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Minimum sum edge colorings of multicycles

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ABSTRACT

In the minimum sum edge coloring problem, we aim to assign natural numbers to edges of a graph, so that adjacent edges receive different numbers, and the sum of the numbers assigned to the edges is minimum. The *chromatic edge strength* of a graph is the minimum number of colors required in a minimum sum edge coloring of this graph. We study the case of multicycles, defined as cycles with parallel edges, and give a closed-form expression for the chromatic edge strength of a multicycle, thereby extending a theorem due to Berge. It is shown that the minimum sum can be achieved with a number of colors equal to the chromatic index. We also propose simple algorithms for finding a minimum sum edge coloring of a multicycle. Finally, these results are generalized to a large family of minimum cost coloring problems.

1. Introduction

During a banquet, n people are sitting around a circular table. The table is large, and each participant can only talk to her/his left and right neighbors. For each pair of neighbors around the table, there is a given number of available discussion topics. If we suppose that each participant can only discuss one topic at a time, and that each topic takes an unsplittable unit amount of time, then what is the minimum duration of the banquet, after which all available topics have been discussed? What is the minimum average elapsed time before a topic is discussed?

In this paper, we show that there always exists a scheduling of the conversations such that these two minima are reached simultaneously. We also propose an algorithm for finding such a scheduling. This amounts to *coloring* edges of a *multicycle* on n vertices.

We first recall some standard definitions. Let $G = (V, E)$ be a finite undirected (multi)graph without loops. A *vertex coloring* of G is a mapping from V to a finite set of colors such that adjacent vertices are assigned different colors. The *chromatic number* $\chi(G)$ of G is the minimum number of colors that can be used in a coloring of G. An *edge coloring* of G is a mapping from E to a finite set of colors such that adjacent edges are assigned different colors. The minimum number of colors in an edge coloring of G is called the *chromatic index* $\chi'(G)$. From now on, we assume that colors are positive integers. The *vertex chromatic sum* of G is defined as $\Sigma(G) = \min\left\{\sum_{v\in V} f(v)\right\}$, where the minimum is taken over all colorings f of G. Similarly, the *edge chromatic sum* of G, denoted by $\Sigma'(G)$, is defined as $\Sigma'(G) = \min\left\{\sum_{e\in E} f(e)\right\}$, where the minimum is taken over all edge colorings. In both cases, a coloring yielding the chromatic sum is called a *minimum sum coloring*.

We also define the minimum number of colors needed in a minimum sum coloring of G. This number is called the *strength* $s(G)$ of the graph G in the case of vertex colorings, and the *edge strength* $s'(G)$ in the case of edge colorings. Clearly, $s(G) \geq \chi(G)$ and $s'(G) \geq \chi'(G)$.

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The chromatic sum is a useful notion in the context of parallel job scheduling. A *conflict graph* between jobs is a graph in which two jobs are adjacent if they share a resource, and therefore cannot be run in parallel. If each job takes one time unit, then a scheduling that minimizes the makespan is a coloring of the conflict graph with a minimum number of colors. On the other hand, a minimum sum coloring of the conflict graph corresponds to a scheduling that minimizes the *average* time before a job is completed. In our example above, jobs are conversations, resources are the banqueters, and the conflict graph is the line graph of a multicycle.

Previous results. Chromatic sums have been introduced by Kubicka in 1989 [1]. The computational complexity of determining the vertex chromatic sum of a simple graph has been studied extensively since then. It is NP-hard even when restricted to some classes of graphs for which finding the chromatic number is easy, such as bipartite or interval graphs [2,3]. A number of approximability results for various classes of graphs were obtained in the last ten years [4–7]. Similarly, computing the edge chromatic sum is NP-hard for bipartite graphs [8], even if the graph is also planar and has maximum degree 3 [9]. Hardness results were also given for the vertex and edge strength of a simple graph by Salavatipour [10], and Marx [11].

Some results concern the relations between the chromatic number $\chi(G)$ and the strength $s(G)$ of a graph. It has been known for long that the vertex strength can be arbitrarily larger than the chromatic number [12]. However, if G is a proper interval graph, then $s(G) = \chi(G)$ [13], and $s(G) \le \min\{n, 2\chi(G) - 1\}$ if G is an interval graph [14]. Hajiabolhassan, Mehrabadi, and Tusserkani [15] proved an analog of Brooks' theorem for the vertex strength of simple graphs: $s(G) \le \Delta(G)$ for every simple graph G that is neither an odd cycle nor a complete graph, where $\Delta(G)$ is the maximum degree in G.

