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Noether symmetry in $f(R)$ cosmology

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ABSTRACT

The Noether symmetry of a generic $f(R)$ cosmological model is investigated by utilizing the behavior of the corresponding Lagrangian under the infinitesimal generators of the desired symmetry. We explicitly calculate the form of $f(R)$ for which such symmetries exist. It is shown that the resulting form of $f(R)$ yields a power law expansion for the cosmological scale factor. We also obtain the effective equation of state parameter for the corresponding cosmology and show that our model can provide a gravitational alternative to the quintessence.

1. Introduction

Observations of type Ia supernovae [1] and cosmic microwave background [2] have revealed that the universe is undergoing an accelerating phase in its expansion. The explanation of this acceleration in the context of general relativity has stimulated a myriad of ideas, the most notable of which is the introduction of a mysterious cosmic fluid, the so-called dark energy [3]. In more recent times, modified theories of gravity, constructed by adding correction terms to the usual Einstein–Hilbert action, have opened a new window to study the accelerated expansion of the universe where it has been shown that such correction terms could give rise to accelerating solutions of the field equations without having to invoke concepts such as dark energy [4].

In a more general setting, one can use a generic function, $f(R)$, instead of the usual Ricci scalar R as the action for the model. Such $f(R)$ gravity theories have been extensively studied in the literature over the past few years. In finding the dynamical equations of motion one can vary the action with respect to the metric (metric formalism), or view the metric and connections as independent dynamical variables and vary the action with respect to both independently (Palatini formalism) [5]. In this theory, the Palatini form of the action is shown to be equivalent to a scalar-tensor type theory from which the scalar field kinetic energy is absent. This is achieved by introducing a conformal transformation in which the conformal factor is taken as an auxiliary scalar field [6]. As is well known, in the usual Einstein–Hilbert action these two approaches give the same field equations. However, in $f(R)$ gravity the Palatini formalism leads to different dynamical equations due to nonlinear terms in the action. There is also a third version of $f(R)$ gravity in which the Lagrangian of the matter depends on the connections of the metric (metric-affine formalism) [7].

In this Letter we consider an $(n+1)$-dimensional flat FRW space–time in the framework of the metric formalism of $f(R)$ gravity. Following [8], we set up an effective Lagrangian in which the scale factor a and Ricci scalar R play the role of independent dynamical variables. This Lagrangian is constructed in such a way that its variation with respect to a and R yields the correct equations of motion as that of an action with a generic $f(R)$ mentioned above. The form of the function $f(R)$ appearing in the modified action is then found by demanding that the Lagrangian admits the desired Noether symmetry [9]. By the Noether symmetry of a given minisuperspace cosmological model we mean that there exists a vector field X, as the infinitesimal generator of the symmetry on the tangent space of the configuration space such that the Lie derivative of the Lagrangian with respect to this vector field vanishes. For a study of the Noether symmetry in various cosmological models see [10]. We shall see that by demanding the Noether symmetry as a feature of the Lagrangian of the model under consideration, we can obtain the explicit form of the function $f(R)$. Since the existence of a symmetry results in a constant of motion, we can integrate the field equations which would then lead to a power law expansion of the universe.

2. The phase space of the model

In this section we consider a spatially flat FRW cosmology within the framework of $f(R)$ gravity. Since our goal is to study models which exhibit Noether symmetry, we do not include any

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matter contribution in the action. Let us start from the $(n+1)$-dimensional action (we work in units where $c = 16\pi G = 1$)

$$\mathcal{S} = \int d^{n+1}x\sqrt{-g}f(R), \tag{1}$$

where R is the scalar curvature and $f(R)$ is an arbitrary function of R. By varying the above action with respect to metric we obtain the equation of motion as

$$\frac{1}{2}g_{\mu\nu}f(R) - R_{\mu\nu}f'(R) + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) = 0, \tag{2}$$

where a prime represents differentiation with respect to R. We assume that the geometry of space–time is described by the flat FRW metric which seems to be consistent with the present cosmological observations

