the argument, but for $\mathfrak{a}=\aleph_0$ this is not necessary).

It is an immediate consequence of Theorem 6 that *for the relational system I consisting of the integers together with all number theoretic predicates there does not exist any countable non-isomorphic arithmetically equivalent relational system; i.e. the first order theory of the system of all number theoretic predicates is categorical in cardibality* $\aleph_0$. This confirms a conjecture of A. ROBINSON (oral communication).

In proving Theorem 5 we use a construction of extensions of models inspired by Skolem's models (for a recent treatment see [4]). Namely, the elements of the extended system are equivalence classes of functions of the original system in the same way that the elements of Skolem's model are equivalence classes of the arithmetical functions. Trying to carry through an argument similar to that on [4, p. 7] one is immediately led to the introduction of the equivalence relation $\sim_F$ (Section 2) between functions of the system. This construction can also be obtained as a special case of the ultraproducts introduced by Łoś [3], especially as formulated by FRAYNE, SCOTT and TARSKI [1].

Complete relational systems as defined in the present paper were very useful in simplifying proofs. They possess some interesting properties (Lemmas 1 and 2, Theorem 7) and thus perhaps deserve a closer study on their own merit.

We would like to thank ABRAHAM ROBINSON for several interesting and helpful conversations on the topics of this paper.

1. *Complete systems*

Definition 1: A system $\mathfrak{R} = \langle A, R_0, \ldots, R_\alpha, \ldots \rangle_{\alpha<\varrho}$, where each R_α is a relation or a function of zero or more variables on the set A, is called a *complete system* if among the R_α appear all relations or all functions on A. A complete system $\mathfrak{R}$ is called a *complete algebra* if each R_α is a function.

As usual, the cardinality $\mathfrak{a}$ of A will be called the cardinality of $\mathfrak{R}$ and ϱ will be called the order of $\mathfrak{R}$.

With $\mathfrak{R}$ we associate a set $\Sigma_{\mathfrak{R}}$ of sentences as follows. We chose a first order language with equality L having, besides the usual logical connectives, quantifiers, and individual variable, a list $P_0, \ldots, P_\alpha, \ldots, \alpha < \varrho$, such that P_α is a relation or function constant of the same order as R_α. The notions of satisfaction of formulas of L by elements of A and of a sentence of L being true in $\mathfrak{R}$ are then defined in a well known manner. It should be kept in mind that in these definitions the symbols P_α of L are always interpreted as the corresponding R_α. The set of all sentences of L which are true in $\mathfrak{R}$ will be denoted by $\Sigma_{\mathfrak{R}}$.

An extension $\mathfrak{R}' = \langle A', R'_0, \ldots, R'_\alpha, \ldots \rangle_{\alpha<\varrho}$ of $\mathfrak{R}$ for which all sentences of $\Sigma_{\mathfrak{R}}$ are true is called an *arithmetically equivalent* extension.

The following theorem is readily verified. For the concept of an *arithmetical extension* see [6].

Theorem 1: *Every arithmetically equivalent extension of a complete system $\mathfrak{R}$ is an arithmetical extension of $\mathfrak{R}$.*

For a *complete* algebra $\mathfrak{R}$ an extension $\mathfrak{R}'$ will be an arithmetically equivalent extension as soon as $\mathfrak{R}'$ is a model of certain subsets much smaller than $\Sigma_{\mathfrak{R}}$; this is the contents of the following two statements.

Theorem 2: *Let $\mathfrak{R}'$ be an extension of the complete algebra $\mathfrak{R}$. If $\mathfrak{R}'$ is a model of all those sentences in $\Sigma_{\mathfrak{R}}$ which are in prenex form and contain only universal quantifiers then $\mathfrak{R}'$ is a model of $\Sigma_{\mathfrak{R}}$ and hence an arithmetically equivalent extension of $\mathfrak{R}$.*

Proof: We have to show that if $\tau \in \Sigma_{\mathfrak{R}}$ then τ is satisfied in $\mathfrak{R}'$. We may clearly assume that τ is in prenex form

$$\tau = (\exists x_1)\ldots(\exists x_n)\,(y_1)\ldots(y_m)\,(\exists x_{n+1})\ldots M(x_1, x_2, \ldots, y_1, y_2, \ldots)$$

where M is a quantifier free formula containing only the variables $x_1, \ldots, x_r, y_1, \ldots, y_q$. Now the satisfaction of τ in $\mathfrak{R}$ implies, since $\mathfrak{R}$ is complete, the existence of q-ary function constants $P_{\alpha_1}, \ldots, P_{\alpha_r}$ such that

$$\tau' = (y_1)\ldots(y_q)M(P_{\alpha_1}(y_1, \ldots, y_q), P_{\alpha_2}(y_1, \ldots, y_q), \ldots, y_1, y_2, \ldots)$$

is satisfied in $\mathfrak{R}$.

