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Generalized L-KKM Type Theorems in L-Convex Spaces with Applications

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Abstract—In this paper, a new notion of generalized L-KKM mappings is introduced. Some generalized L-KKM type theorems for generalized L-KKM mappings with finitely closed (or compactly closed) values and with finitely open (or compactly open) values are established in L-convex spaces. As applications, some Ky Fan type matching theorems, fixed-point theorems and minimax inequality are obtained in L-convex spaces. These results generalize a number of known results in recent literature.

Keywords—L-convex space, Hyperconvex metric space, Finitely closed (open) set, Compactly closed (open) set, Generalized L-KKM mapping.

1. INTRODUCTION AND PRELIMINARIES

Let X and Y be two nonempty sets. We denote by 2^Y and $\mathcal{F}(X)$ the family of all nonempty subsets of Y and the family of all nonempty finite subsets of X, respectively. For any $A \in \mathcal{F}(X)$, we denote by $|A|$ the cardinality of A. Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \dots, e_n$. If J is a nonempty subset of $\{0, 1, \dots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. A topological space X is said to be contractible if the identity mapping I_X of X is homotopic to a constant function.

The notion of a L-convex space was introduced by Ben-El-Mechaiekh *et al.* [1]. An L-convexity structure on a topological space X is given by a nonempty set-valued mapping $\Gamma : \mathcal{F}(X) \to 2^X$ satisfying the following condition.

(1) For each $A \in \mathcal{F}(X)$ with $|A| = n + 1$, there exists a continuous mapping $\phi_A : \Delta_n \to \Gamma(A)$ such that $B \in \mathcal{F}(A)$ with $|B| = J + 1$, implies $\phi_A(\Delta_J) \subset \Gamma(B)$, where Δ_J denotes the face of Δ_n corresponding $B \in \mathcal{F}(A)$. The pair (X, Γ) is then called an L-convex space. A subset D of X is said to be L-convex if for each $A \in \mathcal{F}(D)$, $\Gamma(A) \subset D$. It is clear that each L-convex subset of a L-convex space is also a L-convex space. D is said to be finitely closed (or finitely open) in X if for each $A \in \mathcal{F}(X)$, $D \cap \Gamma(A)$ is closed (or open) in $\Gamma(A)$. D is said to be compactly closed (or compactly open) in X if for each compact subset K of X, $D \cap K$ is closed (or open) in K.

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If an L-convex space (X, Γ) satisfies the additional condition

(2) for each $A, B \in \mathcal{F}(X)$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$,

then (X, Γ) is called by Park and Kim [2,3] a generalized convex (or G-convex) space.

If an L-convex space (X, Γ) satisfies the additional condition

(3) for each $A, B \in \mathcal{F}(X)$, there exists $A_1 \subset A$ such that $A_1 \subset B$ implies $\Gamma(A_1) \subset \Gamma(B)$.

Then (X, Γ) is called by Verma [4,5] a generalized H-space (or G-H-space).

The following notion of H-space, which were introduced by Bardaro and Ceppitelli [6], were motivated by the earlier works of Horvath [7]. A pair (X, Γ) is said to be a H-space if X is a topological space and $\Gamma : \mathcal{F}(X) \to 2^X$ has contractible values such that for any $A, A' \in \mathcal{F}(X)$, $A \subset A'$ implies $\Gamma_A \subset \Gamma_{A'}$.

By Lemma 1 of [8], which is contained in the proof of Theorem 1 of [7], each H-space must be a G-convex space. By the above definitions, it is clear that L-convex space includes G-convex space, G-H-spacev and H-space as very special cases.

In this paper, we first introduce a new notion of generalized L-KKM (in short, $GLKKM$) mappings. Then, some new $GLKKM$ type theorems for set-valued mappings with finitely closed (or compactly closed) values and with finitely open (or compactly open) values are established in L-convex spaces. As applications of our results, some new Ky Fan type matching theorems, fixed-point theorems, and minimax inequality are obtained in L-convex spaces. These theorems unify and generalize many known results in literature.