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{n}\left(dx^i\right)^2. \tag{3}$$

With this background geometry the field equations read

$$(n-1)\dot{H} + \frac{n(n-1)}{2}H^2 = -\frac{1}{f'}\left[f'''\dot{R}^2 + (n-1)H\dot{R}f'' + f''\ddot{R} + \frac{1}{2}(f - Rf')\right], \tag{4}$$

$$H^2 = \frac{1}{n(n-1)f'}\left[(f'R - f) - 2n\dot{R}Hf''\right], \tag{5}$$

where $H = \dot{a}/a$ is the Hubble parameter and a dot represents differentiation with respect to t. To study the symmetries of the minisuperspace under consideration, we need an effective Lagrangian for the model whose variation with respect to its dynamical variables yields the correct equations of motion. Following [8], by considering the action described above as representing a dynamical system in which the scale factor a and scalar curvature R play the role of independent dynamical variables, we can rewrite action (1) as [11]

$$\mathcal{S} = \int dt\,\mathcal{L}(a, \dot{a}, R, \dot{R}) = \int dt\left\{a^n f(R) - \lambda\left[R - n(n-1)\frac{\dot{a}^2}{a^2} - 2n\frac{\ddot{a}}{a}\right]\right\}, \tag{6}$$

where we introduce the definition of R in terms of a and its derivatives as a constraint. This procedure allows us to remove the second order derivatives from action (6). The Lagrange multiplier λ can be obtained by variation with respect to R, that is, $\lambda = a^n f'(R)$. Thus, we obtain the following Lagrangian for the model [8,11]

$$\mathcal{L}(a, \dot{a}, R, \dot{R}) = n(n-1)\dot{a}^2a^{n-2}f' + 2n\dot{a}\dot{R}a^{n-1}f'' + a^n(f'R - f). \tag{7}$$

The momenta conjugate to variables a and R are

$$p_a = \frac{\partial\mathcal{L}}{\partial\dot{a}} = 2n(n-1)\dot{a}a^{n-2}f' + 2na^{n-1}\dot{R}f'', \tag{8}$$

$$p_R = \frac{\partial\mathcal{L}}{\partial\dot{R}} = 2n\dot{a}a^{n-1}f''. \tag{9}$$

The Hamiltonian corresponding to Lagrangian (7) can then be written in terms of a, $\dot{a}$, R and $\dot{R}$ as

$$\mathcal{H}(a, \dot{a}, R, \dot{R}) = n(n-1)\dot{a}^2a^{n-2}f' + 2n\dot{a}\dot{R}a^{n-1}f'' - a^n(f'R - f). \tag{10}$$

Therefore, our cosmological setting is equivalent to a dynamical system where the phase space is spanned by $\{a, R, p_a, p_R\}$ with Lagrangian (7) describing the dynamics with respect to time t. Now, it is easy to see that variation of Lagrangian (7) with respect to R gives the following well-known relation for the scalar curvature

$$R = n(n-1)\frac{\dot{a}^2}{a^2} + 2n\frac{\ddot{a}}{a}, \tag{11}$$

while variation with respect to a yields the field equation (4). Also, Eq. (5) is nothing but the zero energy condition $\mathcal{H} = 0$ (Hamiltonian constraint). A quantum $f(R)$ cosmological model based on Hamiltonian (10) can be found in [12].

The setup for constructing the phase space and writing the Lagrangian is now complete. In the next section we shall use the above Lagrangian to find the form of $f(R)$ that would admit a Noether symmetry.

