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On the spread of the spectrum of a graph[☆]

Bolian Liu[a,*], Liu Mu-huo[b]

[a] *Department of Math., South China Normal University, Guangzhou, 510631, PR China*

[b] *Department of Math., South China Agricultural University, Guangzhou, 510642, PR China*

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ABSTRACT

Some new upper bounds and lower bounds are obtained for the spread $\lambda_1 - \lambda_n$ of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of the adjacency matrix of a simple graph.

1. Introduction

For an $n \times n$ matrix M, the spread, $S(M)$, of M is defined as the diameter of its spectrum, i.e., $S(M) := \max_{i,j} |\lambda_i - \lambda_j|$, where the maximum is taken over all pairs of eigenvalues of M. Suppose M is an adjacency matrix of a simple graph G with n vertices. Since M is real and symmetric, we always assume the eigenvalues of M are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then, $S(M) = \lambda_1 - \lambda_n$. Let $S(G) = S(M)$. The following theorem on $S(G)$ is due to Gregory et al. [2].

Theorem A. *For a graph G with n vertices and e edges,*

$$S(G) \leq \lambda_1 + \sqrt{2e - \lambda_1^2} \leq 2\sqrt{e}.$$

If G has no isolated vertices, then equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if $G = K_{a,b}$ *for some* a, b *with* $e = ab$ *and* $a + b \leq n$.

In this note we obtain some new upper bounds and lower bounds of $S(G)$, which are some improvements of Gregory's bound on $S(G)$ for graphs with additional restrictions.

2. An upper bound on the spread of a graph

Let G be a simple graph of order n with degree sequence $d_1, d_2, \ldots, d_n$, specially let $d(v)$ be the degree of vertex v. We call a graph G with n vertices and e edges an (n, e) graph. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $A(G)$, where $A(G)$ is an adjacency matrix of G. Let $M_1 = \sum_{i=1}^{n} d_i^2$. Then we have

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* Corresponding author.
E-mail address: liubl@scnu.edu.cn (B. Liu).

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Theorem 2.1. *For an* (n, e) *graph* G *with* f *4-cycles,*

$$M_1 = \frac{1}{2}tr(A^4) + e - 4f.$$

Proof. Let $A(G) = (a_{i,j})$ and $A^k = (a_{ij}^k)$. It is known (see [1]) that a_{ii}^4 is the number of all closed walks of length 4 from vertex i to i in G. Let f_i denote the number of 4-cycles located at vertex i. We write $i \sim j$ if vertex i is adjacent to vertex j.

$$\begin{aligned} a_{ii}^4 &= d_i^2 + \sum_{j\sim i}(d_j - 1) + 2f_i \\ &= d_i^2 + \sum_{j\sim i} d_j - d_i + 2f_i. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{tr}\,(A^4) &= \sum_{i=1}^{n} a_{ii}^4 \\ &= \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n}\sum_{j\sim i} d_j - \sum_{i=1}^{n} d_i + 2\sum_{i=1}^{n} f_i \\ &= M_1 + M_1 - 2e + 8f. \end{aligned}$$

We have $M_1 = \frac{1}{2}tr(A^4) + e - 4f$. ■

Notice that $\sum_{i=1}^{n} \lambda_i^4 = tr(A^4)$, trace of A^4 (see [1]), we have

Corollary 2.1. *For an* (n, e) *graph* G *with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$,

$$M_1 = \frac{1}{2}\sum_{i=1}^{n} \lambda_i^4 + e - 4f,$$

where f *is the number of 4-cycles in* G.

We call a graph without $K_4's$ a K_4-free graph.

Theorem 2.2. *For a* K_4*-free* (n, e) *graph* G, $M_1 \le e^2 + e - 4f$. *If* G *is connected and the equality holds, then* $|N(u) \cup N(v)| = n$ *holds for any* $\{u, v\} \in E(G)$.