DEFINITION 1.1. *Let X be nonempty set and (Y, Γ) be a L-convex space. A set-valued mapping $G : X \to 2^Y$ is said to be a $GLKKM$ mapping if for each nonempty finite set $\{x_0, \dots, x_n\} \in \mathcal{F}(X)$ there exists $\{y_0, \dots, y_n\} \in \mathcal{F}(Y)$ (not necessary all different) such that for any nonempty subset $\{y_{i_0}, \dots, y_{i_k}\} \subset \{y_0, \dots, y_n\}$, $0 \le k \le n$, we have*

$$\Gamma(\{y_{i_j} : j = 0, \dots, k\}) \subset \bigcup_{j=0}^{k} G(x_{i_j}).$$

When (Y, Γ) is a H-space or G-convex space or G-H-space, the above definition were given by Chang and Ma [9], Ding [10], Tan [11], and Verma [4,5], respectively.

2. *GLKKM* TYPE THEOREMS

THEOREM 2.1. *Let X be a nonempty set, (Y, Γ) be a L-convex space and $G : X \to 2^Y$ be a set-valued mapping with finitely closed (or, compactly closed) values.*

(1) *If G is a $GLKKM$ mapping, then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

(2) *If the family $\{G(x) : x \in X\}$ has the finite intersection property and $\Gamma(\{y\}) = \{y\}$ for each $y \in Y$, then G is a $GLKKM$ mapping.*

PROOF.

(1) Suppose that the family $\{G(x) : x \in X\}$ has not the finite intersection property. Then there exists a finite set $A = \{x_0, \dots, x_n\} \in \mathcal{F}(X)$ such that $\bigcap_{i=0}^n G(x_i) = \emptyset$. It follows that

$$Y = Y \setminus \bigcap_{i=0}^{n} G(x_i) = \bigcup_{i=0}^{n} (Y \setminus G(x_i)). \tag{2.1}$$

Since G is a $GLKKM$ mapping, there exists $B = \{y_0, \dots, y_n\} \in \mathcal{F}(Y)$ such that for each nonempty subset $\{y_{i_j} : j = 0, \dots, k\} \subset \{y_0, \dots, y_n\}$, we have

$$\Gamma(\{y_{i_j} : j = 0, \dots, k\}) \subset \bigcup_{j=0}^{k} G(x_{i_j}).$$

Since (Y, Γ) is a L-convex space, there exists a continuous mapping $\phi_B : \Delta_n \to \Gamma(B) \subset Y$ such that $\phi_B(\Delta_k) \subset \Gamma(\{y_{i_j} : j = 0, \dots, k\})$ for any nonempty subset $\{y_{i_j} : j = 0, \dots, k\} \subset \{y_0, \dots, y_n\}$. Since $\phi_B(\Delta_n)$ is a compact subset in $\Gamma(B)$, if G has finitely closed values, then for each $x \in X$, $G(x) \cap \Gamma(B)$ is closed in $\Gamma(B)$, and hence, $\phi_B(\Delta_n) \cap G(x) = \phi_B(\Delta_n) \cap G(x) \cap \Gamma(B)$ is closed in $\phi_B(\Delta_n)$ for each $x \in X$; if G has compactly closed values, then it is clear that $\phi_B(\Delta_n) \cap G(x)$ is closed in $\phi_B(\Delta_n)$. It follows from (2.1) that:

$$\phi_B(\Delta_n) = \bigcup_{i=0}^{n} [\phi_B(\Delta_n) \setminus (\phi_B(\Delta_n) \cap G(x_i))] ,$$

and hence, in the two cases, $\{\phi_B(\Delta_n) \setminus (\phi_B(\Delta_n) \cap G(x_i))\}_{i=0}^n$ is a open cover of $\phi_B(\Delta_n)$. Let $\{\psi_i\}_{i=0}^n$ be the continuous partition of unity subordinated to the open convering, then we have that for each $i \in \{0, \dots, n\}$ and $y \in \phi_B(\Delta_n)$,

$$\psi_i(y) \neq 0 \Leftrightarrow y \in \phi_B(\Delta_n) \setminus (\phi_B(\Delta_n) \cap G(x_i)) . \tag{2.2}$$

