On the Exponential Stability and Periodic Solutions of Delayed Cellular Neural Networks

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A set of criteria is presented for the global exponential stability and the existence of periodic solutions of delayed cellular neural networks (DCNNs) by constructing suitable Lyapunov functionals, introducing many parameters and combining with the elementary inequality technique. These criteria have important leading significance in the design and applications of globally stable DCNNs and periodic oscillatory DCNNs. In addition, earlier results are extended and improved; other results are contained. Two examples are given to illustrate the theory.

Key Words: periodic solution; global exponential stability; delayed cellular neural networks; Lyapunov functional; inequality; parameter.

1. INTRODUCTION

The dynamics of cellular neural networks (CNNs) and CNNs with delays (DCNNs) has been deeply investigated in recent years and several important results have been obtained. Most papers dealt with completely stable CNNs and DCNNs that are suitable for image-processing applications. For different aspects of the stability theory of CNNs and DCNNs the interested reader is referred to Refs. [1, 2, 16] and Refs. [3–12, 15, 17, 20, 21], respectively, and the references cited therein. It is well known that CNNs are locally interconnected, regularly repeated, analogue circuits with grid architecture. Each cell(neuron) in a cellular neural network is connected only to its nearest neighbor cells. The circuit diagram and connection pattern implemented for the CNN can be found in Refs. [1, 2]. Processing of moving images requires the introduction of delay in the signals trans-
mitted among the cells [3]. The study of CNNs and DCNNs which exhibit periodic solutions and exponential stability is very hard. To the best of my knowledge, few authors [4, 5, 22, 23] have considered global exponential stability and periodic solutions for the DCNNs. The purpose of this paper is to study further the global exponential stability and periodic solutions of the DCNNs and give a set of criteria for global exponential stability and the existence of periodic solutions of the DCNNs on the basis of Refs. [4, 5, 22, 23] by constructing new Lyapunov functionals, introducing many parameters $q_{ij}$, $r_{ij}$, $q_{ji}$, $r_{ji} \in R$ and $w_i > 0$ ($i, j = 1, 2, \ldots, n$), and combining this with the elementary inequality $2ab \leq a^2 + b^2$ technique. These criteria possess important leading significance in the design and applications of globally stable DCNNs and periodic oscillatory DCNNs since they have infinitely adjustable real parameters and are of great interest in many applications. In addition, the results in [4, 5, 22, 23] are extended and improved; some results in Refs. [6, 10, 12] are contained.

This paper is organized as follows. In Section 2 model description of DCNNs is given. In Sections 3 and 4, a set of new sufficient conditions is derived for the global exponential stability and the existence of periodic solutions of the DCNNs on the parameters $c_i, a_{ij}, b_{ij}, \mu_j, w_i > 0$, $q_{ij}, r_{ij} \in R$ ($i, j = 1, 2, \ldots, n$) by using the Lyapunov functional method and by combining this with the techniques of inequality analysis, respectively. In Section 5 two examples are given to illustrate the theory. In Section 6 we give some concluding remarks of the results.

2. MODEL DESCRIPTION

In this paper, we consider the global exponential stability and periodic solutions of the DCNNs model described by functional differential equations with delays

$$x_i'(t) = -c_ix_i(t) + \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}f_i(x_j(t - \tau_j)) + I_i(t),$$

$$c_i > 0, \quad i = 1, 2, \ldots, n \quad (1)$$

in which $n$ corresponds to the number of units in a neural network; $x_i(t)$ corresponds to the state of the $i$th unit at time $t$; $f_j(x_j(t))$ denotes the output of the $j$th unit at time $t$; $a_{ij}, b_{ij}, c_i$ are constant, $a_{ij}$ denotes the strength of the $j$th unit on the $i$th unit at time $t$; $b_{ij}$ denotes the strength of the $j$th unit on the $i$th unit at time $t - \tau_j$; $I_i(t)$ denotes the external bias on the $i$th unit at time $t$; $\tau_j$ corresponds to the transmission delay along the axon of the $j$th unit and is not negative constant; and $c_i$ represents the
rate with which the \( i \)th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs.

We assume that each of the relations between the output of the cell \( f_i \) 
\((i = 1, 2, \ldots, n)\) and the state of the cell possess the properties:

\((H_1)\) \( f_i \) \((i = 1, 2, \ldots, n)\) is bounded on \( R \);

\((H_2)\) There is a number \( \mu_i > 0 \) such that \(|f_i(u) - f_i(v)| \leq \mu_i |u - v|\) for any \( u, v \in R \).