Proof. For any edge $\{u, v\} \in E(G)$, let f_{uv} denote the number of 4-cycles containing edge $\{u, v\}$. Since G is a K_4-free graph, then $d(u) + d(v) \le e + 1 - f_{uv}$. Thus,

$$\sum_{u\sim v}(d(u) + d(v)) \le \sum_{u\sim v} e + \sum_{u\sim v} 1 - \sum_{u\sim v} f_{uv}. \tag{1}$$

It follows that $M_1 \le e^2 + e - 4f$.

If equality holds in this theorem, by inequality (1), it follows that $d(u) + d(v) = e + 1 - f_{uv}$ holds for any $\{u, v\} \in E(G)$. Let G' denote the subgraph induced by $N(u) \cup N(v)$, and e' denote the number of edges of G'. Since G (and then G') is K_4-free, then $e + 1 - f_{uv} = d(u) + d(v) \le e' + 1 - f_{uv} \le e + 1 - f_{uv}$, implies that $e = e'$. Moreover, since G is connected, then $|N(u) \cup N(v)| = n$. ■

A result by E. Nosal [1] asserts that if $\lambda_1^2 > \sum_{i=2}^{n} \lambda_i^2$, equivalently, if $\lambda_1 > \sqrt{e}$ then G contains at least one K_3. The following Corollary implies that if $\sum_{i=1}^{n} \lambda_i^4 > 2e^2$ then G contains at least one K_4.

Corollary 2.2. *For a* K_4*-free* (n, e) *graph,* $\sum_{i=1}^{n} \lambda_i^4 \le 2e^2$,

where equality holds for $G \cong K_{a,b}$ *or* $G \cong K_3$.

Proof. Combining Theorem 2.2 and Corollary 2.1, we have

$$\frac{1}{2}\sum_{i=1}^{n} \lambda_i^4 + e - 4f \le e^2 + e - 4f.$$

Hence, $\sum_{i=1}^{n} \lambda_i^4 \le 2e^2$.

If $G \cong K_{a,b}$ or $G \cong K_3$, it is easy to check that $d(u) + d(v) = e + 1 - f_{uv}$ holds for any $\{u, v\} \in E(G)$. Thus, equality holds in inequality (1), implies that equality holds in this corollary. ■

We now prove one of our main results.

Theorem 2.3. *For an (n, e) graph G with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$,*

$$S(G) \le \lambda_1 + \sqrt[4]{2M_1 - 2e + 8f - \lambda_1^4} \le 2\sqrt[4]{M_1 - e + 4f}.$$

If G has no isolated vertices, then equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if $G = K_{a,b}$ for some a, b with $e = ab$ and $a + b \le n$.

Proof. By Corollary 2.1, $\sum_{i=1}^n \lambda_i^4 = 2M_1 - 2e + 8f$. Thus, $\lambda_1^4 + \lambda_n^4 \le 2M_1 - 2e + 8f$ and

$$-\sqrt[4]{2M_1 - 2e + 8f - \lambda_1^4} \le \lambda_n \le \sqrt[4]{2M_1 - 2e + 8f - \lambda_1^4}.$$

Therefore,

$$S(G) = \lambda_1 - \lambda_n \le \lambda_1 + \sqrt[4]{2M_1 - 2e + 8f - \lambda_1^4}.$$

Equality holds if and only if $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = 0$, that is, if and only if $A(G) = 0$ or rank$(A) = 2$, equivalently, if and only if the non-isolated vertices of G have at most two distinct neighborhood sets. Thus, equality holds if and only if $e = 0$ or $G = K_{a,b}$ for some a, b with $e = ab$ and $a + b \le n$. If $e = ab$, then $\lambda_1 = -\lambda_n = \sqrt{ab}$. Note that $\lambda_1 + \sqrt[4]{2M_1 - 2e + 8f - \lambda_1^4}$ is a strictly increasing function of λ_1 when $\lambda_1 \le \sqrt[4]{M_1 - e + 4f}$, it is strictly decreasing when $\lambda_1 \ge \sqrt[4]{M_1 - e + 4f}$, we have

$$S(G) \le \lambda_1 + \sqrt[4]{2M_1 - 2e + 8f - \lambda_1^4} \le 2\sqrt[4]{M_1 - e + 4f}.$$