Define a mapping $\psi : \phi_B(\Delta_n) \to \Delta_n$ by $\psi(y) = \sum_{i=0}^n \psi_i(y) e_i$, $\forall y \in \phi_B(\Delta_n)$. Obviously, $\psi \circ \phi_B : \Delta_n \to \Delta_n$ is continuous. By the Brouwer fixed-point theorem, there exists a point $z_0 \in \Delta_n$ such that $z_0 = \psi \circ \phi_B(z_0)$. Let $u_0 = \phi_B(z_0)$, then we have

$$u_0 = \phi_B(z_0) = \phi_B \circ \psi \circ \phi_B(z_0) = \phi_B \circ \psi(u_0) \tag{2.3}$$

and

$$\psi(u_0) = \sum_{j \in J(u_0)} \psi_j(u_0) e_j \in \Delta_J(u_0), \tag{2.4}$$

where $J(u_0) = \{j \in \{0, 1, \dots, n\} : \psi_j(u_0) \neq 0\}$. It follows from (2.2) that $u_0 \in \phi_B(\Delta_n) \setminus (\phi_B(\Delta_n) \cap G(x_j))$ for all $j \in J(u_0)$. Then we have $u_0 \notin G(x_j)$ for all $j \in J(u_0)$. On the other hand, it follows from (2.3) and (2.4) that:

$$u_0 = \phi_B \circ \psi(u_0) \subset \phi_B(\Delta_J(u_0)) \subset \Gamma(\{y_j : j \in J(u_0)\}) \subset \bigcup_{j \in J(u_0)} G(x_j).$$

Thus, there exists $j_0 \in J(u_0)$ such that $u_0 \in G(x_{j_0})$ which contradicts the fact that $u_0 \notin G(x_j)$ for all $j \in J(u_0)$. Therefore, the family $\{G(x) : x \in X\}$ has the finite intersection property.

(2) Suppose that the family $\{G(x) : x \in X\}$ has the finite intersection property and $\Gamma(\{y\}) = \{y\}$ for each $y \in Y$. Then for any $\{x_0, \dots, x_n\} \in \mathcal{F}(X)$, $\bigcap_{i=0}^n G(x_i) \neq \emptyset$. Take any point $y^* \in \bigcap_{i=0}^n G(x_i)$, and let $y_i = y^*$ for $i = 0, \dots n$. Then for any $\{y_{i_j} : j = 0, \dots, k\} \subset \{y_0, \dots, y_n\}$ with $0 \le k \le n$, we have $\Gamma(\{y_{i_j} : j = 0, \dots, k\}) = \Gamma(\{y^*\}) = \{y^*\} \subset \bigcup_{j=0}^k G(x_{i_j})$. Therefore, G is a $GLKKM$ mapping.

REMARK 2.1. Theorem 2.1 generalizes Theorem 3.1 of [10] and Theorem 2.3 of [11] in the following ways:

(1) from G-convex space to L-convex space,
(2) the class of $GLKKM$ mappings includes the class of G-KKM mappings defined in [11] as a proper subclass,
(3) the compactness assumption of G-co(A) for each $A \in \mathcal{F}(Y)$ is dropped.

Obviously, Theorem 2.1 also includes Lemma 2.1 of Verma [4], Theorem 1 of [9] and Lemma 2 of [12] as very special case. If (Y, d) is a hyperconvex metric space (see, [13, p. 159]), define $\Gamma : \mathcal{F}(Y) \to 2^Y$ by $\Gamma(A) = \text{co}(A)$ for each $A \in \mathcal{F}(Y)$ where $\text{co}(A) = \cap\{B \subset Y : B$ is a closed ball containing $A\}$, then (Y, Γ) is a H-space with $\Gamma(\{y\}) = \{y\}$ for each $y \in Y$

(see [13, p. 161]), and hence, (Y, Γ) must be a L-convex space. Therefore, Theorem 2.1 also generalizes Theorem 3 of [14] and Theorem 2.1 of [15] in the following aspects:

(a) from hyperconvex space to L-convex space;
(b) from $GMKKM$ mapping to $GLKKM$ mapping;
(c) from the class of set-valued mappings with finitely metrically closed values to the class of set-valued mappings with finitely closed (or compactly closed) values.

