On lacunary statistically convergent double sequences of fuzzy numbers

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Abstract

In this work, the concepts of double lacunary strongly \( p \)-Cesaro summability and double lacunary statistical convergence of a sequence of fuzzy numbers are introduced. The relationship between double lacunary statistical convergence and double lacunary strongly \( p \)-Cesaro summability is studied.

Keywords: Double sequence; Double lacunary sequence; Fuzzy numbers

1. Introduction

In [3], Nanda studied sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Nuray [4] proved the inclusion relations between the set of statistically convergent and lacunary statistically convergent sequences of fuzzy numbers. Recently, Savaş [6] introduced and discussed double convergent sequences of fuzzy numbers and showed that the set of all double convergent sequences of fuzzy numbers is complete. In [7], Savaş generalized the statistical convergence by using de la Vallee–Poussin mean. Quite recently, Savaş and Mursaleen [5] introduced statistically convergent and statistically Cauchy double sequences of fuzzy numbers.

In this work, we continue to study of the concepts of double lacunary statistical convergence and double lacunary strongly \( p \)-Cesaro summability for sequences of fuzzy numbers.

We begin by introducing some notation and definitions which will be used throughout and we refer the readers to [2,4,5] for more details. A fuzzy number is a function \( X \) from \( \mathbb{R}^n \) to \([0, 1]\), which is normal, fuzzy convex and upper semi-continuous, and where the closure of \( \{x \in \mathbb{R}^n : X(x) > 0\} \) is compact. These properties imply that, for each \( 0 < \alpha \leq 1 \), the \( \alpha \)-level set

\[
X^\alpha = \{x \in \mathbb{R}^n : X(x) \geq \alpha\}
\]

is a nonempty compact convex subset of \( \mathbb{R}^n \), as is the support \( X^0 \). Let \( L(\mathbb{R}^n) \) denote the set of all fuzzy numbers.
Define for each $1 \leq q < \infty$
\[
d_q (X, Y) = \left\{ \int_0^1 \delta_\infty \left( (X_\alpha, Y_\alpha)^q \right) \frac{1}{q} \right\}
\]
and $d_\infty = \sup_{0 < \alpha < 1} \delta_\infty (X_\alpha, Y_\alpha)$ where $d_\infty$ is the Hausdorff metric. Clearly $d_\infty (X, Y) = \lim_{q \to \infty} d_q (X, Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover $d_q$ is a complete, separable and locally compact metric space [1].

Throughout the work, $d$ will denote $d^q$ with $1 \leq q \leq \infty$. We will need the following definitions, (see, [5,8]).

**Definition 1.** A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be convergent in the Pringsheim’s sense or $P$-convergent to a fuzzy number $X_0$ if for every $\epsilon > 0$ there exists $N \in \mathcal{N}$ such that
\[
d (X_{kl}, X_0) < \epsilon \quad \text{for} \quad k, l > N,
\]
and we define $P - \lim X = X_0$. The number $X_0$ is called the Pringsheim limit of $X_{kl}$.

More exactly we say that a double sequence $(X_{kl})$ converges to a finite number $X_0$ if $X_{kl}$ tends to $X_0$ as both $k$ and $l$ tend to $\infty$ independently of one another.

Let $c^2(F)$ denote the set of all double convergent sequences of fuzzy numbers.

**Definition 2.** A double sequence $X = (X_{kl})$ of fuzzy numbers is bounded if there exists a positive number $M$ such that $d (X_{kl}, X_0) < M$ for all $k$ and $l$. We will denote the set of all bounded double sequences by $l^2_\infty(F)$.

Let $K \subseteq \mathcal{N} \times \mathcal{N}$ be a two-dimensional set of positive integers and let $K_{m,n}$ be the numbers of $(k, l)$ in $K$ such that $k \leq n$ and $l \leq m$. Then the lower asymptotic density of $K$ is defined as
\[
P - \lim \inf_{m,n} \frac{K_{m,n}}{mn} = \delta_2 (K).
\]
In the case when the sequence $(\frac{K_{m,n}}{mn})_{m,n=1,1}^{\infty,\infty}$ has a limit then we say that $K$ has a natural density and is defined as
\[
P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2 (K).
\]
For example, let $K = \{(k^2, l^2) : k, l \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of natural numbers. Then
\[
\delta_2 (K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m} \sqrt{n}}{mn} = 0
\]
(i.e. the set $K$ has double natural density zero).