All equalities hold when $\lambda_1 = -\lambda_n = \sqrt{ab}$. ■

For a K_4-free (n, e) graph, by Corollaries 2.1 and 2.2, we have $2M_1 - 2e + 8f - \lambda_1^4 = \sum_{i=1}^n \lambda_i^4 - \lambda_1^4 \le 2e^2 - \lambda_1^4 \le 4e^2 - 4e\lambda_1^2 + \lambda_1^4$. Thus, $\sqrt[4]{2M_1 - 2e + 8f - \lambda_1^4} \le \sqrt{2e - \lambda_1^2}$ holds for K_4-free (n, e) graph. Hence we can conclude that if G is a K_4-free (n, e) graph, the bound of Theorem 2.3 is finer than that of Theorem A.

3. Lower bounds on the spread of a graph

Consider two sequence of real numbers: $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_m$ with $m < n$. The second sequence is said to interlace the first one whenever $\lambda_i \ge \mu_i \ge \lambda_{n-m+i}$ for $i = 1, 2 \cdots m$. The interlacing is called tight if there exists an integer $k \in [1, m]$ such that $\lambda_i = \mu_i$ hold for $1 \le i \le k$ and $\lambda_{n-m+i} = \mu_i$ hold for $k + 1 \le i \le m$. Suppose rows and columns of

$$A_{n\times n} = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \cdots & & \cdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix}$$

are partitioned by a partitioning $X_1 \cdots X_m$ of $\{1, 2, \ldots, n\}$. The quotient matrix is the matrix $B_{m\times m}$ whose entries are the average row sums of the blocks of $A_{n\times n}$, namely the entry $x_{i,j}$ equals the quotient of row sums of $A_{i,j}$ and the $|X_i|$, where $1 \le i, j \le m$. The partition is called regular if each block $A_{i,j}$ of A has constant row (and column) sum.

Lemma 3.1 ([3]). *Suppose B is the quotient matrix of a symmetric partitioned matrix A. Then the eigenvalues of B interlace the eigenvalues of A. If the interlacing is tight, then the partition is regular.*

Interlacing also occurs when one matrix is a principal submatrix of another.

Lemma 3.2 ([3]). *If B is a principal submatrix of a symmetric matrix A, then the eigenvalues of B interlace the eigenvalues of A.*

By Lemma 3.2, we immediately have the following result.

Proposition 3.1 ([2]). *If H is an induced subgraph of G, then $S(G) \ge S(H)$.*

In the following, let α_1 be the maximum size of the independent subset of neighborhoods of vertices of G, namely, $\alpha_1 = \max\{k : K_{1,k}$ is an induced subgraph of $G\}$

Corollary 3.1. *Let ω denote the size of the largest clique of &G&, then $S(G) \ge \omega + \sqrt{\alpha_1} - 1$. Moreover, if G is connected, then equality holds if and only if $G \cong K_\omega$.*

Proof. Suppose G contains a clique $H \cong K_\omega$, the size of which is ω. Note that each vertex in H has degree $\omega - 1$. Then by Lemma 3.2, $\lambda_1(G) \geq \lambda_1(H) = \omega - 1$. Bearing in mind that K_{1,α_1} is an induced subgraph of G, then by Lemma 3.2 it follows that $\lambda_n(G) \leq \lambda_n(K_{1,\alpha_1}) = -\sqrt{\alpha_1}$. Thus, $S(G) = \lambda_1(G) - \lambda_n(G) \geq w + \sqrt{\alpha_1} - 1$.

Moreover, if $S(G) = \omega + \sqrt{\alpha_1} - 1$ holds, then $\lambda_1(G) = \lambda_1(H) = \omega - 1$. This implies that $G \cong H = K_\omega$ because G is connected (see [4], p 17). On converse, if $G \cong K_\omega$, then $S(G) = \omega$. Note that α_1 of K_ω equals to 1, the equality follows. ■

In the following, if not specially indicated, we assume that G is a connected (n, e) graph having an adjacency matrix $A(G)$, short for A, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The matrix $B_{m\times m}$, with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$, will denote a quotient matrix of A.