Concerning the relation between the chromatic index and the edge strength, Mitchem, Morriss, and Schmeichel [16] proved an inequality similar to Vizing's theorem: $s'(G) \le \Delta(G) + 1$ for every simple graph G. Harary and Plantholt [17] have conjectured that $s'(G) = \chi'(G)$ for every simple graph G, but this was later disproved by Mitchem et al. [16], and Hajiabolhassan et al. [15].

Our results. We consider *multigraphs*, in which parallel edges are allowed. In Section 2 we prove that if G is a multicycle, that is, a cycle with parallel edges, then $s'(G) = \chi'(G)$. This statement extends a classical result from Berge.

In Section 3, we give an algorithm of complexity $O(\Delta n)$ for finding a minimum sum coloring of a multicycle G of order n and maximum degree Δ. This algorithm iteratively eliminates edges that will form the color class corresponding to the last color $s'(G)$. For the special case where n is even, we also give a more efficient $O(m)$-time algorithm based on the property of optimal colorings that the first color classes induce a uniform multicycle.

We conclude by generalizing our results to other objective functions.

2. Multicycles

The following well-known result has been proved by König in 1916.

Theorem 1 (*König's Theorem [18]*). *Let $G = (V, E)$ be a bipartite multigraph and let Δ denote its maximum degree. Then $\chi'(G) = \Delta$.*

Hajiabolhassan et al. [15] mention that $s'(G) = \chi'(G)$ for every bipartite graph G. In fact, by using the same technique as in the classical proof of König's theorem, it is easy to deduce that $s'(G) = \chi'(G)$ for every bipartite multigraph G.

Theorem 2. *Let $G = (V, E)$ be a bipartite multigraph and let Δ denote its maximum degree. Then $s'(G) = \chi'(G) = \Delta$.*

Multicycles are cycles in which we can have parallel edges between two consecutive vertices. We consider the chromatic edge strength of multicycles.

The chromatic edge strength $s'(G)$ of a graph G is bounded from below by both Δ and $\lceil \frac{m}{\tau} \rceil$, where Δ is the maximum degree in G and τ is the cardinality of a maximum matching in G. In this section, we show that the lower bound $\max\{\Delta, \lceil \frac{m}{\tau} \rceil\}$ is indeed tight for multicycles. We assume that the multiplicity of each edge in the multicycle is at least one, so that the size τ of a maximum matching is equal to $\lfloor n/2 \rfloor$.

We first give a closed-form expression for the chromatic index of multicycles.

Theorem 3 (*[19]*). *Let $G = (V, E)$ be a multicycle on n vertices with m edges and maximum degree Δ. Let τ denote the maximum cardinality of a matching in G. Then*

$$\chi'(G) = \begin{cases} \Delta, & \text{if } n \text{ is even,} \\ \max\left\{\Delta, \left\lceil \dfrac{m}{\tau} \right\rceil\right\}, & \text{if } n \text{ is odd.} \end{cases}$$

We now introduce some useful notations. Given C the set of colors used in an edge coloring of a multigraph G, we denote by C_x the subset of colors of C assigned to edges incident to vertex x of G. Given two colors α and β, we call a path an (α, β)-path if the colors of its edges alternate between α and β. We also denote by $d_G(x)$ the degree of vertex x in G. We now state our main result.

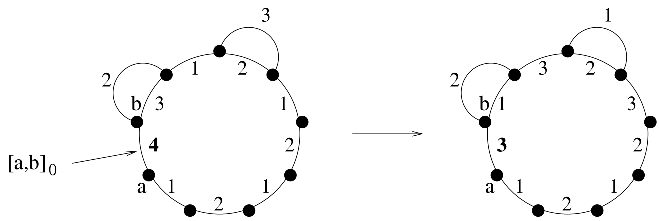


Fig. 1. An illustration of the case $\sigma \in C_b \setminus C_a$ in the proof of Theorem 4, on a multicycle G with $s' = \chi' = 3$. The edge $[a, b]_0$ is the only edge colored with color $s' + 1 = 4$. In this example, color $\sigma = 3$ and color $\alpha \in C_a \setminus C_b$ is equal to 1.

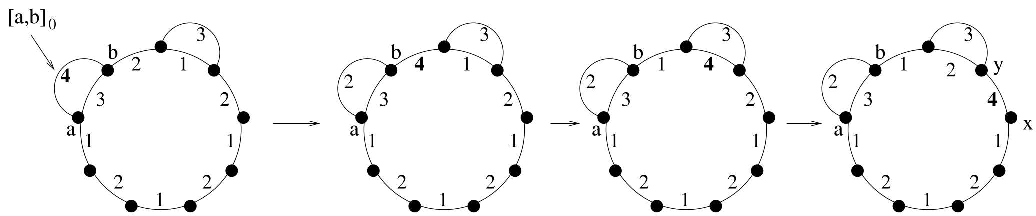


Fig. 2. An illustration of the case $\sigma \in C_a \cap C_b$ in the proof of Theorem 4, with $s' = \chi' = 3$. Again, $[a, b]_0$ is the only edge colored with color $s' + 1 = 4$, and $\sigma = 3$.