It is easy to find from \((H_2)\) that \( f_i \) is a continuous function on \( R \). In particular, if the relation between the output of the cell and the state of the cell is described by a piecewise-linear function \( f_i(x) = \frac{x}{n}([x + 1] - |x - 1|) \), then it is easy to see that the functions \( f_i \) clearly satisfy the hypotheses \((H_1)\) and \((H_2)\) above, and \( \mu_i = 1 \) \((i = 1, 2, \ldots, n)\).

3. GLOBAL EXPONENTIAL STABILITY

Consider the special case of the DCNN model (1) as \( I_i(t) = I_i \), i.e.,

\[
x'_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_j)) + I_i,
\]

\( c_i > 0, i = 1, 2, \ldots, n, \) (2)

where the delays \( \tau_i, i = 1, 2, \ldots, n, \) are non-negative constants; \( I_i, i = 1, 2, \ldots, n \) are constant numbers.

Assume that the nonlinear system (2) is supplemented with initial values of the type

\[
x_i(t) = \phi_i(t), \quad -\tau \leq t \leq 0, \quad \tau = \max_{1 \leq i \leq n} \tau_i, i = 1, 2, \ldots, n
\]

in which the \( \phi_i(t), i = 1, 2, \ldots, n, \) are continuous functions, and the system (2) has a unique equilibrium \( x^* = (x^*_1, x^*_2, \ldots, x^*_n) \). Let \( x^* = (x^*_1, x^*_2, \ldots, x^*_n) \) be the equilibrium of system (2); we denote

\[
\| \phi - x^* \| = \sup_{-\tau \leq t \leq 0} \left[ \sum_{i=1}^{n} (\phi_i(t) - x^*_i)^2 \right].
\]

**DEFINITION 1.** The equilibrium \( x^* = (x^*_1, x^*_2, \ldots, x^*_n) \) is said to be globally exponentially stable, if there exist constants \( \varepsilon > 0 \) and \( M \geq 1 \)
such that

\[ \sum_{i=1}^{n} (x_i(t) - x_i^*)^2 \leq M\|\phi - x^*\|e^{-\varepsilon t} \]

for all \( t \geq 0. \)

**Theorem 1.** For the DCNN (2), suppose that the outputs of the cell \( f_i \) for all \( i = 1, 2, \ldots, n \) satisfy the hypotheses \((H_1)\) and \((H_2)\) above and there exist constants \( q^*, r^*, q^*, r^* \in \mathbb{R} \) and \( w_i > 0, i, j = 1, 2, \ldots, n, \) such that

\[-c_i + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^2 - q_j^2 |a_{ij}|^2 - r_j^2 \right) + \frac{w_j}{w_i} \mu_j^2 |a_{ji}|^2 \]

\[ + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^2 - q_j^2 |b_{ij}|^2 - r_j^2 \right) + \frac{w_j}{w_i} \mu_j^2 |b_{ji}|^2 \leq 0, \quad i = 1, 2, \ldots, n, \]

in which \( \mu_j \) \((j = 1, 2, \ldots, n)\) is constant numbers of the hypotheses \((H_2)\) above. Then the equilibrium \( x^* \) is globally exponentially stable.

**Proof.** Since there exist constants \( q^*, r^*, q^*, r^* \in \mathbb{R} \) and \( w_i > 0, i, j = 1, 2, \ldots, n, \) such that

\[-c_i + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^2 - q_j^2 |a_{ij}|^2 - r_j^2 \right) + \frac{w_j}{w_i} \mu_j^2 |a_{ji}|^2 \]

\[ + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^2 - q_j^2 |b_{ij}|^2 - r_j^2 \right) + \frac{w_j}{w_i} \mu_j^2 |b_{ji}|^2 \leq 0, \quad i = 1, 2, \ldots, n, \]

we can choose a small \( \varepsilon > 0 \) such that

\[ \frac{\varepsilon}{2} - c_i + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^2 - q_j^2 |a_{ij}|^2 - r_j^2 \right) + \frac{w_j}{w_i} \mu_j^2 |a_{ji}|^2 \]

\[ + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^2 - q_j^2 |b_{ij}|^2 - r_j^2 \right) + \frac{w_j}{w_i} \mu_j^2 |b_{ji}|^2 \leq 0, \quad i = 1, 2, \ldots, n. \]