The size of the largest coclique (independent set of vertices) of G is denoted by $\alpha(G)$, short for α. Let $\triangle$ and δ denote the maximum vertex degree and the minimum vertex degree, respectively. Given a graph G, if $V_1 \subseteq V(G)$, by the average degree of V_1, say d_0, we mean that $d_0 = \sum_{v\in V_1} d(v)/|V_1|$. A vertex of degree k will be referred to as a k-vertex. The number of k-vertices in G will be denoted by n_k. In particular, n_1 is the number of pendant vertices. Given a graph G, if $d(v)$ is a constant for each $v \in V(G)$, then we call G a regular graph. Moreover, if the constant is k, we call G a k-regular graph or we say that G is k-regular.

Proposition 3.2. *If G has two induced subgraphs G_1 and G_2, where G_i has n_i vertices and e_i edges for $i = 1, 2$, $V(G_1)\cap V(G_2) = \emptyset$ and $n_1 + n_2 = n$, then*

$$S(G) \geq 2\sqrt{\left(\frac{e_1}{n_1} - \frac{e_2}{n_2}\right)^2 + \frac{(e - e_1 - e_2)^2}{n_1 n_2}}.$$

Moreover, if equality holds, then each vertex in G_i is adjacent to the same number of vertices in G_j, where $1 \leq i \leq j \leq 2$.

Proof. Note that A has B as its quotient matrix, where $B = \begin{pmatrix} \frac{2e_1}{n_1} & \frac{e-e_1-e_2}{n_1} \\ \frac{e-e_1-e_2}{n_2} & \frac{2e_2}{n_2} \end{pmatrix}$.

Obviously, B has two eigenvalues $\mu_1 = \frac{e_1}{n_1} + \frac{e_2}{n_2} + \sqrt{\left(\frac{e_1}{n_1} - \frac{e_2}{n_2}\right)^2 + \frac{(e-e_1-e_2)^2}{n_1 n_2}}$ and $\mu_2 = \frac{e_1}{n_1} + \frac{e_2}{n_2} - \sqrt{\left(\frac{e_1}{n_1} - \frac{e_2}{n_2}\right)^2 + \frac{(e-e_1-e_2)^2}{n_1 n_2}}$. Then Lemma 3.1 yields the results. ■

The join of two vertex disjoint graphs G_1, G_2 is the graph $G_1 \vee G_2$ obtained from their union by including all edges between the vertices in G_1 and the vertices in G_2.

Corollary 3.2 ([2]). *Suppose $G = G_1 \vee G_2$, where each G_i is a graph with n_i vertices and e_i edges for $i = 1, 2$. Then, $S(G) \geq \sqrt{(\frac{2e_1}{n_1} - \frac{2e_2}{n_2})^2 + 4n_1 n_2}$. If equality holds, then G_1 and G_2 are both regular graphs.*

Proof. Note that $G = G_1 \vee G_2$, then $e - e_1 - e_2 = n_1 n_2$. By Proposition 3.2, the conclusions follow. ■

Proposition 3.3. *Suppose G contains $t (t \geq 1)$ independent vertices, say T, the average degree of which is d_0, then*

$$S(G) \geq 2\sqrt{\left(\frac{e - td_0}{n-t}\right)^2 + \frac{td_0^2}{n-t}} \geq 2d_0\sqrt{\frac{t}{n-t}}.$$

If equality holds between the first two expressions, then the vertex degrees are constant on T and also on $V \setminus T$ and each vertex in $V \setminus T$ is adjacent to the same number of vertices in T. If equality holds between the last two expressions, then G is bipartite with vertex parts T and $V \setminus T$.