THEOREM 2.2. *Let X be a nonempty set, (Y, Γ) be a L-convex space and $G : X \to 2^Y$ be a set-valued mapping with compactly closed values and $\bigcap_{x \in M} G(x)$ is compact for some $M \in \mathcal{F}(X)$. Then*

(1) *if G is a $GLKKM$ mapping, then $\bigcap_{x \in X} G(x) \neq \emptyset$. (2) if $\bigcap_{x \in X} G(x) \neq \emptyset$ and $\Gamma(\{y\}) = \{y\}$ for each $y \in Y$, then G is a $GLKKM$ mapping.*

PROOF. It is easy to see that conclusions (1) and (2) follow from Theorem 2.1.

REMARK 2.2. Theorem 2.2 generalizes Theorem 2.3 of [11] and Theorem 2.1 of [4] to L-convex spaces under weaker assumptions. Theorem 2.2 also includes Corollary 3.1 of [10], Lemma 3 of [12], and Theorem 2.11.8 of [13] as very special case.

THEOREM 2.3. *Let X be a nonempty set, (Y, Γ) be a L-convex space and $G : X \to 2^Y$ be a set-valued mapping with nonempty finitely open (or, compactly open) values.*

(1) *If G is a $GLKKM$ mapping, then the family $\{G(x) : x \in X\}$ has the finite intersection property,*
(2) *if the family $\{G(x) : x \in X\}$ has the finite intersection property and $\Gamma(\{y\}) = \{y\}$ for each $y \in Y$, then G is a $GLKKM$ mapping.*

PROOF.

(1) If the family $\{G(x) : x \in X\}$ has not the finite intersection property, then there exists $\{x_0, x_1, \ldots, x_n\} \in \mathcal{F}(X)$ such that $\bigcap_{i=0}^n G(x_i) = \emptyset$. Since G is a $GLKKM$ mapping, there exists $A = \{y_0, y_1, \cdots y_n\} \in \mathcal{F}(Y)$ such that for each nonempty subset $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subset A$ with $0 \leq k \leq n$, we have

$$\Gamma(\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\}) \subset \bigcup_{j=0}^{k} G(x_{i_j}). \tag{2.5}$$

Since (Y, Γ) be a L-convex space, there exists a continuous mapping $\phi_A : \Delta_n \to \Gamma(A)$ such that for each $\{y_{i_j} : j = 0, \ldots, k\} \subset \{y_1, \ldots, y_n\}$ with $0 \leq k \leq n$, $\phi_A(\Delta_k) \subset \Gamma(\{y_{i_j} : j = 0, \ldots, k\})$. From $\bigcap_{i=0}^n G(x_i) = \emptyset$ it follows that $\bigcap_{i=0}^n (G(x_i) \cap \phi_A(\Delta_n)) = \emptyset$, and hence, we have

$$\phi_A(\Delta_n) = \phi_A(\Delta_n) \setminus \bigcap_{i=0}^{n} (\phi_A(\Delta_n) \cap G(x_i)) = \bigcup_{i=0}^{n} [\phi_A(\Delta_n) \setminus (\phi_A(\Delta_n) \cap G(x_i))].$$