Theorem 4. *Let $G = (V, E)$ be a multicycle on n vertices with m edges and maximum degree Δ, and let τ denote the maximum cardinality of a matching in G. Then*

$$s'(G) = \chi'(G) = \begin{cases} \Delta, & \text{if } n \text{ is even,} \\ \max\left\{\Delta, \left\lceil \dfrac{m}{\tau} \right\rceil\right\}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even, then the result follows from Theorem 2. Thus, we assume that $n = 2k + 1$ for a positive integer k and let $s' = s'(G)$. Let $r = \max\left\{\Delta, \lceil \frac{m}{k} \rceil\right\}$. As $\tau = k$, then it is clear that $\chi'(G) = r$. Moreover, as $s' \geq \chi'(G)$ then, it suffices to prove that $s' \leq r$. Assume that $s' > r$ and G is a smallest counterexample. We claim that there exists a minimum sum edge coloring f of G in which there is only one edge colored with color s'. Otherwise, delete one of the edges with color s', say e. From the minimality of G, there exists a minimum sum edge coloring of $G \setminus e$ with χ' colors. Then we obtain the desired edge coloring of G by assigning the color $s' = \chi' + 1 = r + 1$ to e.

Let E_i denote the set of edges of G with color i and let $[a, b]_0$ be the only edge in G colored with color s'. Moreover, let $G' = G \setminus [a, b]_0$. By the minimality of G, we have that $s'(G') = \chi'(G') = \max\left\{\Delta', \lceil \frac{m-1}{k} \rceil\right\} \leq r$. Let $C = \{1, \ldots, r\}$. The following properties for the edge coloring of G' can be easily deduced:

(1) There exists a color $\sigma \in C$ such that $|E_\sigma| < k$.
(2) $|C_a \cup C_b| = r$.
(3) There exist at least two colors α and β in C such that $\alpha \in C_a \setminus C_b$ and $\beta \in C_b \setminus C_a$, with $\alpha \neq \beta$.

For (1), notice that if there is no color $\sigma \in C$ such that $|E_\sigma| < k$, then $m - 1 = \sum_{i=1}^{r} |E_i| = kr$, hence $r = \frac{m-1}{k} < \frac{m}{k}$, contradicting the definition of r. Property (2) holds, otherwise edge $[a, b]_0$ can be colored with a color in C which contradicts the fact that G' is a counterexample. Finally, notice that the degree of vertices a and b in G' is at most equal to $\Delta - 1$. Since $r \geq \Delta$, there is a color $\beta \notin C_a$ and a color $\alpha \notin C_b$ with $\alpha, \beta \in C$. Clearly $\alpha \neq \beta$, otherwise $[a, b]_0$ can be colored with such a color, contradicting the fact that G is a counterexample. Moreover, by (2), we have that $\alpha \in C_a \setminus C_b$ and $\beta \in C_b \setminus C_a$, which proves (3).

By Property (2), it is sufficient to analyze the cases $\sigma \in C_b \setminus C_a$ (or $\sigma \in C_a \setminus C_b$) and $\sigma \in C_a \cap C_b$. The two cases are illustrated on Figs. 1 and 2, respectively.

If $\sigma \in C_b \setminus C_a$ then, by Property (3), there exists a color $\alpha \in C_a \setminus C_b$ with $\alpha \neq \sigma$. Let $G(\alpha, \sigma)$ denote the subgraph of G' induced by the edges of color α and σ. Let $G_b(\alpha, \sigma)$ denote the connected component of $G(\alpha, \sigma)$ containing b. Clearly, $G_b(\alpha, \sigma)$ is a simple (σ, α)-path having b as last vertex and not containing vertex a, otherwise we have a contradiction to Property (1). Hence we can recolor the edges of the path $G_b(\alpha, \sigma)$ by swapping colors α and σ in such a way that $\sigma \notin C_b$. Since $\sigma \notin C_a$, we assign color σ to $[a, b]_0$, and obtain an edge coloring f'' of G using r colors. Fig. 1 provides an example of this case.