By our assumption τ is also satisfied in $\mathfrak{R}'$. But $\tau' \to \tau$ is certainly satisfied in *all* systems similar to $\mathfrak{R}$; the sentence τ is therefore satisfied in $\mathfrak{R}$. This completes the proof.

The sentences mentioned in Theorem 2 have the following form: a number of universal quantifiers followed by an expression built by combining a number of equations and inequations between terms (a term is an expression built from individual variables and function constants by substitutions) by propositional connectives. We may restrict ourselves even further and essentially consider only equations between terms.

Theorem 3: *Let P_α, P_β, and P_γ be function constants such that*

(1) $$\sigma_1 = (x)(y)[x \neq y \leftrightarrow P_\alpha(x, y) = P_\beta(x, y)],$$

(2) $$\sigma_2 = (x)(y)(u)(v)[x = y \wedge u = v \leftrightarrow P_\gamma(x, u) = P_\gamma(y, v)],$$

are true in $\mathfrak{R}$ (since $\mathfrak{R}$ is a complete algebra such functions always exist).

If $\mathfrak{R}'$ is an extension of $\mathfrak{R}$ in which σ_1, σ_2, and all sentences of $\Sigma_{\mathfrak{R}}$ which are of the form $(x_1)\ldots(x_n)$ $[t_1 = t_2]$ where t_1 and t_2 are terms, are satisfied, then $\mathfrak{R}'$ is an arithmetically equivalent extension of $\mathfrak{R}$.

Proof: Let τ be a universal sentence

$$\tau = (x_1)(x_2)\ldots(x_n)\ M(x_1, \ldots, x_n)$$

where $M(x_1, \ldots, x_n)$ does not contain quatifiers. We may assume that $M(x_1, \ldots, x_n)$ is in disjunctive normal form and thus is a disjunction of conjunctions of equations and inequations between terms. Now replace each inequation $t \neq s$ in M by $P_\alpha(t, s) = P_\beta(t, s)$. From the new sentence eliminate all conjunctions by successively replacing $t = s \wedge u = v$ by $P_\gamma(t, u) = P_\gamma(s, v)$. In this way we get a disjunction of one or more equations $v_1 = u_1 \vee \ldots \vee v_m = u_m$. This disjunction is equivalent to

$$\sim [v_1 \neq u_1 \wedge \ldots \wedge v_m \neq u_m].$$

Transform this, as before, into a single equation $t_1 = t_2$. It is easy to see that

(3) $$\sigma_1 \wedge \sigma_2 \to [\tau \leftrightarrow (x_1)\ldots(x_n)\ [t_1 = t_2]].$$

Assume now that $\tau \in \Sigma_{\mathfrak{R}}$. Since $\sigma_1 \in \Sigma_{\mathfrak{R}}$ and $\sigma_2 \in \Sigma_{\mathfrak{R}}$ we have, by (3), $(x_1)\ldots(x_n)$ $[t_1 = t_2] \in \Sigma_{\mathfrak{R}}$. According to our assumptions $(x_1)\ldots(x_n)$ $[t_1 = t_2]$ is therefore true also in $\mathfrak{R}'$. But σ_1 and σ_2 are true in $\mathfrak{R}'$. Hence, again by (3), τ is true in $\mathfrak{R}'$.

Since $\mathfrak{R}'$ is a model of every universal sentence τ true in $\mathfrak{R}$ it is an arithmetically equivalent extension of $\mathfrak{R}$ by Theorem 2.

2. *A construction of extensions*

A collection **F** of subsets of a set A will be called an *ultrafilter* if

(i) $\phi \notin \mathbf{F}$,

(ii) $B \in \mathbf{F}$ and $C \in \mathbf{F}$ implies $B \cap C \in \mathbf{F}$,

(iii) $B \in \mathbf{F}$ and $B \subseteq C \subseteq A$ implies $C \in \mathbf{F}$,

(iv) for all $B \subseteq A$ either $B \in \mathbf{F}$ or $A - B \in \mathbf{F}$.