Note $\phi_A(\Delta_n)$ is a compact set in $\Gamma(A)$, if G has finitely open values, then for each $x \in X$, $G(x) \cap \Gamma(A)$ is open in $\Gamma(A)$, and hence, $\phi_A(\Delta) \cap G(x) = \phi_A(\Delta_n) \cap G(x) \cap \Gamma(A)$ is open in $\phi_A(\Delta_n)$; if G has compactly open values, it is clear that $\phi_A(\Delta_n) \cap G(x)$ is open in $\phi_A(\Delta_n)$ for each $x \in X$. Hence, in the two cases, $\phi_A(\Delta_n) \cap G(x)$ is open in $\phi_A(\Delta_n)$ for each $x \in X$. For each $z \in \Delta_n$, let

$$I(z) = \{i \in \{0, \ldots, n\} : \phi_A(z) \notin G(x_i)\} \quad \text{and} \quad S(z) = \text{co}(\{e_i : i \in I(z)\}).$$

If for some $z \in \Delta_n$, $I(z) = \emptyset$ then we have $\phi_A(z) \in G(x_i)$ for all $i \in \{0, \ldots, n\}$ which contradicts the assumption $\bigcap_{i=0}^n G(x_i) = \emptyset$. Therefore, we can assume that $I(z) \neq \emptyset$ for each $z \in \Delta_n$, and hence, $S(z)$ is a nonempty compact convex subset of Δ_n for each $z \in \Delta_n$. Since $\bigcup_{i \notin I(z)}[\phi_A(\Delta_n) \setminus (\phi_A(\Delta_n) \cap G(x_i))]$ is closed in $\phi_A(\Delta_n)$ and ϕ_A is continuous, we have

$$U = \Delta_n \setminus \phi_A^{-1}\left(\bigcup_{i \notin I(z)} [\phi_A(\Delta_n) \setminus (\phi_A(\Delta_n) \cap G(x_i))]\right.$$

is a open neighborhood of z in Δ_n. For each $z' \in U$, we have $\phi_A(z') \in G(x_i)$ for all $i \notin I(z)$, and hence, $I(z') \subset I(z)$. It follows that $S(z') \subset S(z)$ for all $z' \in U$. This shows that $S : \Delta_n \to 2^{\Delta_n}$ is a upper semicontinuous set-valued mapping with nonempty compact convex values. By Kakutani fixed point, there exists a $z_0 \in \Delta_n$ such that $z_0 \in S(z_0)$. It follows that:

$$\phi_A(z_0) \in \phi_A(S(z_0)) \subset \Gamma(\{y_i : i \in I(z_0)) \subset \bigcup_{i \in I(z_0)} G(x_i).$$

Hence, there exists a $i_0 \in I(z_0)$ such that $\phi_A(z_0) \in G(x_{i_0})$. By the definition of $I(z_0)$, we have

$$\phi_A(z_0) \notin G(x_i), \qquad \forall\, i \in I(z_0),$$

which is a contradiction. Therefore, the family $\{G(x) : x \in X\}$ has the finite intersection property.

(2) The proof is same as that in Theorem 2.1, we omit.

REMARK 2.3. Theorem 2.3 generalizes Theorem 2.2 of [15] and Theorem 2.11.18 of [13] in the following aspects:

(a) from hyperconvex space to L-convex space;
(b) from the class of $GMKKM$ mappings to the class of $GLKKM$ mappings;
(c) from G being finitely metrically open-valued to G being finitely open-valued or compactly open-valued;
(d) for each $A \in \mathcal{F}(M)$, the compactness assumption of $\mathrm{co}(A)$ is dropped.

3. APPLICATIONS

In this section, by applying our $GLKKM$ principle in the above section, some new matching theorems, fixed-point theorems, and minimax inequality are obtained.