Double statistical convergence of the sequences of fuzzy numbers was first deduced by Savas and Mursaleen [5]. They defined the statistical analogue for double sequences $X = (X_{k,l})$ of fuzzy numbers as follows.

**Definition 3.** A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be statistically convergent to $X_0$ provided that for each $\epsilon > 0$
\[
P - \lim \frac{1}{m,n} |\{(k, l) ; k \leq m \text{ and } l \leq n : d(X_{kl}, X_0) \geq \epsilon\}| = 0.
\]
In this case we write $st_2 - \lim_{k,l} X_{k,l} = X_0$ and we denote the set of all double statistically convergent sequences of fuzzy numbers by $st_2^2(F)$.

**Definition 4.** Let $X = (X_{kl})$ be a double sequence of fuzzy numbers and let $p$ be a positive real number. The double sequence $X$ is said to be strongly double $p$-Cesaro summable to $X_0$ such that
\[
P - \lim_{m,n \to \infty} \frac{1}{mn} \sum_{k=1}^m \sum_{l=1}^n d(X_{kl}, X_0)^p = 0.
\]
That is,

$$|\sigma_{1,1}|_p(F) = \left\{ X = (X_{k,l}) : \text{for some fuzzy number } X_0, P - \lim_{m,n \to \infty} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} d(X_{kl}, X_0)^p = 0 \right\}. $$

In this case, we may say that $X$ is strongly double $p$-Cesaro summable to $X_0$. The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad \text{as } r \to \infty$$

and

$$l_0 = 0, \quad \tilde{h}_s = l_s - l_{s-1} \to \infty \quad \text{as } s \to \infty.$$

Notation: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \tilde{h}_s$, $\theta_{r,s}$ is determined by $I_{r,s} = \{(k, l) : k \leq k_r \& l \leq l_s\}$, $q_r = \frac{k_r}{k_r - 1}$, $\tilde{q}_s = l_s - l_{s-1}$, and $q_{r,s} = q_r \tilde{q}_s$.

**Definition 5.** Let $\theta_{r,s}$ be a double lacunary sequence; the double sequence $X = (X_{k,l})$ is said to be double lacunary strongly $p$-Cesaro summable if there is a fuzzy number $X_0$ such that

$$N_{\theta_{r,s}}(F) = \left\{ X = (X_{k,l}) : \text{for some fuzzy number } X_0, P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} d(X_{k,l}, X_0)^p = 0 \right\}. $$

We now consider the double lacunary statistical convergence.

**Definition 6.** Let $\theta_{r,s}$ be a double lacunary sequence; the double fuzzy sequence $X$ is said to be double lacunary $\theta_{r,s}$-statistically convergent to a fuzzy number $X_0$ provided that for every $\epsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left| \{(k, l) \in I_{r,s} : d(X_{k,l}, X_0) \geq \epsilon\} \right| = 0.$$ 

In this case, we write $S_{\theta_{r,s}} \to X = X_0$ or $X_{k,l} \overset{P}{\to} X_0(S_{\theta_{r,s}})$ and we denote the set of all double $S_{\theta_{r,s}}$-statistically convergent sequences of fuzzy numbers by $S_{\theta_{r,s}}(F)$.

There is a strong connection, which we will study in this work, between $|\sigma_{1,1}|_p(F)$ and the sequence space $N_{\theta_{r,s}}(F)$.

2. Main results

In the following theorem we show some relations between $N_{\theta_{r,s}}(F)$ and $S_{\theta_{r,s}}(F)$-convergence and show that $N_{\theta_{r,s}}(F)$ and $S_{\theta_{r,s}}(F)$-convergence are equivalent for bounded double sequences.

**Theorem 1.** Let $\theta_{r,s}$ be a double lacunary sequence and let $X = \{X_{k,l}\}$ a double sequence of fuzzy numbers. Then,

A. $N_{\theta_{r,s}}(F)$ is a subset of $S_{\theta_{r,s}}(F)$,

B. $l_{\infty}^2(F) \cap S_{\theta_{r,s}}(F) \subseteq N_{\theta_{r,s}}(F)$,

C. $S_{\theta_{r,s}}(F) \cap l_{\infty}^2 = N_{\theta_{r,s}}(F) \cap l_{\infty}^2(F)$.

**Proof.** (A) We have

$$\sum_{(k,l) \in I_{r,s}} d(X_{k,l}, X_0) \geq \sum_{(k,l) \in I_{r,s} \& d(X_{k,l}, X_0) \geq \epsilon} d(X_{k,l}, X_0) \geq \epsilon \left| \{(k, l) \in I_{r,s} : d(X_{k,l}, X_0) \geq \epsilon\} \right|,$$

and

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} d(X_{k,l}, X_0) = 0.$$
This implies that
\[ P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k, l) \in I_{r,s} : d(X_{k,l}, X_0) \geq \epsilon\}| = 0. \]

This completes the proof of (A).