We now want to show that

$$\sum_{e \in E} f''(e) < \sum_{e \in E} f(e), \qquad (*)$$

contradicting $s'(G) > r$. If the length of the path $G_b(\alpha, \sigma)$ is even, then $\sum_{e \in E} f''(e) - \sum_{e \in E} f(e) = \sigma - r - 1 \leq r - r - 1 < 0$. If the length of the path $G_b(\alpha, \sigma)$ is odd, say $2s+1$, with $s \geq 0$, then the difference is $(\sigma + (s+1)\alpha + s\sigma) - (r + 1 + (s+1)\sigma + s\alpha) = \alpha - r - 1 \leq r - r - 1 < 0$. Thus, inequality $(*)$ always holds.

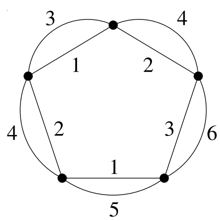


Fig. 3. An example for which the natural greedy algorithm fails. The graph is a 2-uniform 5-cycle, and has chromatic edge strength $\max\{\Delta, \lceil m/k\rceil\} = \max\{4, \lceil 10/2\rceil\} = 5$. However, iteratively removing maximum matchings can yield a 6-coloring.

The other case is when $\sigma \in C_a \cap C_b$. By Property (3), there exists a color $\beta \in C_b \setminus C_a$. Let us assume that vertices are ordered clockwise and let b be the clockwise vertex of edge $[a, b]_0$. Recolor edge $[a, b]_0$ with color β and the edge of color β incident to b with color $s' = r + 1$. This recoloring does change neither the value of the sum nor the number of colors. Let $[x, y]_0$ be the edge that is recolored with color s', with x being its counterclockwise vertex.

By Property (3) again, a color β_y such that $\beta_y \in C_y \setminus C_x$ exists. We can therefore repeat the above procedure until the edge $[x, y]_0$ is such that $\sigma \in C_x \setminus C_y$ or $\sigma \in C_y \setminus C_x$. This is always possible, because the cycle is odd, and $|E_\sigma| < k$; hence by moving around the cycle this way, we will eventually find an edge $[x, y]_0$ that is adjacent to only one edge of color σ. Assume, without loss of generality, that $\sigma \in C_y \setminus C_x$. Then letting $a = x$ and $b = y$ leads us back to the first case. Fig. 2 gives an example of this case. □

3. Algorithms

We now present algorithms for minimum sum coloring of multicycles. The complexity of our algorithms will be a function of both m and n. Hence if the input consists of the number of parallel edges between every pair of consecutive vertices, our algorithms will only be pseudopolynomial. This is natural since we expect a coloring to be represented by an encoding of size $\Theta(m)$.

The line graph of a multicycle is a proper circular arc graph. Hence the problem of coloring edges of multicycles is a special case of proper circular arc graph coloring. It is easy to realize that not all proper circular arc graphs are line graphs of multicycles, though. Proper circular arc graphs were shown by Orlin, Bonuccelli, and Bovet [20] to admit *equitable colorings*, that is, colorings in which the sizes of any two color classes differ by at most one, that only use χ colors. Therefore, a corollary of our results is that multicycles admit both equitable and minimum sum edge colorings with the same, minimum, number of colors, and that both types of colorings can be computed efficiently.

We first present a general algorithm, then focus on the case where n is even.

3.1. The general case

A natural idea for solving minimum cost coloring problems is to use a greedy algorithm that iteratively removes maximum independent sets (or maximum matchings in the case of edge coloring) [4,7]. It can be shown that this approach fails here (see Fig. 3). Instead we use an algorithm in which the smallest color class, corresponding to color s', is removed iteratively.

We first consider the case where $\lceil m/k\rceil \geq \Delta$ and k divides m. Then the number of colors must be equal to m/k. But since each color class can contain at most k edges, every color class in a minimum sum coloring must have size exactly k. Such a coloring can be easily found in linear time by a sweeping algorithm that assigns each color $i \bmod \chi'$ in turn. This is a special case of the algorithm of Orlin et al. (Lemma 2, [20]) for circular arc graph coloring. In the remainder of this section, we refer to this case as the "easy case".