THEOREM 3.1. *Let X be a nonempty set, (Y,Γ) be a L-convex space, $\{A_i\}_{i=0}^n$ be a family of finitely closed (or compactly closed) subsets of Y such that $\bigcup_{i=0}^n A_i = Y$ and $x_0, x_1, \ldots, x_n$ be $n+1$ points of X. Then for any $n+1$ points $y_0, \ldots, y_n$ in Y there exists $\{y_{i_0}, \ldots, y_{i_k}\} \subset \{y_0, \ldots y_n\}$ with $0 \le k \le n$ such that*

$$\Gamma(\{y_{i_j} : j = 0, \ldots, k\}) \cap \left(\bigcap_{j=0}^{k} A_{i_j}\right) \neq \emptyset.$$

PROOF. Let $X_0 = \{x_0, x_1, \ldots, x_n\}$. Since (Y,Γ) is a L-convex space, for any given $A = \{y_0, \ldots, y_n\} \subset Y$, define a set-valued mapping $G : X_0 \to 2^Y$ by $G(x_i) = Y \setminus A_i$ for each $i = 0, 1, \ldots, n$. Since each A_i is finitely closed (or, compactly closed), then G is a set-valued mapping with finitely open (or, compactly open) values. Suppose that the conclusion is false, then for any $B = \{y_{i_j} : j = 0, 1, \ldots, k\} \subset \{y_0, \ldots, y_n\}$, we have $\Gamma(B) \cap (\bigcap_{j=0}^k A_{i_j}) = \emptyset$ and so

$$\Gamma(B) \subset \bigcup_{j=0}^{k}(Y \setminus A_{i_j}) = \bigcup_{i=0}^{k} G(x_{i_j}),$$

i.e, $G : X_0 \to 2^Y$ is a $GLKKM$ mapping. By Theorem 2.3, $\bigcap_{i=0}^n G(x_i) \neq \emptyset$. It follows that $Y \neq \bigcup_{i=0}^n A_i$ which contradicts the assumption $Y = \bigcup_{i=0}^n A_i$. Hence, the conclusion holds.

REMARK 3.1. Theorem 3.1 generalizes Theorem 2.7 of [15] and Theorem 2.11.19 of [13] in several aspects.

THEOREM 3.2. *Let X be a nonempty set, (Y,Γ) be a L-convex space and $\{A_i\}_{i=0}^n$ be a family of finitely open (or compactly open) subsets of Y such that $Y = \bigcup_{i=0}^n A_i$ and $x_0, x_1, \dots, x_n$ be $n+1$ points of X. Then for any $n+1$ points $y_0, \dots, y_n$ in Y, there exists $\{y_{i_j} : j = 0,1,\dots,k\} \subset \{y_0,\dots,y_n\}$ with $0 \le k \le n$ such that*

$$\Gamma(\{y_{i_j} : j = 0,1,\dots,k\}) \cap \left(\bigcap_{j=0}^k A_{i_j}\right) \neq \emptyset.$$

PROOF. Let $X_0 = \{x_0, x_1, \dots, x_n\}$. Define a set-valued mapping $G : X_0 \to 2^Y$ by $G(x_i) = Y \setminus A_i$ for each $i = 0,1,\dots n$. Since each A_i is finitely open (or compactly open) in Y, each $G(x_i)$ is finitely closed (or compactly closed) in Y. Suppose that the conclusion is false, then for any $\{y_{i_j} : j = 0,1,\dots,k\} \subset \{y_0,\dots,y_n\}$ with $0 \le k \le n$, we have $\Gamma(\{y_{i_j} : j=0,1,\dots,k\}) \cap (\bigcap_{j=0}^k A_{i_j}) = \emptyset$, and so

$$\Gamma(\{y_{i_j} : j = 0,1,\dots,k\}) \subset \bigcup_{j=0}^k (Y \setminus A_{i_j}) = \bigcup_{j=0}^k G(x_{i_j}),$$

i.e., $G : X_0 \to 2^Y$ is a *GLKKM* mapping. By Theorem 2.1, $\bigcap_{i=0}^n G(x_i) \neq \emptyset$. It follows that $Y \neq \bigcup_{i=0}^n A_i$ which contradicts the assumption $Y = \bigcup_{i=0}^n A_i$. Hence, the conclusion holds.

REMARK 3.2. Theorem 3.2 generalizes Theorem 2.8 of [15] and Theorem 2.11.20 of [13] in the following aspects:

(a) from hyperconvex space to L-convex space;
(b) from the family of open subsets to the family of finitely open (or, compactly open) subsets.