With an ultrafilter **F** of A one can associate a relation $\sim_{\mathbf{F}}$ between functions from A into A (i.e. elements of A^A) as follows:

$$f \sim_{\mathbf{F}} g \text{ if and only if } \{x \mid f(x) = g(x)\} \in \mathbf{F}.$$

Conditions (ii) and (iii) imply that $\sim_{\mathbf{F}}$ is an equivalence relation. If $f \sim_{\mathbf{F}} g$ we shall say that f and g are equivalent mod **F**. Condition (i) guarantees that if f and g are constant functions, say $f(x) = b$, $g(x) = c$, then $f \sim_{\mathbf{F}} g$ only if $b = c$. The set A^A of all functions is thus decomposed into equivalence classes and the classes containing a constant function stand in one to one correspondence with the elements of A.

Definition 2: Let $\mathfrak{R} = \langle A, R_0, \ldots, R_\alpha, \ldots \rangle_{\alpha<\varrho}$ be an algebra and let **F** be an ultrafilter of A. The algebra $\mathfrak{R}_{\mathbf{F}} = \langle A', R'_0, \ldots, R'_\alpha, \ldots \rangle_{\alpha<\varrho}$ is defined as follows. The set A' is the collection of all equivalence classes of A^A mod **F**. If R_α is a function of n variables then R'_α is the function from $(A')^n$ into A' defined by

$$R'_\alpha(\bar{f}_1, \ldots, \bar{f}_n) = \bar{R}_\alpha(f_1, \ldots, f_n), \tag{4}$$

where $f_1 \in A^A, \ldots, f_n \in A^A$, and $\bar{g}$ denotes the equivalence class mod **F** of the function g.

Remark: Conditions (ii) and (iii) ensure that (4) does indeed define a function.

By identifying the equivalence classes mod **F** of the constant functions with the corresponding elements of A, we can consider A as a subset of A'. The functions R'_α then become extensions of the corresponding functions R_α, the whole system $\mathfrak{R}_{\mathbf{F}}$ may thus be considered as an extension of $\mathfrak{R}$.

Theorem 4: *If $\mathfrak{R}$ is a complete algebra and* **F** *is an ultrafilter of A then $\mathfrak{R}_{\mathbf{F}}$ is an arithmetically equivalent extension of $\mathfrak{R}$.*

Proof: By Theorem 3 we just have to show that all equations between terms which are true in $\mathfrak{R}$ are true in $\mathfrak{R}_{\mathbf{F}}$; and furthermore that two sentences of the forms (1) and (2) which are true in $\mathfrak{R}$ are also true in $\mathfrak{R}_{\mathbf{F}}$.

The first part is obvious.

Let now P_α, P_β, and P_γ be function constants for which σ_1 and σ_2 are true in $\mathfrak{R}$. Let $f(t)$ and $g(t)$ be functions from A to A. If $\bar{f} \neq \bar{g}$ ($\bar{f}, \bar{g}$, etc.

are the equivalence classes mod F of f, g, etc.) then the set $\{t \mid f(t) = g(t)\} = S$ is not in $\mathbf{F}$. But then, by (iv), $A - S \in \mathbf{F}$. By (1)

$$A - S = \{t \mid R_\alpha(f(t), g(t)) = R_\beta(f(t), g(t))\}.$$

Hence $R'_\alpha(\bar{f}, \bar{g}) = R'_\beta(\bar{f}, \bar{g})$. Similarly if $\bar{f} = \bar{g}$ then $R'_\alpha(\bar{f}, \bar{g}) \neq R'_\beta(\bar{f}, \bar{g})$. The sentence σ_1 is thus true in $\mathfrak{R}_\mathbf{F}$. One should notice that this is the only point where we use condition (iv).

It is easily verified that σ_2 is also true in $\mathfrak{R}_\mathbf{F}$. This completes the proof.

The previous result can also be deduced as a special case of Theorem (2.6) of Łoś [3, p. 105] or Theorem II announced by Frayne and Scott [2].