Rewrite (2) as

\[ (x_i(t) - x_i^*)' = -c_i(x_i(t) - x_i^*) + \sum_{j=1}^{n} a_{ij}[f_j(x_j(t)) - f_j(x_j^*)] \]

\[ + \sum_{j=1}^{n} b_{ij}[f_j(x_j(t - \tau_j)) - f_j(x_j^*)]. \]
Now consider the Lyapunov functional
\[
V(t) = \frac{1}{2} \sum_{i=1}^{n} w_i \left[ (x_i(t) - x^*_i)^2 e^{\alpha t} \right. \\
+ \left. \sum_{j=1}^{n} \mu^{q_j} \int_{t-\tau_j}^{t} (x_j(s) - x^*_j)^2 e^{\alpha(s+\tau_j)} ds \right].
\]
Calculating the rate of change of \( V \) along the solution of (3), we have
\[
\frac{dV(t)}{dt} = \sum_{i=1}^{n} w_i \left[ \frac{1}{2} (x_i(t) - x^*_i)^2 e^{\alpha t} + (x_i(t) - x^*_i)(x_i(t) - x^*_i) e^{\alpha t} \right. \\
+ \frac{1}{2} \sum_{j=1}^{n} \mu^{q_j} |b_{ij}| r_i (x_j(t) - x^*_j)^2 e^{\alpha(t+\tau_j)} \\
- \frac{1}{2} \sum_{j=1}^{n} \mu^{q_j} |b_{ij}| r_i (x_j(t - \tau_j) - x^*_j)^2 e^{\alpha t} \right].
\] (4)
Estimating the right side of (4) using elementary inequality \( 2ab \leq a^2 + b^2 \) and (H2), we get
\[
\frac{dV(t)}{dt} \bigg|_{(3)} \leq \sum_{i=1}^{n} w_i \left[ \frac{1}{2} (x_i(t) - x^*_i)^2 e^{\alpha t} + (x_i(t) - x^*_i) \left( x_i(t) - x^*_i \right) e^{\alpha t} \right. \\
+ \frac{1}{2} \sum_{j=1}^{n} \mu^{q_j} |b_{ij}| r_i (x_j(t) - x^*_j)^2 e^{\alpha t} e^{\alpha t} \\
- \frac{1}{2} \sum_{j=1}^{n} \mu^{q_j} |b_{ij}| r_i (x_j(t - \tau_j) - x^*_j)^2 e^{\alpha t} \right] \\
\leq e^{\alpha t} \sum_{i=1}^{n} w_i \left[ \left( \frac{\mu}{2} - c_i \right) (x_i(t) - x^*_i)^2 \right. \\
+ \sum_{j=1}^{n} |a_{ij}| \mu_j |x_j(t) - x^*_j||x_j(t) - x^*_j| \\
+ \sum_{j=1}^{n} |b_{ij}| \mu_j |x_j(t) - x^*_j||x_j(t - \tau_j) - x^*_j| \\
+ \frac{1}{2} \sum_{j=1}^{n} \mu^{q_j} |b_{ij}| r_i (x_j(t) - x^*_j)^2 e^{\alpha t} \\
- \frac{1}{2} \sum_{j=1}^{n} \mu^{q_j} |b_{ij}| r_i (x_j(t - \tau_j) - x^*_j)^2 e^{\alpha t} \right].
\]
\[ \begin{align*}
&= e^{et} \sum_{i=1}^{n} w_i \left[ \left( \frac{\epsilon}{2} - c_i \right)(x_i(t) - x^*_i)^2 \\
&+ \sum_{j=1}^{n} \left( |a_{ij}|^{(2-\tau^*_j)/2} \mu_j^{(2-\eta^*_j)/2} |x_j(t) - x^*_j| \right) \\
&\times \left( |a_{ij}|^{\tau^*_j/2} \mu_j^{\eta^*_j/2} |x_j(t) - x^*_j| \right) \\
&+ \sum_{j=1}^{n} \left( |b_{ij}|^{(2-\tau_j)/2} \mu_j^{(2-\eta_j)/2} |x_j(t) - x^*_j| \right) \\
&\times \left( |b_{ij}|^{\tau_j/2} \mu_j^{\eta_j/2} |x_j(t) - x^*_j| \right) \\
&+ \frac{1}{2} \sum_{j=1}^{n} \mu_j^{\eta_j} |b_{ij}|^{\eta_j} (x_j(t) - x^*_j)^2 e^{et} \\
&- \frac{1}{2} \sum_{j=1}^{n} \mu_j^{\eta_j} |b_{ij}|^{\eta_j} (x_j(t) - x^*_j)^2 \right] \\
&\leq e^{et} \sum_{i=1}^{n} w_i \left[ \left( \frac{\epsilon}{2} - c_i \right)(x_i(t) - x^*_i)^2 \\
&+ \frac{1}{2} \sum_{j=1}^{n} \left( |a_{ij}|^{(2-\tau^*_j)/2} \mu_j^{(2-\eta^*_j)/2} |x_j(t) - x^*_j| \right)^2 \\
&+ \left( |a_{ij}|^{\tau^*_j/2} \mu_j^{\eta^*_j/2} |x_j(t) - x^*_j| \right)^2 \\
&+ \frac{1}{2} \sum_{j=1}^{n} \left( |b_{ij}|^{(2-\tau_j)/2} \mu_j^{(2-\eta_j)/2} |x_j(t) - x^*_j| \right)^2 \\
&+ \left( |b_{ij}|^{\tau_j/2} \mu_j^{\eta_j/2} |x_j(t) - x^*_j| \right)^2 \\
&+ \frac{1}{2} \sum_{j=1}^{n} \mu_j^{\eta_j} |b_{ij}|^{\eta_j} (x_j(t) - x^*_j)^2 e^{et} \\
&- \frac{1}{2} \sum_{j=1}^{n} \mu_j^{\eta_j} |b_{ij}|^{\eta_j} (x_j(t) - x^*_j)^2 \right] \\
&= e^{et} \sum_{i=1}^{n} w_i \left[ \left( \frac{\epsilon}{2} - c_i \right)(x_i(t) - x^*_i)^2 \\
&+ \frac{1}{2} \sum_{j=1}^{n} |a_{ij}|^{2-\tau^*_j} \mu_j^{2-\eta^*_j} (x_j(t) - x^*_j)^2 \\
&+ \frac{1}{2} \sum_{j=1}^{n} |b_{ij}|^{2-\tau_j} \mu_j^{2-\eta_j} (x_j(t) - x^*_j)^2 \right] \\
&\end{align*} \]
\[ + \frac{1}{2} \sum_{j=1}^{n} |a_{ij}|^{2} \mu_{j}^{\gamma_{j}} (x_{i}(t) - x^{*})^{2} \]
\[ + \frac{1}{2} \sum_{j=1}^{n} |b_{ij}|^{2} \mu_{j}^{2 - q_{j}} (x_{i}(t) - x^{*})^{2} \]
\[ + \frac{1}{2} e^{\epsilon t} \sum_{j=1}^{n} \mu_{j}^{\gamma_{j}} |b_{ij}|^{\nu_{j}} (x_{i}(t) - x^{*})^{2} \]
\[ = e^{\epsilon t} \sum_{i=1}^{n} w_{i} \left[ \frac{e}{2} - c_{i} + \frac{1}{2} \sum_{j=1}^{n} \left( |a_{ij}|^{2} \mu_{j}^{2 - q_{j}} + \frac{w_{j}^{2}}{w_{i}} |a_{ij}|^{2} \mu_{j}^{\gamma_{j}} \right) \right] \]
\[ + \frac{1}{2} \sum_{j=1}^{n} \left( |b_{ij}|^{2} \mu_{j}^{2 - q_{j}} + e^{\epsilon t} \frac{w_{j}}{w_{i}} |b_{ij}|^{\nu_{j}} \mu_{j}^{\gamma_{j}} \right) (x_{i}(t) - x^{*})^{2} \]
\[ \leq 0 \]  \( (5) \)