3. The Noether symmetry

As is well known, Noether symmetry approach is a powerful tool in finding the solution to a given Lagrangian, including the one presented above. In this approach, one is concerned with finding the cyclic variables related to conserved quantities and consequently reducing the dynamics of the system to a manageable one. The investigation of Noether symmetry in the model presented above is therefore the goal we shall pursue in this section. Here our aim is to find the function $f(R)$ such that the corresponding Lagrangian exhibits the desired symmetry. Following [9], we define the Noether symmetry induced on the model by a vector field X on the tangent space $TQ = (a, R, \dot{a}, \dot{R})$ of the configuration space $Q = (a, R)$ of Lagrangian (7) through

$$X = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial R} + \frac{d\alpha}{dt}\frac{\partial}{\partial\dot{a}} + \frac{d\beta}{dt}\frac{\partial}{\partial\dot{R}}, \tag{12}$$

such that the Lie derivative of the Lagrangian with respect to this vector field vanishes

$$L_X\mathcal{L} = 0. \tag{13}$$

In (12), α and β are functions of a and R and $\frac{d}{dt}$ represents the Lie derivative along the dynamical vector field, that is,

$$\frac{d}{dt} = \dot{a}\frac{\partial}{\partial a} + \dot{R}\frac{\partial}{\partial R}. \tag{14}$$

It is easy to find the constants of motion corresponding to such a symmetry. Indeed, Eq. (13) can be rewritten as

$$L_X\mathcal{L} = \left(\alpha\frac{\partial\mathcal{L}}{\partial a} + \frac{d\alpha}{dt}\frac{\partial\mathcal{L}}{\partial\dot{a}}\right) + \left(\beta\frac{\partial\mathcal{L}}{\partial R} + \frac{d\beta}{dt}\frac{\partial\mathcal{L}}{\partial\dot{R}}\right) = 0. \tag{15}$$

Noting that $\frac{\partial\mathcal{L}}{\partial q} = \frac{dp_q}{dt}$, we have

$$\left(\alpha\frac{dp_a}{dt} + \frac{d\alpha}{dt}p_a\right) + \left(\beta\frac{dp_R}{dt} + \frac{d\beta}{dt}p_R\right) = 0, \tag{16}$$

which yields

$$\frac{d}{dt}(\alpha p_a + \beta p_R) = 0. \tag{17}$$

Thus, the constants of motion are

$$Q = \alpha p_a + \beta p_R. \tag{18}$$

In order to obtain the functions α and β we use Eq. (15). In general this equation gives a quadratic polynomial in terms of $\dot{a}$ and $\dot{R}$ with coefficients being partial derivatives of α and β with respect to the configuration variables a and R. Thus, the resulting expression is identically equal to zero if and only if these coefficients are zero. This leads to a system of partial differential equations for α and β. For Lagrangian (7), condition (15) results in

$$na^{n-1}(f'R-f)\alpha+a^n f''R\beta+2n\dot{R}^2\left(a^{n-1}f''\frac{\partial\alpha}{\partial R}\right)$$
$$+n\dot{a}^2\left[(n-1)(n-2)a^{n-3}f'\alpha+(n-1)a^{n-2}f''\beta\right.$$
$$\left.+2(n-1)a^{n-2}f'\frac{\partial\alpha}{\partial a}+2a^{n-1}f''\frac{\partial\beta}{\partial a}\right]$$
$$+2n\dot{R}\dot{a}\left[(n-1)a^{n-2}f''\alpha+a^{n-1}f'''\beta+a^{n-1}f''\frac{\partial\alpha}{\partial a}\right.$$
$$\left.+(n-1)a^{n-2}f'\frac{\partial\alpha}{\partial R}+a^{n-1}f''\frac{\partial\beta}{\partial R}\right]=0, \tag{19}$$

which leads to the following system of equations

$$n(f'R-f)\alpha+af''R\beta=0, \tag{20}$$

$$f''\frac{\partial\alpha}{\partial R}=0, \tag{21}$$

$$(n-1)(n-2)f'\alpha+(n-1)af''\beta+2(n-1)af'\frac{\partial\alpha}{\partial a}$$
$$+2a^2f''\frac{\partial\beta}{\partial a}=0, \tag{22}$$

$$(n-1)f''\alpha+af'''\beta+af''\frac{\partial\alpha}{\partial a}+(n-1)f'\frac{\partial\alpha}{\partial R}$$
$$+af''\frac{\partial\beta}{\partial R}=0. \tag{23}$$