When is the extension $\mathfrak{R}_\mathbf{F}$ a proper extension? If $\mathbf{F}$ contains a set consisting of a single element, say x_0, then for each $f \in A^A$ we have $f \sim_\mathbf{F} f(x_0)$. Thus each function is equivalent to some constant function and we do not have a proper extension. The above ultrafilter then contains $\{x_0\}$ and we shall call such an ultrafilter a *trivial ultrafilter*. If, on the other hand, $\mathbf{F}$ is non-trivial then there are functions not equivalent to any constant function (e.g. the identity function $f(x) = x$). The domain A' contains elements other than those in A and thus we have a proper extension.

If S is any infinite subset of A there exsists a non-trivial ultrafilter $\mathbf{F}$ such that $S \in \mathbf{F}$. This is a simple consequence of Zorn's Lemma.

Let now $\mathfrak{R} = \langle A, R_0, \ldots, R_\alpha, \ldots \rangle_{\alpha < \varrho}$ be a complete algebra of cardinality $\bar{\bar{A}} = \mathfrak{a}$ and let $\mathfrak{a}^{\aleph_0} = \mathfrak{a}$. Take a *countable* subset $S \subseteq A$. Let $\mathbf{F}$ be a non-trivial ultrafilter containing S. The algebra $\mathfrak{R}_\mathbf{F}$ is an arithmetically equivalent proper extension of $\mathfrak{R}$.

If $f, g \in A^A$ then $f \sim_\mathbf{F} g$ as soon as f and g coincide on the countable set $S \in \mathbf{F}$. Thus there are at most $\mathfrak{a}^{\aleph_0}$ different equivalence classes mod $\mathbf{F}$. Hence $\bar{\bar{A}}' = \mathfrak{a} = \bar{\bar{A}}$. We thus have proved

Theorem 5: *If* $\mathfrak{a}^{\aleph_0} = \mathfrak{a}$ *then every complete algebra of cardibality* $\mathfrak{a}$ *has a proper arithmetically equivalent extension of the same cardinality.*

Corollary: *If* $\mathfrak{a}^{\aleph_0} = \mathfrak{a}$ *then every relational system of cardinality* $\mathfrak{a}$ *has a proper arithmetically equivalent extension of the same cardinality.*

3. *Non existence of certain extensions*

The information which we can gather about the case $\mathfrak{a}^{\aleph_0} > \mathfrak{a}$ rests upon a result due to Tarski [5]. If $\mathfrak{a}^{\aleph_0} > \mathfrak{a}$ and $\bar{\bar{A}} = \mathfrak{a}$ then there exists a collection C of countable subsets of A such that $\bar{\bar{C}} > \mathfrak{a}$ (namely, $\bar{\bar{C}} = \mathfrak{a}^{\aleph_0}$) and the intersection of every two elements of C is finite.

Theorem 6: *A complete system over a countable domain has no proper countable arithmetically equivalent extension.*

Proof: Let $\mathfrak{R} = \langle A, R_0, R_1, \ldots \rangle$, $\bar{\bar{A}} = \aleph_0$, be a complete system.

Let $\{R_\gamma\}_{\gamma \in C}$ be a collection of *one to one* functions from A into A such that $\bar{\bar{C}} = \aleph_0^{\aleph_0}$ and the ranges of every two functions have a finite intersection. Thus for every pair $\alpha \in C$, $\beta \in C$, there exsists an integer n ($n = 0$ is not excluded) such that the sentence $\tau_{\alpha\beta}$ expressing the fact that the ranges of R_α and R_β have exactly n elements in common is true in $\mathfrak{R}$; this sentence is of course elementary.

If $\mathfrak{R}' = \langle A', R_0', R_1', \ldots \rangle$ is a proper arithmetically equivalent extension of $\mathfrak{R}$ then $R_\gamma(A) \subset R_\gamma(A')$ (proper inclusion) for all $\gamma \in C$. For R_γ is a one to one function and since this can be expressed elementarily so is its extension R_γ'; hence if $a' \in A' - A$ then $R_\gamma'(a') \notin R_\gamma(A)$. But for any two α and β the sentence $\tau_{\alpha\beta}$ remains true in $\mathfrak{R}'$, thus $R_\alpha'(A')$ and $R_\beta'(A')$ have the same number of common elements as $R_\alpha(A)$ and $R_\beta(A)$. Hence $R_\alpha(A) \cap R_\beta(A) = R_\alpha'(A') \cap R_\beta'(A')$. This means that the non empty sets $R_\gamma'(A') - R_\gamma(A)$ are pairwise disjoint. Thus A' contains at least $\bar{\bar{C}}$ different elements, in particular A' cannot be countable. This completes the proof.