and so
\[ V(t) \leq V(0), \quad t \geq 0 \]

since
\[
e^{\epsilon t} \left( \min_{1 \leq j \leq n} w_{j} \right)^{\frac{1}{2}} \sum_{i=1}^{n} (x_{i}(t) - x^{*})^{2} \leq V(t), \quad t \geq 0
\]
\[ V(0) = \frac{1}{2} \sum_{i=1}^{n} w_{i} \left[ (\phi_{i}(0) - x^{*})^{2} + \sum_{j=1}^{n} \mu_{j}^{\gamma_{j}} |b_{ij}|^{\nu_{j}} \right. \]
\[ \times \int_{-\tau}^{0} (x_{i}(s) - x^{*})^{2} e^{\epsilon (s + \tau_{i})} ds \]
\[ \leq \frac{1}{2} \left[ \max_{1 \leq i \leq n} \alpha_{i} + \tau e^{\epsilon t} \sum_{i=1}^{n} w_{i} \max_{1 \leq i \leq n} \left( \mu_{j}^{\gamma_{j}} |b_{ij}|^{\nu_{j}} \right) \| \phi - x^{*} \|. \right. \]

Then we easily get
\[ \sum_{i=1}^{n} (x_{i}(t) - x^{*})^{2} \leq M \| \phi - x^{*} \| e^{-\epsilon t} \]

for all \( t \geq 0 \), where \( M \geq 1 \) is a constant. This implies that the equilibrium \( x^{*} = (x^{*}_{1}, x^{*}_{2}, \ldots, x^{*}_{n}) \) is globally exponentially stable.
Applying Theorem 1 above, we easily prove the following corollaries:

**COROLLARY 1.** For the DCNN (2), suppose that the outputs of the cell $f_i$ $(i = 1, 2, \ldots, n)$ satisfy the hypotheses $(H_1)$ and $(H_2)$ above and there exist constants $q_{ij}^0, r_{ij}^0, q_{ij}, r_{ij} \in R$ $(i, j = 1, 2, \ldots, n)$ such that

$$\frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^{2-q_j^0} \lvert a_{ij} \rvert^{2-r_j^0} + \mu_j^{q_j^0} \lvert a_{ij} \rvert^{r_j^0} \right) + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^{2-q_j^0} \lvert b_{ij} \rvert^{2-r_j^0} + \mu_j^{q_j^0} \lvert b_{ij} \rvert^{r_j^0} \right)$$

$$< c_i, \quad i = 1, 2, \ldots, n$$

in which $\mu_j$ $(i = 1, 2, \ldots, n)$ is constant numbers of the hypotheses $(H_2)$ above. Then the equilibrium $x^*$ is also globally exponentially stable.

**COROLLARY 2.** If the relation between the output of the cell and the state of the cell is described by a piecewise-linear function $f_i(x) = \frac{1}{2}(|x + 1| - |x - 1|)$ and there exist constants $r_{ij}^0, r_{ij} \in R$ $(i, j = 1, 2, \ldots, n)$ such that

$$\frac{1}{2} \sum_{j=1}^{n} \left( \lvert a_{ij} \rvert^{2-r_j^0} + \lvert a_{ij} \rvert^{r_j^0} \right) + \frac{1}{2} \sum_{j=1}^{n} \left( \lvert b_{ij} \rvert^{2-r_j^0} + \lvert b_{ij} \rvert^{r_j^0} \right) < c_i,$n$$

$$i = 1, 2, \ldots, n,$$

then the equilibrium $x^*$ is also globally exponentially stable.

**Remark 1.** In Corollary 1,

(1) Take

(i) $a_{ij}^* = 2(1 - \alpha_i^*) = 2 \beta_i^*, \quad r_{ij}^* = 2(1 - \eta_i^*), \quad q_{ij} = 2(1 - \alpha_i)$

(ii) $a_{ij}^* = 2(1 - \alpha_i^*), \quad r_{ij}^* = 2(1 - \eta_i^*), \quad q_{ij} = 2(1 - \alpha_i)$

(iii) $a_{ij}^* = 2(1 - \alpha_i^*), \quad r_{ij}^* = 2(1 - \eta_i^*), \quad q_{ij} = 2(1 - \alpha_i)$

(iv) $a_{ij}^* = 2(1 - \alpha_i^*), \quad r_{ij}^* = 2(1 - \eta_i^*), \quad q_{ij} = 2(1 - \alpha_i)$

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(vi) $a_{ij}^* = 2(1 - \alpha_i^*), \quad r_{ij}^* = 2(1 - \eta_i^*), \quad q_{ij} = 2(1 - \alpha_i)$

(vii) $a_{ij}^* = 2(1 - \alpha_i^*), \quad r_{ij}^* = 2(1 - \eta_i^*), \quad q_{ij} = 2(1 - \alpha_i)$

(viii) $a_{ij}^* = 2(1 - \alpha_i^*), \quad r_{ij}^* = 2(1 - \eta_i^*), \quad q_{ij} = 2(1 - \alpha_i)$

respectively, in
which \( \alpha_i^* + \beta_i^* = 1 \), \( \eta_i^* + \xi_i^* = 1 \), \( \alpha_i + \beta_i = 1 \), \( \eta_i + \xi_i = 1 \) \((i = 1, 2, \ldots, n)\). Then we have derived all the results in Ref. [6] and our conclusions are globally exponentially stable better than those in Ref. [6]; i.e., our results contain the results of [6].

(2) Take (i) \( r_{ij}^* = q_{ij}^* = r_{ij} = q_{ij} = 1 \); (ii) \( r_{ij}^* = r_{ij} = 1 \), \( q_{ij}^* = 0 \), \( q_{ij} = 1 \); (iii) \( r_{ij} = r_{ij} = 1 \), \( q_{ij}^* = 2 \), \( q_{ij} = 1 \); (iv) \( r_{ij}^* = r_{ij} = 1 \), \( q_{ij}^* = 1 \), \( q_{ij} = 0 \) \((i, j = 1, 2, \ldots, n)\), respectively. Thus we have also derived the main results in Ref. [10] and our conclusions are globally exponentially stable better than those in Ref. [10]; i.e., the main results of [10] are a special case of Corollary 1.