(B) Assume that \( X_{k,l} \xrightarrow{p} X_0(S_{\theta_{r,s}}(F)) \). Let \( M \) be such that \( d(X_{k,l}, X_0) \leq M \) for all \( k \) and \( l \). Also for given \( \epsilon > 0 \) we obtain the following:
\[ \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} d(X_{k,l}, X_0) = \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s} \cap \{X_{k,l} : d(X_{k,l}, X_0) \geq \epsilon\}} d(X_{k,l}, X_0) + \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s} \cap \{X_{k,l} : d(X_{k,l}, X_0) < \epsilon\}} d(X_{k,l}, X_0) \leq \frac{M}{h_{r,s}} |\{(k, l) \in I_{r,s} : d(X_{k,l}, X_0) \geq \epsilon\}| + \epsilon. \]

Therefore \( X \in l_{\infty}^p(F) \) and \( X_{k,l} \xrightarrow{p} X_0(S_{\theta_{r,s}}(F)) \) implies \( X_{k,l} \xrightarrow{p} X_0(N_{\theta_{r,s}}(F)) \). This completes the proof of (B).

(C) \( N_{\theta_{r,s}}(F) \cap l_{\infty}^p(F) = S_{\theta_{r,s}} \cap l_{\infty}^p(F) \) follows directly from (A), (B). \( \square \)

**Theorem 2.** Let \( \theta_{r,s} = \{k_r, l_s\} \) be double lacunary sequences. In order to have \( |\sigma_{1,1}|_p(F) \subset N_{\theta_{r,s}}(F) \) it is necessary and sufficient that \( \lim \inf_r q_r > 1 \) and \( \lim \inf_s \tilde{q}_s > 1 \).

**Proof.** Suppose that \( \lim \inf_r q_r > 1 \) and \( \lim \inf_s \tilde{q}_s > 1 \); then there exist \( \delta > 0 \) and \( \delta_1 > 0 \) such that \( \delta + 1 < q_r \) and \( \delta_1 + 1 < \tilde{q}_s \) for all \( r, s \geq 1 \). This implies that \( \frac{h_r}{k_r} > \frac{\delta}{\delta + 1} \) and \( \frac{h_s}{l_s} > \frac{\delta_1}{\delta_1 + 1} \). Suppose that \( X = (X_{kl}) \in |\sigma_{1,1}|_p(F) \); then we can write
\[ T_{r,s} = \frac{1}{h_{r,s}} \sum_{k \in I_{r,s}} \sum_{l \in I_{l,s}} d(X_{kl}, X_0)^p \]
\[ = \frac{1}{h_{r,s}} \left[ \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p - \sum_{k=1}^{k_r-l_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p - \sum_{k=1}^{k_r-l_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p + \sum_{k=1}^{k_r-l_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p \right]. \]
\[ T_{r,s} = \frac{k_r l_s}{h_{r,s}} \left( \frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p \right) - \frac{k_r l_s}{h_{r,s}} \left( \frac{1}{k_r l_s} \sum_{k=1}^{k_r-l_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p \right) \]
\[ - \frac{k_r l_s}{h_{r,s}} \left( \frac{1}{k_r l_s} \sum_{k=1}^{k_r-l_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p \right) + \frac{k_r l_s}{h_{r,s}} \left( \frac{1}{k_r l_s} \sum_{k=1}^{k_r-l_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p \right). \]

Since \( X \in |\sigma_{1,1}|_p(F) \), each of the four double sums above converges to zero in the Pringsheim sense. The terms
\[ \frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p, \quad \frac{1}{k_r l_s} \sum_{k=1}^{k_r-l_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p \]
and
\[ \frac{1}{k_r l_s} \sum_{k=1}^{k_r-l_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p, \quad \frac{1}{k_r l_s} \sum_{k=1}^{k_r-l_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p \]
are all Pringsheim null sequences. Thus \( T_{r,s} \) is a null Pringsheim sequence. Therefore \( X \) is in \( N_{\theta_{r,s}}(F) \).