To deal with complete systems of cardinality greater than $\aleph_0$ certain preliminaries are necessary. We first make the observation that for a complete system $\mathfrak{R} = \langle A, R_0, R_1, \ldots \rangle$ and an arithmetically equivalent extension $\mathfrak{R}' = \langle A', R_0', R_1', \ldots \rangle$, if there exists a predicate (set) R_α of cardinality $\mathfrak{b}$ which is enlarged in passing from $\mathfrak{R}$ to $\mathfrak{R}'$ (i.e. there exists an element $x \in A'$ such that $R_\alpha'(x)$ but either $x \notin A$ or not $R_\alpha(x)$) then every predicate R_β of cardinality $\mathfrak{b}$ is enlarged in passing from $\mathfrak{R}$ to $\mathfrak{R}'$.

We shall say that $\mathfrak{R}'$ is *non-standard with respect to cardinality* $\mathfrak{b}$ if the predicates of $\mathfrak{R}$ of cardinality $\mathfrak{b}$ are enlarged in passing to $\mathfrak{R}'$, otherwise we shall call $\mathfrak{R}'$ *standard with respect to* $\mathfrak{b}$.

We have the following two results concerning standardness with respect to a cardinal of an arithmetically equivalent extension $\mathfrak{R}'$ of a complete system $\mathfrak{R}$ of cardinality $\mathfrak{a}$.

Lemma 1: *If* $\mathfrak{b} = \Sigma_{\tau \in T} \mathfrak{a}_\tau$ *and if* $\mathfrak{R}'$ *is standard with respect to all* $\mathfrak{a}_\tau$ *and* $\bar{\bar{T}}$ *then* $\mathfrak{R}'$ *is standard with respect to* $\mathfrak{b}$.

Proof: Let $R(x, y)$ be a binary predicate of $\mathfrak{R}$ with the following properties. The set $S = \{x \mid (\exists y)R(x, y)\}$ has cardinality $\bar{\bar{T}}$. If $x \neq x'$ then $\sim [R^{\tau}(x, y) \wedge R(x', y)]$. There exists a one to one correspondence $x_\tau \leftrightarrow \tau$ between S and T such that $\{y \mid R(x_\tau, y)\}$ has cardinality $\mathfrak{a}_\tau$. From our assumptions it follows that $R(x, y)$ and $R'(x, y)$ are the same predicates. Therefore the predicate $Q(y) \equiv (\exists x)R(x, y)$ does not change when passing to $\mathfrak{R}'$. But the cardinality of $\{y \mid Q(y)\}$ is $\mathfrak{b}$.

Lemma 2: *If* $2^{\mathfrak{b}} \leqslant \mathfrak{a}$ *and* $\mathfrak{R}'$ *is standard with respect to* $\mathfrak{b}$ *then* $\mathfrak{R}'$ *is standard with respect to* $2^{\mathfrak{b}}$.

Proof: Let $P(x)$ be a predicate of $\mathfrak{R}$ of cardinality $\mathfrak{b}$. Let $Q(x, y)$ be a predicate of $\mathfrak{R}$ with the following properties. $Q(x, y) \to P(x)$. Extensionality: $(x)[Q(x, y) \leftrightarrow Q(x, y')] \to y = y'$. For every subset of the predicate $P(x)$ there exists a y bearing the relation $Q(x, y)$ to exactly those x in the subset.

It is immediately seen that since $P(x)$ does not change when passing to $\mathfrak{R}'$ and since the first two properties of $Q(x, y)$ are elementarily expressible and hence preserved in $\mathfrak{R}'$, $Q(x, y)$ does not change when passing to $\mathfrak{R}'$ (this depends in an essential way also on the third property of Q even though it is not elementarily expressible). But the cardinality of $\{y \mid (\exists x)Q(x, y)\}$ is $2^{\mathfrak{b}}$.

Theorem 7: *Assuming* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. *If the cardinality* $\mathfrak{a}$ *of a complete system* $\mathfrak{R}$ *is less or equal to the first weakly inaccessible number and if* $\mathfrak{R}'$ *is a proper arithmetically equivalent extension of* $\mathfrak{R}$, *then* $\mathfrak{R}'$ *is non-standard with respect to all infinite cardinals.*

Proof: Assume by way of contradiction that $\mathfrak{R}'$ is standard with respect to some infinite cardinal. If $\mathfrak{R}'$ is standard with respect to a cardinal then it is clearly standard with respect to all smaller cardinals. Thus $\mathfrak{R}'$ is standard with respect to $\aleph_0$. On the other hand $\mathfrak{R}'$ is not standard with respect to the cardinality of $\mathfrak{R}$.