THEOREM 3.3. *Let X be a nonempty L-convex subset of a L-convex space (Y,Γ) and $A : X \to 2^Y$ be a set-valued mapping with finitely closed (or, compactly closed) values. Suppose that there exist $n+1$ points $x_0, x_1, \dots, x_n$ in X such that $Y = \bigcup_{i=0}^n G(x_i)$ and for each $y \in X$, $A^{-1}(y) = \{x \in X : y \in A(x)\}$ is a L-convex subset of Y. Then A has a fixed point in X.*

PROOF. By Theorem 3.2 with $x_i = y_i$, $i = 0,\dots n$, there exists a subset $\{x_{i_j} : j = 0,1,\dots,k\}$ of $\{x_0,\dots,x_n\}$ such that $\Gamma(\{x_{i_j} : j = 0,1,\dots,k\}) \cap (\bigcap_{j=0}^k A(x_{i_j})) \neq \emptyset$. Take any $x^* \in \Gamma(\{x_{i_j} : j = 0,1,\dots,x_k\}) \cap (\bigcap_{j=0}^k A(x_{i_j}))$, then $x_{i_j} \in A^{-1}(x^*)$ for all $j = 0,1,\dots k$. Since $A^{-1}(x^*)$ is L-convex, we have $x^* \in \Gamma(\{x_{i_j} : j = 0,1,\dots,k\}) \subset A^{-1}(x^*)$, i.e., $x^* \in X$ is a fixed point of A.

REMARK 3.3. Theorem 3.3 generalizes Theorem 3.1 of [15] and Theorem 2.11.21 of [13] in following ways:

(a) from hyperconvex space to L-convex space;
(b) from the class of set-valued mappings with closed values to the class of set-valued mappings with finitely closed (or compactly closed) values;
(c) the compactness assumption of X is dropped.

THEOREM 3.4. *Let X be a nonempty L-convex subset of a L-convex space (Y,Γ) and $A : X \to 2^Y$ be a set-valued mapping with finitely open (or compactly open) values. Suppose that there exist $n+1$ points $x_0, x_1, \dots, x_n$ of X such that $Y = \bigcup_{i=0}^n A(x_i)$ and for each $y \in Y$, the set $A^{-1}(y)$ is L-convex. Then A has a fixed point in X.*

PROOF. By using Theorem 3.1 and the similar argument as in the proof of Theorem 3.3, it is easy to prove that the conclusion holds.

REMARK 3.4. Theorem 3.4 generalizes Theorem 3.2 of [15] and Theorem 1.11.22 of [13] in several aspects.

DEFINITION 3.1. *Let X be a nonempty set and (Y,Γ) be a L-convex space. A function $\phi : Y \times X \to \mathbf{R} \cup \{\pm\infty\}$ is said to be generalized γ-L-diagonally quasiconcave in $x \in X$ for some $\gamma \in \mathbf{R}$ if for each $A = \{x_0, x_1, \dots, x_n\} \in \mathcal{F}(X)$, there exists $B = \{y_0, y_1, \dots, y_n\} \in \mathcal{F}(Y)$ such that for any $\{y_{i_j} : j = 0,1,\dots,k\} \subset \{y_0, y_1, \dots, y_n\}$ and for any $y^* \in \Gamma(\{y_{i_j} : j = 0,1,\dots,k\})$, $\min_{0 \le j \le k} \phi(y^*, x_{i_j}) \le \gamma$.*

LEMMA 3.1. *Let X be a nonempty set, (Y, Γ) be a L-convex space, $\gamma \in \mathbf{R}$ be a given real number and $\phi : Y \times X \to \mathbf{R} \cup \{\pm\infty\}$ be a function. Then the following conditions are equivalent:*

(1) *the mapping $G : X \to 2^Y$ defined by $G(x) = \{y \in Y : \phi(y, x) \leq \gamma\}$ for each $x \in X$ is a $GLKKM$ mapping,*