Conversely, suppose that \( \lim \inf_r q_r = 1 \) and \( \lim \inf_s \tilde{q}_s = 1 \). Since \( \theta_{r,s} = \{k_r, l_s\} \), we can choose sequences \( \{k_{r_j}, l_{s_i}\} \) satisfying
\[ \frac{k_{r_j}}{k_{r_{j-1}}} > j, \quad \frac{k_{r_j}}{k_{r_{j-1}}} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{l_{s_i}}{l_{s_{i-1}}} > i, \quad \frac{l_{s_i}}{l_{s_{i-1}}} < 1 + \frac{1}{i} \]
where \( r_j \geq r_{j-1} + 2 \) and \( s_i \geq s_{i-1} + 2 \). Let \( A \) and \( B \) denote two distinct fuzzy numbers. Define \( X = (X_{kl}) \) by \( X_{kl} = A \) if \((k, l) \in (I_r, I_s)\) for some \( j = i = 1, 2, 3, \ldots \), \( X_{kl} = B \) otherwise. Then for any fuzzy numbers \( F \),

\[
\frac{1}{h_{r_j, s_i}} \sum_{k \in I_r} \sum_{l \in I_s} d(X_{kl}, F) = d(A, F),
\]

and

\[
\frac{1}{h_{rs}} \sum_{k \in I_r} \sum_{l \in I_s} d(X_{kl}, F) = d(B, F),
\]

for \( r \neq r_j \) and \( s \neq s_i \). That is \( X \not\in \mathcal{N}_{\theta, r,s} \). If \( m \) and \( n \) are any sufficiently large integers, we can find the unique \( j \) and \( i \) for which

\[
k_{r_j-1} < m \leq k_{r_j} \quad \text{and} \quad l_{s_i-1} < n \leq l_{s_i}
\]

\[
\frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} d(B, X_{kl}) \leq \frac{k_{r_j-1}l_{s_i-1} + h_{r_j, s_i}}{k_{r_j-1}l_{s_i-1}} \left( \frac{2}{ji} \right)
\]

as \( m, n \to \infty \); it follows that also \( j, i \to \infty \). Hence \( X \in |\sigma_{1,1}|_p(F) \). This is a contradiction and so completes the proof. □

**Theorem 3.** Let \( \theta_{r,s} = \{k_r, l_s\} \) be double lacunary sequences. \( \mathcal{N}_{\theta, r,s}(F) \subset |\sigma_{1,1}|_p(F) \) if and only if \( \limsup_r q_r < \infty \) and \( \limsup_s \tilde{q}_s < \infty \).

**Proof.** Suppose that \( \limsup_r q_r < \infty \) and \( \limsup_s \tilde{q}_s < \infty \); there exists an \( H > 0 \) such that \( q_r < H \) and \( \tilde{q}_s < H \) for all \( r \) and \( s \). Let \( X \in \mathcal{N}_{\theta, r,s}(F) \) and \( \varepsilon > 0 \). Also there exist \( r_0 > 0 \) and \( s_0 > 0 \) such that for every \( i \geq r_0 \) and \( j \geq s_0 \)

\[
T_{r,s} = \frac{1}{h_{rs}} \sum_{k \in I_r} \sum_{l \in I_s} d(X_{kl}, X_0)^p < \varepsilon.
\]

Let \( M = \max \{ T_{r,s} : 1 \leq r \leq r_0 \text{ and } 1 \leq s \leq s_0 \} \), and \( m \) and \( n \) be such that \( k_{r-1} < m \leq k_r \) and \( l_{s-1} < n \leq l_s \). Thus we obtain the following:

\[
\frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} d(X_{kl}, X_0)^p \leq \frac{1}{k_{r-1}l_{s-1}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p
\]

\[
\leq \frac{1}{k_{r-1}l_{s-1}} \sum_{p,u=1}^{r_0,s_0} \left( \sum_{(k,l) \in (I_p,I_u)} d(X_{kl}, X_0)^p \right)
\]

\[
= \frac{1}{k_{r-1}l_{s-1}} \sum_{p,u=1}^{r_0,s_0} h_{p,u}A_{p,u} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < p \leq r) \cup (s_0 < u \leq s_0)} h_{p,u}A_{p,u}
\]