(3) Take \( r_{ij}^* = r_{ij} = 1 \), \( q_{ij}^* = 2(1 - \alpha_j^*) = 2\beta_j^* \), \( q_{ij} = 2(1 - \alpha_j) = 2\beta_j \) where \( \alpha_j^* + \beta_j^* = 1 \), \( \alpha_j + \beta_j = 1 \) \((i = 1, 2, \ldots, n)\). Thus Theorem 1 in Ref. [12] has been derived.

In addition, one can easily see that Theorem 1 and Corollaries 1–2 above are independent of some results in Refs. [4, 5, 12, 17, 20–23] in the sense that for any one of them there exists a network which satisfies it but does not the others.

4. PERIODIC SOLUTIONS OF THE DCNNs

In this section, we study the periodic solutions of the DCNN of the type

\[
x_i'(t) = -c_ix_i(t) + \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}f_j(x_j(t - \tau_i)) + I_i(t),
\]

\[c_i > 0, \quad i = 1, 2, \ldots, n\] (6)

in which \( I_i : \mathbb{R}^+ \rightarrow \mathbb{R} \), \( i = 1, 2, \ldots, n \), are continuously periodic functions with period \( \omega \), i.e., \( I_i(t + \omega) = I_i(t) \). Other symbols possess the same meaning as that of (2).

**Theorem 2.** For the DCNNs (6), suppose that the outputs of the cell \( f_i \) \((i = 1, 2, \ldots, n)\) satisfy the hypotheses \( (H_1) \) and \( (H_2) \) above and there exist constants \( q_{ij}^* \), \( r_{ij}^* \), \( q_{ij} \), \( r_{ij} \) \( \in \mathbb{R} \) and \( w_i > 0 \), \( i, j = 1, 2, \ldots, n \) such that

\[
-c_i + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^{2-q_i^*} |a_{ij}|^{2-r_i^*} + \frac{w_j}{w_i} \mu_j^{2-q_i^*} |a_{ji}|^{r_i^*} \right)
\]

\[
+ \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^{2-q_i} |b_{ij}|^{2-r_i} + \frac{w_j}{w_i} \mu_j^{2-q_i} |b_{ji}|^{r_i} \right) < 0, \quad i = 1, 2, \ldots, n
\]
in which \( \mu_j \) \((j = 1, 2, \ldots, n)\) is constant numbers of the hypotheses \((H_j)\) above. Then there exists exactly one \( \omega \)-periodic solution of \((6)\) and all other solutions of \((6)\) converge exponentially to it as \( t \to +\infty \).

Proof. Let \( C = C([-\tau, 0], \mathbb{R}^n) \) be the Banach space of continuous functions which map \([-\tau, 0]\) into \( \mathbb{R}^n \) with the topology of uniform convergence. For any \( \varphi \in C \), we define

\[
\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|,
\]

in which \( |\varphi(\theta)| = \sum_{i=1}^{n} |\varphi_i(\theta)|^2 \).

For \( \forall \phi, \psi \in C \), we denote the solutions of \((6)\) through \((0, \phi)\) and \((0, \psi)\) as \( x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi))' \), \( x(t, \psi) = (x_1(t, \psi), x_2(t, \psi), \ldots, x_n(t, \psi))' \), respectively.

Define

\[
x_i(\phi) = x(t + \theta, \phi), \quad \theta \in [-\tau, 0], t \geq 0.
\]

Then \( x_i(\phi) \in C \) for \( \forall t \geq 0 \).

Thus it follows from system \((6)\) that

\[
(x_i(t, \phi) - x_i(t, \psi))' = -c_i(x_i(t, \phi) - x_i(t, \psi))
+ \sum_{j=1}^{n} a_{ij}^c [f_j(x_i(t, \phi)) - f_j(x_i(t, \psi))]
+ \sum_{j=1}^{n} b_{ij} c_j [f_j(x_i(t - \tau_j, \phi)) - f_j(x_i(t - \tau_j, \psi))]
\]

for \( t \geq 0 \), \( i = 1, 2, \ldots, n \). Choose a small \( \varepsilon > 0 \) such that

\[
\frac{\varepsilon}{2} - c_i + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^{2q_j} |a_{ij}|^{2r_j} + \frac{w_j^{r_j}}{w_i} \mu_j^{2q_j} |a_{ij}|^{r_j} \right)
+ \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^{2q_j} |b_{ij}|^{2r_j} + \frac{w_j^{r_j}}{w_i} e^\varepsilon \mu_j^{2q_j} |b_{ij}|^{r_j} \right) < 0,
\quad i = 1, 2, \ldots, n.
\]