Eq. (21) gives $f''=0$ or $\frac{\partial\alpha}{\partial R}=0$. In the case where $f''=0$, we obtain

$$f(R)=c_1R+c_2. \tag{24}$$

Substituting this result into Eq. (20) we get $c_2=0$ and recover the usual Einstein–Hilbert gravity without cosmological constant for which $f(R)=R$. Also, since Eq. (23) results in $\frac{\partial\alpha}{\partial R}=0$, from Eq. (22) we obtain the following differential equation for $\alpha(a)$

$$(n-2)\alpha(a)+2a\frac{d\alpha}{da}=0, \tag{25}$$

with solution

$$\alpha(a)=a^{-\frac{n-2}{2}}. \tag{26}$$

Therefore, for any arbitrary function $\beta(a,R)$, the following vector field represents a Noether symmetry for the minisuperspace of the flat FRW cosmology in Einstein–Hilbert gravity

$$X=a^{-\frac{n-2}{2}}\frac{\partial}{\partial a}-\frac{n-2}{2}\dot{a}a^{-\frac{n}{2}}\frac{\partial}{\partial\dot{a}}+\beta\frac{\partial}{\partial R}+\dot{\beta}\frac{\partial}{\partial\dot{R}}. \tag{27}$$

Since from (9) we have $p_R=0$, the corresponding constant of motion is $Q=\alpha p_a$, which yields

$$a^{-\frac{n-2}{2}}\dot{a}a^{n-2}=\text{const}\quad\Rightarrow\quad a(t)\sim t^{2/n}. \tag{28}$$

Although, this solution for the scale factor satisfies the field equation (4), it does not satisfy the Hamiltonian constraint (5). This is not surprising since it is well known that the flat FRW cosmology with Einstein–Hilbert action (without the cosmological constant) has no vacuum solution. Thus, we remove the case $f''=0$ from our consideration. When $f''\neq0$, from Eq. (21) we obtain

$$\frac{\partial\alpha}{\partial R}=0. \tag{29}$$

Under this condition Eq. (20) results in

$$\alpha(a)=\frac{af''R}{n(f-f'R)}\beta(a,R). \tag{30}$$

If one uses this expression in Eq. (22) to eliminate $\alpha(a)$, one obtains

$$\frac{\partial\beta}{\partial a}=\frac{n(n-1)f}{2a(f'R-nf)}\beta(a,R). \tag{31}$$

To solve this equation we assume that the function $\beta(a,R)$ can be written in the form $\beta(a,R)=A(a)B(R)$, where A and B are functions only of a and R, respectively. Substituting this ansatz for $\beta(a,R)$ into Eq. (31), we obtain

$$\frac{2a}{A}\frac{dA}{da}=\frac{n(n-1)f}{f'R-nf}. \tag{32}$$

Since the left-hand side of this equation is a function of a only while the right-hand side is a function of R, we should have

$$\frac{n(n-1)f}{f'R-nf}=c=\text{Const}, \tag{33}$$

which results in

$$f(R)=R^{\frac{n(n+c-1)}{c}}, \tag{34}$$

where the constant $c\neq-n$ is required to have $f''\neq0$. On the other hand Eq. (32), with its right hand-side equal to c, has the solution

$$\frac{2a}{a}\frac{dA}{da}=c\quad\Rightarrow\quad A(a)=a^{c/2}. \tag{35}$$

Now, using (34) and $\beta(a,R)=a^{c/2}B(R)$ in Eq. (30), we find

$$\alpha(a)=-\frac{n+c-1}{c}R^{-1}a^{\frac{c}{2}+1}B(R). \tag{36}$$

Since $\alpha(a)$ should be a function of a only, from the above expression for $\alpha(a)$ we conclude that $B(R)=R$ and thus obtain

$$\beta(a,R)=Ra^{c/2},\qquad\alpha(a)=-\frac{n+c-1}{c}a^{\frac{c}{2}+1}. \tag{37}$$