\[
\leq \frac{M}{k_{r-1}l_{s-1}} \sum_{p,u=1}^{r_0,s_0} h_{p,u} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < p \leq r) \cup (s_0 < u \leq s_0)} h_{p,u}
\]

\[
\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{\sup_{(p \geq r_0) \cup (u \geq s_0)} A_{p,u}}{k_{r-1}l_{s-1}} \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < p \leq r) \cup (s_0 < u \leq s_0)} h_{p,u}
\]

\[
\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < p \leq r) \cup (s_0 < u \leq s_0)} h_{p,u}
\]

\[
\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \varepsilon \frac{k_{r-1}l_{s-1}}{k_{r-1}l_{s-1}} \sum_{(r_0 < p \leq r) \cup (s_0 < u \leq s_0)} h_{p,u}
\]

\[
\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \varepsilon H^2.
\]
Now \( k_r \) and \( l_s \) both approach infinity as both \( m \) and \( n \) approach infinity. Thus
\[
\frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} d(X_{kl}, X_0)^p \to 0.
\]

Therefore \( X \in |\sigma_{1,1}|_p(F) \).

Conversely, we shall assume that \( \limsup q_r = \infty \) or \( \limsup \bar{q}_s = \infty \). We shall prove that there is a bounded \( N_{\theta,s}^q(F) \)-convergent sequence that is not \( |\sigma_{1,1}|_p(F) \). Now \( \theta_r,s \) is double lacunary; we could construct two subsequences \( k_r \) and \( \bar{q}_s \) of \( \theta_r,s \) satisfying \( q_r > j, \bar{q}_s > i \) and let \( A \) and \( B \) be distinct fuzzy numbers. Define \( X = (X_{kl}) \) by
\[
X_{kl} = A, \text{ if } k_r - 1 < k < 2k_r - 1 \text{ and } l_{s-1} < l \leq 2l_{s-1}; \quad \text{and } X_{kl} = B, \text{ otherwise.}
\]

Let us write
\[
T_{r,j,s} = \frac{1}{h_{r,j,s}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, B) < \left( \frac{1}{j-1} \right) \left( \frac{1}{i-1} \right).
\]

This implies that \( \lim_{i,j} T_{r,j,s} = 0 \) if \( r \neq r_j \) and \( s \neq s_j \). Therefore \( X \in N_{\theta,s}^q(F) \). On the other hand, for the double sequence \( \{X_{kl}\} \) above and for a fuzzy number \( F \), we can write
\[
\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, F) \geq \frac{1}{k_r l_s} \sum_{k=1}^{2k_r - 1} \sum_{l=1}^{l_s} d(A, F) + \frac{1}{k_r l_s} \sum_{k=2k_r - 1 + 1}^{k_r} \sum_{l=1}^{l_s} d(B, F)
\]
\[
= d(A, F) \left( \frac{k_r - 1}{k_r} \right) \left( \frac{l_{s-1}}{l_s} \right) + d(B, F) \left( \frac{k_r - 2k_{r-1}}{k_r} \right) \left( \frac{l_{s-1}}{l_s} \right)
\]
\[
= d(A, F) \left( \frac{k_r - 1}{k_r} \right) \left( \frac{l_{s-1}}{l_s} \right) + d(B, F) \left( 1 - \frac{2}{q_{r_j}} \right) \left( 1 - \frac{2}{q_{s_j}} \right).
\]

Since \( \frac{1}{q_{r_j}} < \frac{1}{j} \) and \( \frac{1}{q_{s_j}} < \frac{1}{i} \), we obtain
\[
\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, F) \geq d(A, F) \left( \frac{k_r - 1}{k_r} \right) \left( \frac{l_{s-1}}{l_s} \right) + d(B, F) \left( 1 - \frac{2}{j} \right) \left( 1 - \frac{2}{i} \right) \to d(B, F).
\]

Therefore \( P - \lim_{j,m} \frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, F) \geq d(B, F) \). On the other hand, for \( k = 1, \ldots, 2k_r - 1 \) and \( l = 1, \ldots, 2l_{s-1} \)
\[
\frac{1}{2k_r l_{s-1}} \sum_{k=1}^{2k_r - 1} \sum_{l=1}^{2l_{s-1}} d(X_{kl}, F) \geq \frac{1}{2k_r l_{s-1}} \sum_{k=1}^{2k_r - 1} \sum_{l=1}^{2l_{s-1}} d(X_{kl}, F)
\]
\[
\geq \frac{d(B, F)}{4}
\]

which implies that \( X \notin |\sigma_{1,1}|_p(F) \). This completes the proof. \( \square \)