Proof. The t independent vertices give rise to a partition of A with quotient matrix $B = \begin{pmatrix} 0 & d_0 \\ \frac{td_0}{n-t} & \frac{2(e-td_0)}{n-t} \end{pmatrix}$. Then B has two eigenvalues $\mu_1 = \frac{e-td_0}{n-t} + \sqrt{\left(\frac{e-td_0}{n-t}\right)^2 + \frac{td_0^2}{n-t}}$ and $\mu_2 = \frac{e-td_0}{n-t} - \sqrt{\left(\frac{e-td_0}{n-t}\right)^2 + \frac{td_0^2}{n-t}}$. By Lemma 3.1, $\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_n$, which implies the first inequality. The second inequality is obvious to the first one.

If equality holds between the first two expressions, then $\lambda_1 = \mu_1$ and $\lambda_n = \mu_2$, this implies that the interlacing is tight. By Lemma 3.1, the corresponding statement follows.

If equality holds between the last two expressions, then $e = td_0$, therefore G is bipartite with vertex parts T and $V \setminus T$. ■

Here we will give an illustration of the use of Proposition 3.3.

Example 3.1. Let G_0 be the graph, which is depicted in Fig. 1. Here we choose the two 4-vertices and the four 1-vertices as V_1. Obviously, V_1 are the independent vertices set with $d_0 = 2$, where d_0 is the average degree of V_1. Note that $e = 12$, $n = 10$ and $|V_1| = 6$. By Proposition 3.3, we have $S(G_0) > 2\sqrt{6}$ because the conditions necessary for equality are not satisfied.

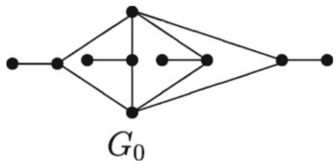

Fig. 1.

Corollary 3.3. *If $n > n_1 \geq 1$ in G, then $S(G) \geq 2\sqrt{(\frac{e-n_1}{n-n_1})^2 + \frac{n_1}{n-n_1}}$. Moreover, if equality holds, then each non-pendant vertex has the same degree and is adjacent to the same number of pendant vertices.*

By a semi-regular bipartite graph, we mean a bipartite graph $G = (X, Y)$ with the vertex degree being constant on X and also on Y.

Corollary 3.4. $S(G) \geq 2\delta\sqrt{\frac{\alpha}{n-\alpha}}$. *If equality holds, then the graph is a semi-regular bipartite graph.*

In the proof of Proposition 3.3 we show that $\lambda_1 \geq \mu_1$, the greatest eigenvalue of the matrix B. If we replace the matrix B with the similar matrix $\widetilde{B} = \begin{pmatrix} 0 & d_0\sqrt{\frac{t}{n-t}} \\ d_0\sqrt{\frac{t}{n-t}} & \frac{2(e-td_0)}{n-t} \end{pmatrix}$. Then $\mu_1(B) = \mu_1(\widetilde{B}) \geq \frac{x^T\widetilde{B}x}{x^Tx} = \frac{2e}{n}$, where $x^T = [\sqrt{t}, \sqrt{n-t}]$. It turns out that $\mu_1 \geq \frac{2e}{n}$. Thus, we have the following remark

Remark 3.1. Suppose G contains $t(t \geq 0)$ independent vertices, the average degree of which is d_0, then

$$\lambda_1 \geq \frac{e-td_0}{n-t} + \sqrt{\left(\frac{e-td_0}{n-t}\right)^2 + \frac{td_0^2}{n-t}} \geq \frac{2e}{n}.$$

Proposition 3.4. *If G (not necessary connected) has $k(1 \leq k \leq n-1)$ positive eigenvalues and $2e \geq k\lambda_1^2$, then*

$$S(G) \geq \lambda_1 + \sqrt{\frac{2e-k\lambda_1^2}{n-k}}.$$

Equality holds if and only if G has two distinct eigenvalues λ_1 and λ_n, where λ_1 has multiplicity k and λ_n has multiplicity $n-k$.

Proof. Note that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$. Then, $\lambda_i^2 \leq \lambda_1^2$ hold for $1 \leq i \leq k$ and $\lambda_i^2 \leq \lambda_n^2$ hold for $k+1 \leq i \leq n$. Bearing in mind that $2e = \sum_{i=1}^n \lambda_i^2$(see [4],p10), then $2e \leq k\lambda_1^2 + (n-k)\lambda_n^2$, which implies that $\lambda_n \leq -\sqrt{\frac{2e-k\lambda_1^2}{n-k}}$. The required inequality follows.