To determine the constant c, we note that relations (34) and (37) should satisfy Eq. (23). Thus, by substituting these results into Eq. (23) we find

$$c=-1-n, \tag{38}$$

which completes our solutions as

$$\alpha(a)=-\frac{2}{n+1}a^{-\frac{n-1}{2}},\qquad\beta(a,R)=Ra^{-\frac{n+1}{2}}, \tag{39}$$

and

$$f(R)=R^{\frac{2n}{n+1}}. \tag{40}$$

Therefore, in the context of $f(R)$ cosmology, a flat FRW metric has Noether symmetry if the corresponding action is given by Eq. (40) and the Noether symmetry is generated by the following vector field

$$X=-\frac{2}{n+1}a^{-\frac{n-1}{2}}\frac{\partial}{\partial a}+Ra^{-\frac{n+1}{2}}\frac{\partial}{\partial R}+\frac{n-1}{n+1}\dot{a}a^{-\frac{n+1}{2}}\frac{\partial}{\partial\dot{a}}$$
$$+\left(\dot{R}a^{-\frac{n+1}{2}}-\frac{n+1}{2}R\dot{a}a^{-\frac{n+3}{2}}\right)\frac{\partial}{\partial\dot{R}}. \tag{41}$$

To obtain the corresponding cosmology resulting from this type of $f(R)$, we note that the existence of Noether symmetry (41) implies the existence of a constant of motion $Q=\alpha p_a+\beta p_R$. Hence, using Eqs. (8) and (9) we have

$$Q=-\frac{4n^2(n-1)}{(n+1)^2}\dot{a}a^{\frac{n-3}{2}}R^{\frac{n-1}{n+1}}-\frac{8n^2(n-1)}{(n+1)^3}a^{\frac{n-1}{2}}\dot{R}R^{-\frac{2}{n+1}}, \tag{42}$$

which may be rewritten in the form

$$Q=-\frac{8n^2}{(n+1)^2}\frac{d}{dt}\left(a^{\frac{n-1}{2}}R^{\frac{n-1}{n+1}}\right), \tag{43}$$

and can be immediately integrated with the result, assuming $a(t=0)=0$

$$a^{\frac{n-1}{2}}R^{\frac{n-1}{n+1}}=-\frac{(n+1)^2}{8n^2}Qt. \tag{44}$$

On the other hand, the Hamiltonian constraint $\mathcal{H}=0$ with $\mathcal{H}$ given by (10) gives

$$2n^2\dot{a}^2R^{-1}+\frac{4n^2}{n+1}\dot{a}\dot{R}aR^{-2}=a^2. \tag{45}$$

To obtain the scale factor from the above relation, note that equation (44) yields

$$\dot{R}=\left[-\frac{(n+1)^2}{8n^2}Q\right]^{\frac{n+1}{n-1}}\left(\frac{n+1}{n-1}t^{\frac{2}{n-1}}a^{-\frac{n+1}{2}} - \frac{n+1}{2}t^{\frac{n+1}{n-1}}\dot{a}a^{-\frac{n+3}{2}}\right), \tag{46}$$

which, upon substitution into relation (45), gives

$$\frac{4n^2}{n-1}\left[-\frac{(n+1)^2}{8n^2}Q\right]^{-\frac{n+1}{n-1}}\dot{a}a^{\frac{n-1}{2}}=t^{\frac{2n}{n-1}}, \tag{47}$$

where, after integration we obtain

$$a(t)\sim t^{\frac{2(3n-1)}{n^2-1}}. \tag{48}$$

Therefore, in the context of our $f(R)=R^{\frac{2n}{n+1}}$ gravity, the universe evolves with a power law expansion (note that for any $n>1$ the power of the scale factor is positive). It is remarkable from (48) that the condition under which the universe would accelerate is $\frac{2(3n-1)}{n^2-1}>1$, that is, $n\leqslant 5$. This means that models with spatial dimension $n\leqslant 5$ obey an accelerated power law expansion while for $n>5$ a decelerated expansion occurs.