Algorithm (MulticycleColor)**.**

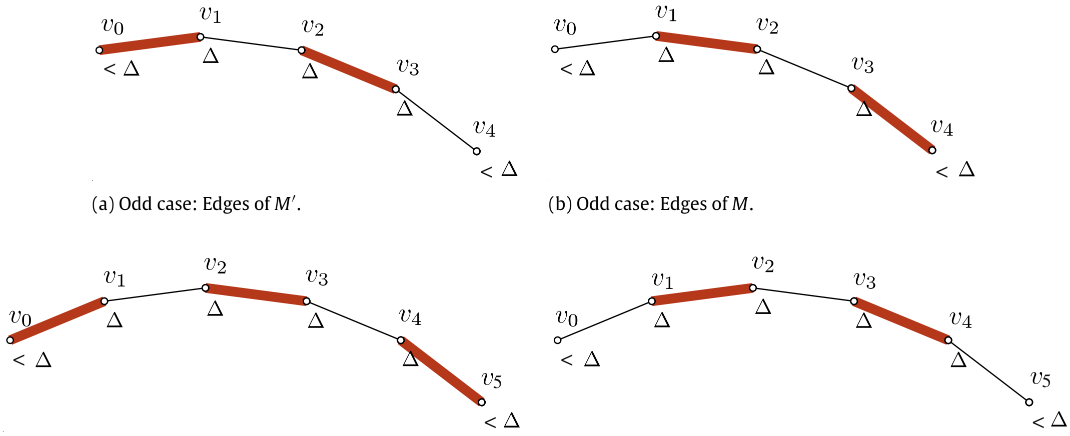
1. $i \leftarrow s'(G), G_i \leftarrow G$
2. **if** $\lceil |E(G_i)|/k\rceil \geq \Delta(G_i)$ and k divides $|E(G_i)|$ **then** apply the "easy case" algorithm and terminate
3. **else**
   (a) Find a matching M of minimum size such that $s'(G_i \setminus M) = s'(G_i) - 1$
   (b) color the edges of M with color i
   (c) $G_{i-1} \leftarrow G_i \setminus M, i \leftarrow i - 1$
   (d) **if** $G_i \neq \emptyset$ **then** go to step 2.

The correctness of the algorithm relies on the following lemma.

Lemma 1. *Given a matching M in a multicycle G such that*

1. $s'(G \setminus M) = s'(G) - 1$,
2. *M has minimum size among all matchings satisfying condition 1,*

there exists a minimum sum edge coloring of G such that M is the set of edges colored with color $s'(G)$.



(a) Odd case: Edges of M'.

(b) Odd case: Edges of M.

(c) Even case: Edges of M'.

(d) Even case: Edges of M.

Fig. 4. Illustration of the proof of Lemma 1.

Proof. We distinguish three cases, (a), (b), and (c), depending on the relative values of $\lceil m/k \rceil$ and Δ.

Case (a) We first assume that $\lceil m/k \rceil > \Delta$ and k does not divide m, thus $m = \lfloor m/k \rfloor \cdot k + q$, with $q > 0$. In that case, M has size exactly q. To find a minimum sum coloring, we color the edges of M with color $\lceil m/k \rceil$. The remaining edges are colored using the "easy case" algorithm, which applies since $\lfloor m/k \rfloor \geq \Delta$ and the number of remaining edges is a multiple of k. This coloring must have minimum sum, because only one color class has not size k.

Case (b) When $\Delta > \lceil m/k \rceil$, we have $s'(G) = \Delta$ from Theorem 4. We claim that in that case, M is a minimum matching that hits all vertices of degree Δ. To prove this, suppose otherwise. Then $(G \setminus M)$ has maximum degree Δ, and thus from Theorem 4, $s'(G \setminus M) = s'(G)$, contradicting condition 1. Now we have to ensure that there exists a minimum sum coloring such that M is the color class $s'(G) = \Delta$.

We consider a minimum sum coloring and the color class Δ in this coloring. This class, say M', must also be a matching hitting all vertices of degree Δ. We now describe a recoloring algorithm that, starting with this coloring, produces a coloring whose sum is not greater and whose color class Δ is exactly M. We define a *block* as a maximal sequence of adjacent vertices of degree Δ. The algorithm examines each block, and shifts the edges of M' if they do not match with those of M. Two cases can occur, depending on the parity of the block length.

The first case is when a block contains an odd number of vertices of degree Δ, say $v_1, v_2, \ldots, v_{2t+1}$ for some integer t. In that case, the only way in which M and M' can disagree is, without loss of generality, when M' contains edges of the form $v_0v_1, v_2v_3, \ldots, v_{2t}v_{2t+1}$, while M contains $v_1v_2, v_3v_4, \ldots, v_{2t+1}v_{2t+2}$ (see Fig. 4(a)–Fig. 4(b)), where v_0 and v_{2t+2} are the predecessor of v_1 and the successor of v_{2t+1}, respectively. Since the degree of v_0 is, by definition of a block, strictly less than Δ, there must exist a color $\alpha \in C_{v_1} \setminus C_{v_0}$. Furthermore, since all vertices within the block have degree Δ, the color class for color α contains $t+1$ edges of the form $v_{2i+1}v_{2i+2}$ for $0 \leq i \leq t$. Hence we can recolor the edges of M' of the form $v_{2i}v_{2i+1}$ for $0 \leq i \leq t$ with color α, and the $t+1$ edges of color α within the block with color Δ. Note that at this point, the coloring might be not proper anymore, as two edges colored Δ might be incident to v_{2t+2}.