We consider another Lyapunov functional

\[
V(t) = \frac{1}{2} \sum_{i=1}^{n} \alpha_i \left[ (x_i(t, \phi) - x_i(t, \psi))^2 e^{\varepsilon t} + \sum_{j=1}^{n} \mu_j^{2q_j} |b_{ij}|^{r_j} \right]
\times \int_{t-\tau_j}^{t} (x_j(s, \phi) - x_j(s, \psi))^2 e^{\varepsilon (t+s)} ds.
\]
By a minor modification of the proof of Theorem 1, we can easily get
\[
\sum_{i=1}^{n} \left( x_i(t, \phi) - x_i(t, \psi) \right)^2 \leq ke^{-\varepsilon t} \| \phi - \psi \|
\]
for \( \forall t \geq 0 \), where \( k \geq 1 \) is a constant. One can easily obtain from the formula above that
\[
\| x_i(\phi) - x_i(\psi) \| \leq ke^{-\varepsilon (t-\tau)} \| \phi - \psi \|. \tag{7}
\]
We can choose a positive integer \( m \) such that
\[
k e^{-\varepsilon (m \omega - \tau)} \leq \frac{1}{m}. \tag{8}
\]
Now define a Poincare mapping \( P : C \to C \) by \( P\phi = x_\omega(\phi) \). Then we can derive from (6)–(8) that
\[
\| P^m\phi - P^m\psi \| \leq \frac{1}{m} \| \phi - \psi \|.
\]
This implies that \( P^m \) is a contraction mapping, hence there exists a unique fixed point \( \phi^* \in C \) such that \( P^m\phi^* = \phi^* \). Note that
\[
P^m(P\phi^*) = P(P^m\phi^*) = P\phi^*.
\]
This shows that \( P\phi^* \in C \) is also a fixed point of \( P^m \), so \( P\phi^* = \phi^* \), i.e.,
\[
x_\omega(\phi^*) = \phi^*.
\]
Let \( x(t, \phi^*) \) be the solution of (6) through \((0, \phi^*)\). Obviously, \( x(t + \omega, \phi^*) \) is also a solution of (6), and note that
\[
x_{t+\omega}(\phi^*) = x_{t}(x_\omega(\phi^*)) = x_{t}(\phi^*)
\]
for \( t \geq 0 \); therefore
\[
x(t + \omega, \phi^*) = x(t, \phi^*)
\]
for \( t \geq 0 \).

This shows that \( x(t, \phi^*) \) is exactly one \( \omega \)-periodic solution of (6), and it is easy to see that all other solutions of (6) converge exponentially to it as \( t \to +\infty \).
Applying Theorem 2 above, we can prove the following corollaries:

**COROLLARY 3.** For the DCNNs (6), suppose that the outputs of the cell \( f_i \) \((i = 1, 2, \ldots, n)\) satisfy the hypotheses \((H_1)\) and \((H_2)\) above and there exist constants \( q_i^*, r_i, q_i, r_i \in R \) \((i, j = 1, 2, \ldots, n)\) such that

\[
\frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^2 q_i^* |a_{ij}|^{2-r_{ij}} + \mu_j^{q_i} |a_{ij}|^{r_{ij}} \right) + \frac{1}{2} \sum_{j=1}^{n} \left( \mu_j^2 q_i^* |b_{ij}|^{2-r_{ij}} + \mu_j^{q_i} |b_{ij}|^{r_{ij}} \right) < c_i, \quad i = 1, 2, \ldots, n
\]

in which \( \mu_j \) \((j = 1, 2, \ldots, n)\) is constant numbers of the hypotheses \((H_2)\) above. Then there exists exactly one \( \omega \)-periodic solution of (6) and all other solutions of (6) converge exponentially to it as \( t \to +\infty \).

**COROLLARY 4.** If the relation between the output of the cell and the state of the cell is described by a piecewise-linear function \( f(x) = \frac{1}{2}(x + 1 - |x - 1|) \) and there exist constants \( r_i, r_i \in R \) \((i, j = 1, 2, \ldots, n)\) such that

\[
\frac{1}{2} \sum_{j=1}^{n} (|a_{ij}|^{2-r_{ij}} + |a_{ij}|^{r_{ij}}) + \frac{1}{2} \sum_{j=1}^{n} (|b_{ij}|^{2-r_{ij}} + |b_{ij}|^{r_{ij}}) < c_i, \quad i = 1, 2, \ldots, n,
\]

then there exists exactly one \( \omega \)-periodic solution of (6) and all other solutions of (6) converge exponentially to it as \( t \to +\infty \).