Let $\mathfrak{c}$ be the smallest cardinal with respect to which $\mathfrak{R}'$ is not standard; thus $\mathfrak{c} > \aleph_0$. The cardinal $\mathfrak{c}$ must have an immediate predecessor. For if $\mathfrak{c} = \aleph_\alpha$ where α is a limit number then, since $\mathfrak{c}$ is smaller than the first inaccessible number and $\mathfrak{c} \neq \aleph_0$, we have $\mathfrak{c} = \Sigma_{\tau \in T}\, \mathfrak{a}_\tau$ for some T and cardinals $\mathfrak{a}_\tau$ such that $\bar{\bar{T}} < \mathfrak{c}$ and $a_\tau < \mathfrak{c}$. Thus $\mathfrak{R}'$ is standard with respect to $\bar{\bar{T}}$ and all $\mathfrak{a}_\tau$. This implies, by Lemma 1, that $\mathfrak{R}'$ is standard with respect to $\mathfrak{c}$ which it cannot be.

Hence $\mathfrak{c} = \aleph_\alpha$ where α is not a limit number. Now $\mathfrak{R}'$ is standard with respect to $\aleph_{\alpha-1}$. Assuming the generalized contininium hypothesis we have $2^{\aleph_{\alpha-1}} = \mathfrak{c}$ so that by Lemma 2 $\mathfrak{R}'$ is standard with respect to $\mathfrak{c}$, a contradiction.

Theorem 8: *Assuming* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. *If the cardinality* $\mathfrak{a}$ *of a complete system is less than the first weakly inaccessible number and if* $\mathfrak{a}^{\aleph_0} > \mathfrak{a}$, *then* $\mathfrak{R}$ *has no proper arithmetically equivalent extension of the same cardinality.*

Proof: Assume $\mathfrak{R}'$ to be an arithmetically equivalent extension of $\mathfrak{R}$. By Theorem 7 $\mathfrak{R}'$ is non-standard with respect to all infinite cardinals, in particular with respect to $\aleph_0$.

There exists now by Tarski's theorem a collection of countable predicates $\{R_\tau(x)\}_{\tau \in T}$ of $\mathfrak{R}$ such that $\bar{\bar{T}} = \mathfrak{a}^{\aleph_0}$, and the intersection of each pair of these predicates is finite. Each of the sets defined by the $R_\tau(x)$ is enlarged when passing to $\mathfrak{R}'$. The argument of Theorem 6 therefore applies to show that the cardinality of $\mathfrak{R}'$ is at least $\mathfrak{a}^{\aleph_0} > \mathfrak{a}$.

It would be interesting to determine whether it follows from a suitable axiom of existence of inaccessible cardinals that even for some cardinals $\mathfrak{b}$ satisfying $\mathfrak{b}^{\aleph_0} > \mathfrak{b}$, every relational system of cardinality $\mathfrak{b}$ has a proper arithmetical extension of the same cardinality $\mathfrak{b}$.

Hebrew University, Jerusalem

REFERENCES

1. Frayne, T. E., D. S. Scott and A. Tarski, Reduced products, A.M.S. Notices, 5, 673–674 (Abstract 550–7) (1958).
2. ——— and ———, Model theoretic properties of reduced products, Ibid., 675 (Abstract 550–10) (1958).
3. Łoś, J., Les classes definissables d'algebras, Mathematical Interpretations of Formal Systems (North Holland Publishing Co. Amsterdam), 98–113 (1955).
4. Skolem, Th., Peano's axioms and models of arithmetic, Ibid., 1–14 (1955).
5. Tarski, A., Sur la decomposition des ensembles en sous-ensembles presque disjoints, Fund. Math., 12, 188–205 (1928).
6. ——— and R. L. Vaught, Arithmetical extensions of relational systems, Comp. Math., 13, 82–101.

Added in proof:

7. Rabin, M. O., Arithmetical extensions with prescribed cardinality, A.M.S. Notices, 6, 259–260 (Abstract 555–48) (1959).