The other case is when a block contains an even number of vertices of degree Δ, say $v_1, v_2, \ldots, v_{2t}$ for some integer t. In that case, since M is minimum, it contains edges of the form $v_1v_2, v_3v_4, \ldots, v_{2t-1}v_{2t}$. The only way in which M' can disagree with M is by containing edges $v_0v_1, v_2v_3, \ldots, v_{2t}v_{2t+1}$ (see Fig. 4(c)–Fig. 4(d)). Like in the previous case, there must be a color $\alpha \notin C_{v_0}$, so we can recolor the edges of M' of the form $v_{2i}v_{2i+1}$ for $0 \leq i < t$ with color α, and the edges of color α within the block with color Δ. Note that the edge $v_{2t}v_{2t+1}$ of color Δ has not been recolored, thus $v_{2t-1}v_{2t}$ and $v_{2t}v_{2t+1}$ both have color Δ, and at this point the coloring is not proper anymore.

We proceed in this way for each block. Notice that the sum of the coloring is unaltered, and that the set of edges of color Δ is now a superset of M. Also, while G is not necessarily properly colored anymore, the graph $G \setminus M$ is properly colored with at most Δ colors. But since removing M decreases the strength, we know that we can recolor $G \setminus M$ with $\Delta - 1$ colors without increasing the sum. Doing that and gluing back the edges of M colored with color Δ, we obtain a minimum sum coloring where only the edges of M have color Δ, as claimed.

Case (c) Finally, in the case where $\lceil m/k \rceil = \Delta$, with $m = \lfloor m/k \rfloor \cdot k + q$, the matching M consists of at least q edges that together hit all vertices of degree Δ. If M has exactly size q, then case (a) above applies, since we know that by removing M, we also decrease the maximum degree. Otherwise case (b) applies. □

We have to make sure that the main step of the algorithm can be implemented efficiently.

Lemma 2. *Finding a matching M in a multicycle G such that $s'(G \setminus M) = s'(G) - 1$ and M has minimum size can be done in $O(n)$ time.*

Proof. The three cases of the previous proof must be checked. In the case where $\lceil m/k \rceil > \Delta$ and $m = \lfloor m/k \rfloor \cdot k + q$, we can pick any matching of size q, which can clearly be done in linear time. In the second case, when $\lceil m/k \rceil < \Delta$, we need to find a minimum matching hitting all vertices of degree Δ. This can be achieved in linear time as well by proceeding in a clockwise greedy fashion.

Finally, in the last case, we need to find a minimum set of at least q edges that together hit all vertices of degree Δ. This can also be achieved in $O(n)$ time as follows. We first find the minimum matching hitting all maximum degree vertices. If the resulting matching has size at least q, then we are done and back to the previous case. Otherwise, we need to include additional edges. For that purpose, we can proceed in the clockwise direction and iteratively extend each block in order to include the exact number of additional edges. This can take linear time as well if we took care to count the size of each block and of the gaps between them in the previous pass. □

Theorem 5. *Algorithm* MULTICYCLECOLOR *finds a minimum sum coloring of a multicycle on n vertices and with maximum degree Δ in time $O(\Delta n)$.*

Proof. The number of iterations of the algorithm is at most $s'(G) = \max\{\lceil m/k \rceil, \Delta\}$. Hence the running time is $O(\max\{\lceil m/k \rceil n, \Delta n\}) = O(\max\{m, \Delta n\}) = O(\Delta n)$. □

We deliberately ignored the situation in which after some iterations, the multicyle G_i does not contain a full cycle anymore, that is, one of the edge multiplicity m_i drops to 0. We are then left with a collection of disjoint *multipaths*, for which the minimum sum coloring problem becomes easier. This special case is described in the following section.

3.2. A linear time algorithm for even length multicycles

We turn to the special case $n = 2k$, that is, the number of vertices is even. We show that in that case, minimum sum colorings have a convenient property that can be exploited in a fast algorithm. This algorithm first colors a uniform multicycle contained in G such that the remaining edges of G form a (possibly unconnected) multipath. This multipath is then colored separately.

We begin this section by the following result on multipaths. We consider multipaths with vertices labelled $\{1, 2, \dots, n\}$, such that edges are only between vertices of the form $i, i + 1$.