If equality holds, then $\lambda_i = \lambda_1$ hold for $1 \leq i \leq k$ and $\lambda_i = \lambda_n$ hold for $k+1 \leq i \leq n$, i.e. G has two distinct eigenvalues λ_1 and λ_n, where λ_1 has multiplicity k and λ_n has multiplicity $n-k$. Conversely, if $\lambda_1 = \lambda_2 = \cdots = \lambda_k$ and $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n$, then $2e = k\lambda_1^2 + (n-k)\lambda_n^2$, which implies that $\lambda_n = -\sqrt{\frac{2e-k\lambda_1^2}{n-k}}$, then the equality holds. ■

With some observation to the proof of Proposition 3.4, we also can obtain another form of lower bound with respect to λ_n.

Corollary 3.5. *If G (not necessary connected) has $k(1 \leq k \leq n-1)$ negative eigenvalues and $2e \geq k\lambda_n^2$, then*

$$S(G) \geq \sqrt{\frac{2e-k\lambda_n^2}{n-k}} - \lambda_n.$$

Equality holds if and only if G has two distinct eigenvalues λ_1 and λ_n, where λ_1 has multiplicity $n-k$ and λ_n has multiplicity k.

In [2], the upper bound for regular graph is given. Here we will give some lower bounds with the same methods present in the former discussion.

Proposition 3.5. *If G is a k-regular graph, then $S(G) \geq \frac{nk}{n-\alpha}$. Moreover, if equality holds, then there exists a largest coclique C such that every vertex not in C is adjacent to precisely $\frac{k\alpha}{n-\alpha}$ vertices of C.*

Proof. Note that A has B as its quotient matrix, where $B = \begin{pmatrix} 0 & k \\ \frac{k\alpha}{n-\alpha} & k - \frac{k\alpha}{n-\alpha} \end{pmatrix}$. Since B has two eigenvalues $\mu_1 = k$ and $\mu_2 = -\frac{k\alpha}{n-\alpha}$, then Lemma 3.1 gives the required inequality. If equality holds, then $\lambda_1 = \mu_1$ and $\lambda_n = \mu_2$. The interlacing is tight and hence the partition is regular from Lemma 3.1. ■

Proposition 3.6. *If G is a k-regular graph with two induced subgraphs G_1 and G_2, where G_i has n_i vertices and e_i edges for $i = 1, 2$, $V(G_1) \cap V(G_2) = \emptyset$ and $n_1 + n_2 = n$, then*

$$S(G) \geq 2\left(k - \frac{e_1}{n_1} - \frac{e_2}{n_2}\right).$$

If the equality holds, then G_1 and G_2 are both regular graphs.

Proof. Note that A has B as its quotient matrix, where $B = \begin{pmatrix} \frac{2e_1}{n_1} & k - \frac{2e_1}{n_1} \\ k - \frac{2e_2}{n_2} & \frac{2e_2}{n_2} \end{pmatrix}$. Since B has two eigenvalues $\mu_1 = k$ and $\mu_2 = 2\frac{e_1}{n_1} + 2\frac{e_2}{n_2} - k$, then Lemma 3.1implies the required inequality. If equality holds, then $\lambda_1 = \mu_1$ and $\lambda_n = \mu_2$. The interlacing is tight and hence the partition is regular from Lemma 3.1. ■

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References

[1] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1979.
[2] D.A. Gregory, D. Hershkowitz, S.J. Kirkland, The spread of the spectrum of a graph, Linear Algebra and its Applications 332–334 (2001) 23–35.
[3] W.H. Haemers, Interlacing eigenvalues and graphs, Linear Algebra and its Applications 227–228 (1995) 593–616.
[4] Bolian Liu, Combinatorial Matrix Theory, Science Press, Beijing, 2005 (in Chinese).