(2) *the function $\phi(y, x)$ is generalized γ-L-diagonally quasiconcave in x.*

PROOF. (1) $\Rightarrow$ (2). As G is $GLKKM$ mapping, for each $A = \{x_0, x_1, \dots, x_n\} \in \mathcal{F}(X)$, there exists $B = \{y_0, y_1 \dots, y_n\} \in \mathcal{F}(Y)$ such that for any $\{y_{i_j} : j = 0, 1, \dots, k\} \subset B$ with $0 \leq k \leq n$, we have $\Gamma(\{y_{i_j} : j = 0, 1, \dots, k\}) \subset \bigcup_{j=0}^{k} G(x_{i_j})$. Hence, for any $y^* \in \Gamma(\{y_{i_j} : j = 0, 1, \dots, k\})$, there exists $0 \leq m \leq k$ such that $y^* \in G(x_{i_m})$, and hence, $\phi(y^*, x_{i_m}) \leq \gamma$. Therefore, we have that $\min_{0 \leq j \leq k} \phi(y^*, x_{i_j}) \leq \gamma$ which implies that $\phi(y, x)$ is generalized γ-L-diagonally quasiconcave in x.

(2) $\Rightarrow$ (1). As $\phi(y, x)$ is generalized γ-L-diagonally quasiconcave in x, for any $\{x_0, x_1, \dots, x_n\} \in \mathcal{F}(X)$, there exists $B = \{y_0, y_1, \dots, y_n\} \in \mathcal{F}(Y)$ such that for any $\{y_{i_j} : j = 0, 1, \dots, k\} \subset B$ and for any $y^* \in \Gamma(\{y_{i_j} : j = 0, 1, \dots, k\})$, $\min_{0 \leq j \leq k} \phi(y^*, x_{i_j}) \leq \gamma$. This implies that there exists $m \in \{0, 1, \dots, k\}$ such that $y^* \in G(x_{i_m})$. From the arbitrariness of $y^* \in \Gamma(\{y_{i_j} : j = 0, 1, \dots, k\})$, it follows that $\Gamma(\{y_{i_j} : j = 0, 1, \dots, k\}) \subset \bigcup_{j=0}^{k} G(x_{i_j})$, i.e., G is a $GLKKM$ mapping.

REMARK 3.5. Lemma 3.1 generalizes Lemma 4.1 of [16] in several aspects.

THEOREM 3.5. *Let X be nonempty L-convex subset of a L-convex space (Y, Γ) and $\phi : Y \times X \to \mathbf{R} \cup \{\pm\infty\}$ be a function such that*

(1) *$\phi(y, x)$ is generalized 0-L-diagonally quasiconcave in x,*

(2) *for each $x \in X$, the function $y \mapsto \phi(y, x)$ is lower semicontinuous on each compact subset of Y,*

(3) *there exists $M \in \mathcal{F}(X)$ such that the set $\bigcap_{x \in M}\{y \in Y : \phi(y, x) \leq 0\}$ is compact.*

Then there exists $y^* \in Y$ such that $\sup_{x \in X}\phi(y^*, x) \leq 0$.

PROOF. Define a set-valued mapping $G : X \to 2^Y$ by $G(x) = \{y \in Y : \phi(y, x) \leq 0\}$ for each $x \in X$. By (1) and Lemma 3.1, G is a $GLKKM$ mapping. Condition (2) implies that G has compactly closed values and condition (3) implies that there exists $M \in \mathcal{F}(X)$ such that $\bigcap_{x \in M} G(x)$ is compact. By Theorem 2.3, $\bigcap_{x \in X} G(x) \neq \emptyset$. Take any $y^* \in \bigcap_{x \in X} G(x)$, then we obtain $\sup_{x \in X}\phi(y^*, x) \leq 0$.

REMARK 3.6. Theorem 3.5 generalizes Theorem 2.11.15 of [13] in several aspects which is a version of Fan's minimax inequality principle in L-convex spaces.

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