Lemma 3. *There always exists a minimum sum edge coloring of a multipath H, such that its color classes E_i are maximum matchings in the graphs $H_i = H \setminus \cup_{j=1}^{i-1} E_j$; furthermore these matchings contain all the edges appearing in odd position from left to right in each connected component of H_i.*

Proof. Suppose H has been colored optimally. We want to transform such an edge coloring into another one that verifies the hypothesis of the theorem. Let i be the minimum positive integer for which E_i does not verify the hypothesis. Note that the color of every edge in H_i is at least i.

We can assume that H_i is connected, the following reasoning being applicable to each connected component. We first remark that E_i is a maximal matching, otherwise one edge can be recolored with color i, contradicting the optimality of the given edge coloring. Thus E_i can be partitioned into *blocks*, defined as maximal sequences of consecutive vertices $\{y_1, y_2, \dots, y_{2t}\}$ such that one of the edges between y_{2j-1} and y_{2j} has color i, for $1 \leq j \leq t$. Two consecutive blocks are separated by a single vertex whose incident edges have colors strictly greater than i. We now show that if a block starts at an even vertex, it can be recolored without decreasing the color sum. We let y_0 be the vertex preceding y_1.

Recoloring: Let $\alpha_1 \neq i$ be any color appearing on the edges between y_0 and y_1. We recolor an edge of color α_1 with color i and color the edge between vertices y_1 and y_2 of color i with color α_1. Now, for each j, with $1 < j \leq t$, we recolor the edge $y_{2j-1}y_{2j}$ of color i with a color α_j appearing on the edges $y_{2j-2}y_{2j-1}$, and color the edge $y_{2j-2}y_{2j-1}$ of color α_j with color i. The color α_j is chosen such that $\alpha_j = \alpha_{j-1}$ if color α_{j-1} appears on edges $y_{2j-2}y_{2j-1}$, and it is any color appearing on edges $y_{2j-2}y_{2j-1}$ otherwise. At the end, we have two cases. Either y_{2t} is the last vertex of the path, and we are done, or there exist edges between y_{2t} and, say, y_{2t+1}. One of these edges may be of color α_t, and can be recolored with color i. Otherwise, any such edge can be recolored with color i. Since, by definition, y_{2t+1} was not incident to any edge with color i, this yields a proper coloring whose sum is not greater than the original one.

Now, it is clear that we can assume that every block starts at an odd vertex. This implies that there is only one block. Furthermore, this block must start with the first vertex of the path. Hence E_i is a maximum matching containing all the edges appearing in odd position. □

From Lemma 3, we can deduce the following result, that settles the case of multipaths.

Theorem 6. *The greedy algorithm that iteratively picks a maximum matching formed by all edges appearing in odd position in each connected component of a multipath H, computes a minimum edge sum coloring of H in time $O(m)$.*

We now consider the case of even multicycles. We assume that the vertices in the multicycle G on $n = 2k$ vertices are labelled clockwise with integers $0, 1, \dots, n - 1$, and arithmetic operations are taken modulo n. For each $0 \leq i < n$, let m_i denote the number of parallel edges between two consecutive vertices i and $i + 1$ in G. Let p be a positive integer. A multicycle G with $m = pn$ edges is called *p-uniform* if $m_i = p$ for every i such that $0 \leq i < n$.

Lemma 4. *Let G be a multicycle of even length and let $p = \min_i m_i$. Let f be any minimum sum edge coloring of G. Then, f can be transformed into another minimum sum edge coloring f' such that the first $2p$ color classes E_i induced by f', with $1 \leq i \leq 2p$, are such that $|E_i| = k$ and their union induces a p-uniform multicycle.*

Proof. Let G be a multicycle on n vertices, with $n = 2k$ for some integer $k > 1$. Let f be any minimum sum edge coloring of G. Clearly, as f is minimum, we have that $|E_1| \geq |E_2| \geq \cdots \geq |E_{\chi'}|$. Let us consider the following claim.

Claim 1. *The coloring f can be transformed into a minimum sum edge coloring f' having the property that the edges colored with colors 1 and 2 induce a subgraph of G isomorphic to a cycle.*

Notice that, by using Claim 1, the lemma follows directly by induction on p. So, in order to prove Claim 1, first notice that, by using a similar recoloring argument as in the proof of Lemma 3, we can deduce that $|E_1| = k$.

Now, without loss of generality, assume that f is such that there is an edge colored with color 1 between vertices $2j$ and $2j + 1$ for each j with $0 \leq j < k$. Moreover, let $c \geq 2$ be the minimum color appearing on the edges between vertices $2j + 1$ and $2j + 2$, for all $0 \leq j < k$.