**Theorem 4.** Let \( \theta_r,s = \{k_r, l_s\} \) be a double lacunary sequence. \( N_{\theta,s}^q(F) = |\sigma_{1,1}|_p(F) \) if and only if \( 1 < \liminf q_r \leq \limsup q_r < \infty \) and \( 1 < \liminf \bar{q}_s \leq \limsup \bar{q}_s < \infty \).

**Proof.** Theorem 4 follows from Theorems 2 and 3. \( \square \)

Now we are going to ask whether both \( N_{\theta,s}^q(F) \) and \( |\sigma_{1,1}|_p(F) \) limits of fuzzy numbers are the same. We will give the answer in the following theorem.

**Theorem 5.** If \( X \in N_{\theta,s}^q(F) \cap |\sigma_{1,1}|_p(F) \), and \( p \leq 1 \), then \( N_{\theta,s}^q(F) - \lim X_{kl} = |\sigma_{1,1}|_p(F) - \lim X_{kl} \).
Proof. Let 
\[ N_{\theta,r,s}(F) - \lim X_{kl} = X_0 \]
and \(|\sigma_{1,1}\rho(F) - X_{kl} = X'_0\) and suppose that \(X_0 \neq X'_0\). Then write
\[
T_{rs} + T'_{rs} = \frac{1}{h_{rs}} \sum_{k \in I_r, l \in I_s} d(X_{kl}, X_0)^p + \frac{1}{h_{rs}} \sum_{k \in I_r, l \in I_s} d(X_{kl}, X'_0)^p \\
\geq \frac{1}{h_{rs}} \sum_{k \in I_r, l \in I_s} d(X_0, X'_0)^p \\
\geq d(X_0, X'_0)^p
\]
where we used the condition \(p \leq 1\). Now since \(X \in N_{\theta,r,s}\), \(P - \lim_{r,s \to \infty} T'_{rs} = 0\). Thus, for \(\min(r, p) > N\), we have
\[
T_{rp} > \frac{1}{2} d(X_0, X'_0).
\]
Now
\[
\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p = \frac{1}{k_r l_s} \sum_{r,s} \left( \sum_{k \in I_r, l \in I_s} d(X_{kl}, X_0)^p \right) \\
\geq \frac{1}{k_r l_s} \sum_{k \in I_r, l \in I_s} d(X_{kl}, X_0)^p \\
= \left( 1 - \frac{1}{q_r} \right) \left( 1 - \frac{1}{q_s} \right) T_{rs} \\
\geq \frac{1}{2} \left( 1 - \frac{1}{q_r} \right) \left( 1 - \frac{1}{q_s} \right) d(X_0, X'_0)^p.
\]
Since \(X \in |\sigma_{1,1}\rho(F)\), the left side converges to zero in Pringsheim’s sense, i.e.,
\[
\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d(X_{kl}, X_0)^p \to 0
\]
as \(r \to \infty\) and \(s \to \infty\). So we must have \(q_r \to 1\) and \(\bar{q}_s \to 1\). But these statements imply, by the proof of Theorem 2, \(N_{\theta,r,s}(F) \subset |\sigma_{1,1}\rho(F)\). Thus, since \(N_{\theta,r,s}(F) - \lim X_{kl} = X'_0\), it follows that \(|\sigma_{1,1}\rho(F) - X_{kl} = X'_0\). Therefore
\[
P - \lim_{m n \to \infty} \frac{1}{m n} \sum_{k=1}^{m} \sum_{l=1}^{n} d(X_{kl}, X_0)^p = 0.
\]
But,
\[
\frac{1}{m n} \sum_{k=1}^{m} \sum_{l=1}^{n} d(X_{kl}, X_0)^p + \frac{1}{m n} \sum_{k=1}^{m} \sum_{l=1}^{n} d(X_{kl}, X'_0)^p \geq d(X_0, X'_0)^p \geq 0
\]
which yields a contradiction since both terms on the left converge to 0. This completes the proof. \(\Box\)

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References


Further reading