**Remark 2.** Similarly, one can also see that Theorem 2 and Corollaries 3–4 above are independent of the results in Refs. [4, 5, 22, 23] in the sense that for any one of them there exists a network which satisfies it but does not the others.

5. TWO EXAMPLES

**EXAMPLE 1.** Consider the cellular neural networks with delays

\[
\begin{align*}
x'_1(t) &= -9x_1(t) + 2f(x_1(t)) - f(x_2(t)) + 2f(x_1(t - \tau_1)) + f(x_2(t - \tau_2)) + I_1, \\
x'_2(t) &= -9x_2(t) - f(x_1(t)) + 2f(x_2(t)) + f(x_1(t - \tau_1)) + 2f(x_2(t - \tau_2)) + I_2.
\end{align*}
\]
where the relation between the output of the cell and the state of the cell is described by a piecewise-linear function $f(x) = f(x) = \frac{1}{2}(|x + 1| - |x - 1|)$, $\tau_1 > 0$, $\tau_2 > 0$. It is easy to prove that Example 1 has unique equilibrium. By taking $r_{ij}^* = r_{ij} = 1$, $c_1 = c_2 = 9$, $a_{11} = b_{11} = 2$, $a_{12} = -1$, $b_{12} = 1$, $a_{22} = b_{22} = 2$, $a_{21} = -1$, $b_{21} = 1$, $I_1 = 14$, $I_2 = 5$ in Corollary 2, then

$$c_1 > |a_{11}| + \frac{1}{2}(|a_{21}| + |a_{12}|) + |b_{11}| + \frac{1}{2}(|b_{21}| + |b_{12}|);$$

$$c_2 > \frac{3}{2}(|a_{21}| + |a_{12}|) + |a_{22}| + \frac{3}{2}(|b_{21}| + |b_{12}|) + |b_{22}|$$

and so the unique equilibrium $(2, 1)$ is globally exponential stable.

**EXAMPLE 2.** Consider the cellular neural networks with delays

$$\begin{cases}
x'_1(t) = -7x_1(t) + f(x_1(t)) + f(x_2(t)) + 2f(x_1(t - \tau_1)) + f(x_2(t - \tau_2)) + 3\cos t,
\end{cases}$$

$$\begin{cases}
x'_2(t) = -8x_2(t) - f(x_1(t)) + 2f(x_2(t)) - f(x_1(t - \tau_1)) + f(x_2(t - \tau_2)) + 6\sin t,
\end{cases}$$

(10)

where the relation between the output of the cell and the state of the cell is described by a piecewise-linear function $f(x) = f(x) = \frac{1}{2}(|x + 1| - |x - 1|)$, $\tau_1 > 0$, $\tau_2 > 0$. By taking $r_{ij}^* = r_{ij} = 1$, $c_1 = 7$, $c_2 = 8$, $a_{11} = 1$, $b_{11} = 2$, $a_{12} = 1$, $b_{12} = 1$, $a_{22} = 2$, $b_{22} = 1$, $a_{21} = -1$, $b_{21} = -1$ in Corollary 4, we see that

$$c_1 > |a_{11}| + \frac{1}{2}(|a_{21}| + |a_{12}|) + |b_{11}| + \frac{1}{2}(|b_{21}| + |b_{12}|);$$

$$c_2 > \frac{3}{2}(|a_{21}| + |a_{12}|) + |a_{22}| + \frac{3}{2}(|b_{21}| + |b_{12}|) + |b_{22}|.$$

Thus by Corollary 4, Eq. (10) has a unique $2\pi$-periodic solution, and all other solutions of Eq. (10) converge exponentially to it as $t \to +\infty$.

6. CONCLUSIONS

A set of criteria has been given ensuring the global exponential stability and periodic solutions of DCNNs by introducing ingeniously many real parameters $q_{ij}^*, r_{ij}^*, q_{ij}, r_{ij} \in R$ and $w_i > 0$ $(i, j = 1, 2, \ldots, n)$, constructing suitable Lyapunov functionals and applying some inequality techniques. These criteria are of prime importance and great interest in many application fields and the design of networks since they possess infinitely ad-
justable real parameters. In addition, these criteria are easy to verify and apply in practice; for instance, they can be applied to design globally exponentially stable DCNNs and periodic oscillatory DCNNs. The methods of this paper may be applied to some other systems such as the systems given in Refs. [13, 18, 19], and so on.

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REFERENCES