Suppose that there exists a maximal sequence $i_1, \ldots, i_{2t}$ of consecutive vertices in G, such that colors 1 and c belong to the set of colors assigned by f to the edges between vertices i_{2q-1} and i_{2q}, with $1 \leq q \leq t$. Then by using the same recoloring argument as in the proof of Lemma 3, we can move color c in order to transform such a sequence into a $(c, 1)$-path. Moreover, again by using the same recoloring argument as in the proof of Lemma 3, we can deduce that $|E_c| = k$.

So, if $c = 2$ we are done, otherwise, we can swap the colors 2 and c so that $|E_2| = k$ and $E_1 \cup E_2$ induce a cycle. □

Theorem 7. *There exists an $O(m)$-time algorithm for computing a minimum sum edge coloring of a multicycle G of even length with m edges.*

Proof. Let $n = 2k$ be the number of vertices in G and let $p = \min_i\{m_i\}$, for $0 \leq i < n$. For each $0 \leq j < k$, assign to p edges between vertices $2j$ and $2j + 1$ the odd colors $1, 3, \ldots, 2p - 1$ and assign to p edges between vertices $2j + 1$ and $2j + 2$ the even colors $2, 4, \ldots, 2p$.

The previous pn colored edges induce a subgraph of G isomorphic to a p-uniform multicycle. When removing this p-uniform multicycle from G, we obtain a multipath or a set of disjoint multipaths, the edges of which can be colored with colors in $\{2p + 1, \ldots, s'(G)\}$, from Theorem 6.

Such a coloring can be computed in $O(m)$ time, and by Lemmas 3 and 4, it is a minimum sum edge coloring of G. □

4. Conclusion

The question of whether minimum sum colorings always use a minimum number of colors, as is the case for multicycles, can be asked for other classes of graphs or multigraphs. We can also consider other types of colorings. In this conclusion, we outline a generalization of our results to a large family of coloring problems.

In the generalized optimal cost chromatic partition problem [21], each color has an integer cost, but this cost is not necessarily equal to the color itself. The cost of a vertex coloring is $\sum_{v \in V} c(f(v))$, where $c(i)$ is the cost of color i. For any set of costs, our proofs can be generalized to show that on the one hand, the minimum number of colors needed in a minimum cost edge coloring of G is equal to $\chi'(G)$ when G is bipartite or a multicycle, and on the other hand that a minimum cost coloring can be computed in $O(\Delta n)$ time for multicycles.

In fact, our results can be generalized to an even broader class of edge coloring problems. Given an edge coloring $f : E \mapsto \mathbb{N}$, we define a cost $C(f)$ of the form:

$$C(f) = \sum_i c(i, |f^{-1}(i)|),$$

where $c : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$ is a real function of a color i and an integer k, and $f^{-1}(i)$ is the set of edges e such that $f(e) = i$. Hence the cost to minimize is a sum of the cost of each color class, itself defined as some function of the color and the size of the color class.

In the minimum sum coloring problem, the function c is defined by

$$c(i, k) = i \cdot k.$$

We further suppose that the functions $c(i, k)$ satisfy the following property:

Given two nonincreasing integer sequences $a_1 \geq a_2 \ldots \geq a_n$ and $b_1 \geq b_2 \ldots \geq b_n$ such that

$$\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i, \quad \forall j = 1, \ldots, n,$$

we have

$$\sum_{i=1}^{n} c(i, a_i) \leq \sum_{i=1}^{n} c(i, b_i). \tag{1}$$

This property clearly holds in the minimum sum coloring problem. It formalizes the fact that when minimizing the cost $C(f)$, we are looking for a distribution of the color class sizes that is as nonuniform as possible. In particular, when an element (edge or vertex) in a color class i is recolored with a color $j < i$, whose class is larger, then the objective function decreases. This is the argument that we implicitly used in our proof of Theorem 4. It is also the argument that ensures the correctness of the algorithms.

Property (1) can also be shown to hold (see [22]) when the following two conditions are satisfied:

1. $c(i, k) = c(j, k)\ \forall i, j$, that is, when the cost of a class only depends on its size, in which case we will say that the functions are *separable*,
2. the functions $c(i, k) = c(k)$ are concave.

This is the case for instance in the minimum entropy edge coloring problem [23], for which $c(k) = -\frac{k}{m} \log \frac{k}{m}$. A number of other coloring problems falling in that class were recently studied by Fukunaga, Halldórsson, and Nagamochi [22].

For all minimum cost edge coloring problems whose objective function satisfies (1), all our results apply. In fact, the colorings that we compute are *robust* colorings, in the sense that they minimize every objective function satisfying the above property.

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