4. The equation of state parameter

One of the advantages of $f(R)$ theories of gravity is that they can describe and provide gravitational alternative for dark energy. Indeed, the equations of motion resulting from $f(R)$ gravity admit such solutions which predict the same accelerated expansion as those resulting from the usual Einstein–Hilbert gravity with dark energy. In other words, in the context of $f(R)$ cosmology one can describe the accelerated expansion of the universe without the necessity to introduce the exotic fluid with a negative equation of state (EoS) parameter w. To introduce an effective EoS parameter in our model, one may compare the field equations (4) and (5) with the usual field equations of a flat FRW universe filled with a perfect fluid with EoS $P=w\rho$, that is, with $3H^2\sim\rho$ and $-2\dot{H}-3H^2\sim P$. This comparison shows that in our model one may introduce the effective energy density and pressure as follows

$$\rho=\frac{1}{2}(Rf'-f)-nH\dot{R}f'', \tag{49}$$

$$P=f'''\dot{R}^2+(n-1)H\dot{R}f''+\ddot{R}f''+\frac{1}{2}(f-Rf'). \tag{50}$$

Therefore, we can define the effective EoS parameter w_{eff} as

$$w_{\text{eff}}=\frac{P}{\rho}=\frac{f'''\dot{R}^2+(n-1)H\dot{R}f''+\ddot{R}f''+\frac{1}{2}(f-Rf')}{\frac{1}{2}(Rf'-f)-nH\dot{R}f''}. \tag{51}$$

To transform the above expression to a more manageable form, consider the quantity $\mathcal{F}=f'(R)$ [8]. In terms of $\mathcal{F}$, Eq. (51) takes the form

$$w_{\text{eff}}=-1+\frac{2}{n(n-1)}\frac{\ddot{\mathcal{F}}-H\dot{\mathcal{F}}}{\mathcal{F}H^2}. \tag{52}$$

For the problem at hand, taking into account that

$$f'(R)=\frac{2n}{n+1}R^{\frac{n-1}{n+1}}, \tag{53}$$

and with the help of Eq. (44) we obtain

$$\mathcal{F}=-\frac{n+1}{4n}Qt^{-2\frac{n-1}{n+1}}. \tag{54}$$

Thus, substituting (54) and $H=\dot{a}/a$ from (48) we are led to the following EoS parameter

$$w_{\text{eff}}=-1+\frac{(n-1)(n+1)}{n(3n-1)}. \tag{55}$$

Now, it is easy to see that $-1<w_{\text{eff}}<0$, which is the characteristic of one type of dark energy, the so-called quintessence. Note that when w_{eff} is less than -1, the EoS describes another type of dark energy known as phantom. As is clear from Eq. (55), the effective EoS parameter is always greater than -1 and thus in the model under consideration here, the phantom phase cannot be accounted for.

5. Conclusions

In this Letter we have studied a generic $f(R)$ cosmological model with an eye to Noether symmetry. For the background geometry, we have considered a flat $(n+1)$-dimensional FRW metric and derived the general equations of motion in this background. The phase space was then constructed by taking the scale factor a and Ricci scalar R as the independent dynamical variables. The Lagrangian of the model in the configuration space spanned by $\{a,R\}$ is so constructed such that its variation with respect to these dynamical variables yields the correct field equations. The existence of Noether symmetry implies that the Lie derivative of this Lagrangian with respect to the infinitesimal generator of the desired symmetry vanishes. By applying this condition to the Lagrangian of the model, we have obtained the explicit form of the corresponding $f(R)$ function. We have shown that this form of $f(R)$ results in a power law expansion for the scale factor of the universe and the expansion is accelerating in the case of $n\leqslant 5$. We have also presented an effective EoS parameter for our $f(R)$ cosmology model. Our analysis shows that the EoS parameter is always greater than -1 and thus the so-called phantom dark energy cannot be described in this kind of modified gravity. On the other hand, the EoS parameter is restricted to the interval $-1<w_{\text{eff}}<0$. Since this interval is characteristic of the quintessence dark energy, our $f(R)$ model can provide a natural gravitational alternative for this kind of dark energy without the necessity to introduce an exotic fluid with a negative EoS parameter.

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