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Partial immersions and partially free maps

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ABSTRACT

In a recent paper D'Ambra et al. (2011) [2] we studied basic properties of partial immersions and partially free maps, a generalization of free maps introduced first by Gromov (1970) in [4]. In this short note we show how to build partially free maps out of partial immersions and use this fact to prove that the partially free maps in critical dimension introduced in Theorems 1.1–1.3 of D'Ambra et al. (2011) [2] for three important types of distributions can actually be built out of partial immersions. Finally, we show that the canonical contact structure on $\mathbb{R}^{2n+1}$ admits partial immersions in critical dimension for every n.

1. Introduction

In a joint paper with D'Ambra and Loi [2], in analogy with the theory of C^k, $k \geqslant 3$, isometric immersions of Nash and Gromov [8,5], we studied basic properties of *partial* isometric immersions (also called $\mathcal{H}$-*immersions*), namely C^1 maps of a manifold M into the Euclidean space $\mathbb{R}^q$ which induce a metric on some vector subbundle $\mathcal{H}$ of TM. In particular we proved (see Theorems 1.1–1.3 in [2]), by an explicit construction, the existence of $\mathcal{H}$-free maps, the analog of free maps for partial isometric immersions, in critical dimension for three types of distributions: 1-dimensional planar distributions which are either Hamiltonian or of finite type; n-dimensional Lagrangian distributions of completely integrable systems in a $2n$-dimensional symplectic manifold; 1-dimensional Hamiltonian distributions in a Riemann–Poisson manifold.

In this note we show that $\mathcal{H}$-free maps can be canonically built out of an $\mathcal{H}$-immersion. Accordingly, we show that Theorems 1.1–1.3 of [2] ultimately depend on the fact that those distributions admit an $\mathcal{H}$-immersion in critical dimension. Moreover, we add to the list the canonical contact distributions in $\mathbb{R}^{2n+1}$.

2. $\mathcal{H}$-immersions and $\mathcal{H}$-free maps

Throughout the paper, we denote by $\mathbb{E}^q$ the q-dimensional Euclidean space, namely the linear space $\mathbb{R}^q$ endowed with the Euclidean metric $e_q = \delta_{ij}\, dy^i\, dy^j$, where (y^i), $i = 1, \dots, q$, are linear coordinates on $\mathbb{R}^q$.

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Let $\mathcal{H}$ be a k-dimensional distribution on a smooth m-dimensional manifold M, namely a vector subbundle of TM such that $\dim \mathcal{H}_x = k$ for all $x \in M$; recall that, locally, $\mathcal{H}$ is the span of k vector fields $\{\xi_1, \dots, \xi_k\}$ and the $\{\xi_a\}_{a=1,\dots,k}$ are called a *local trivialization* of $\mathcal{H}$.

Definition 1. We say that a C^1 map $f = (f^i) : M \to \mathbb{E}^q$ is an $\mathcal{H}$*-immersion* when the restriction of its tangent map Tf to $\mathcal{H}$ is injective, i.e. when $\dim Tf(\mathcal{H}) = k$.

Equivalently, f is an $\mathcal{H}$-immersion when the k vectors $\{Tf(\xi_a)\}_{a=1,\dots,k}$ are linearly independent (namely when the $k \times q$ matrix

$$D_1(f) = \big(\xi_a\big(f^i\big)\big)$$

is full-rank) at every point and for every local trivialization.

Before the next definition we recall that, given a C^2 immersion $f = (f^i) : M \to \mathbb{E}^q$, a second order approximation of f is given by the map $J^2_0(\mathbb{R}, f) : J^2_0(\mathbb{R}, M) \to J^2_0(\mathbb{R}, \mathbb{R}^q)$, where $J^2_0(\mathbb{R}, M)$ denotes the space of 2-jets at 0 of maps $\mathbb{R} \to M$ and similarly for $J^2_0(\mathbb{R}, \mathbb{R}^q)$. This map writes in coordinates as

$$J^2_0(\mathbb{R}, f)\big(x^\alpha, v^\alpha, a^\alpha\big) = \big(f^i(x), v^\alpha \partial_\alpha f^i(x), a^\alpha \partial_\alpha f^i(x) + v^\alpha v^\beta \partial^2_{\alpha\beta} f^i(x)\big).$$

The linear subspace of the q-dimensional fiber of the bundle $J^2_0(\mathbb{R}, \mathbb{R}^q) \to J^1_0(\mathbb{R}, \mathbb{R}^q) \simeq T\mathbb{R}^q$ containing, at a given point $x \in M$, the graph of

$$\big(v^\alpha, a^\alpha\big) \mapsto \big(a^\alpha \partial_\alpha f^i(x) + v^\alpha v^\beta \partial^2_{\alpha\beta} f^i(x)\big),$$

is clearly at most $n + s_n$ dimensional, where $s_n = n(n+1)/2$, for it is spanned by the $n + s_n$ vectors $\{\partial_\alpha f^i \partial_i, \partial^2_{\alpha\beta} f^i \partial_i\}$. This linear subspace is called the *osculating space* of order 2 to f at x and f is called *free* when its dimension is maximal at every $x \in M$, namely when the first and second derivatives of f are linearly independent at every point.

Now consider a distribution $\mathcal{H} \subset TM \simeq J^1_0(\mathbb{R}, M)$ and its prolongation $J^2_0(\mathbb{R}, M; \mathcal{H}) \subset J^2_0(\mathbb{R}, M)$, namely the set of all 2-jets at 0 of maps $\mathbb{R} \to M$ whose 1-jet is contained in $\mathcal{H}$. Given a local trivialization $\{\xi_a\}$, coordinates in $J^2_0(\mathbb{R}, M; \mathcal{H})$ are given by (x^α, μ^a, ν^a), where μ^a and ν^a are vector components with respect to the ξ_a. A direct calculation shows that

$$J^2_0(\mathbb{R}, f)\big(x^\alpha, \mu^a, \nu^a\big) = \big(f^i(x), \mu^a\big(\xi_a f^i\big)(x), \nu^a\big(\xi_a f^i\big)(x) + \mu^a \mu^b\big(\{\xi_a, \xi_b\} f^i\big)(x)\big),$$

where $\{\xi_a, \xi_b\} f^i = \xi_a(\xi_b f^i) + \xi_b(\xi_a f^i)$. At every given $x \in M$, the linear subspace containing the graph of the map

$$\big(\mu^a, \nu^a\big) \to \big(\nu^a\big(\xi_a f^i\big)(x) + \mu^a \mu^b\big(\{\xi_a, \xi_b\} f^i\big)(x)\big)$$

is at most $k + s_k$ dimensional, since it is generated by the $k + s_k$ vectors $\{\xi_a f^i \partial_i, \{\xi_a, \xi_b\} f^i \partial_i\}$. We call this linear subspace the $\mathcal{H}$*-osculating space* of order 2 to f at x.

Definition 2. We say that a C^2 map $f = (f^i) : M \to \mathbb{E}^q$ is $\mathcal{H}$*-free* when its $\mathcal{H}$*-osculating space* of order 2 has maximal dimension at every $x \in M$, namely when the vectors $\{\xi_a f^i \partial_i, \{\xi_a, \xi_b\} f^i \partial_i\}$ are linearly independent at every point.

Equivalently, f is $\mathcal{H}$-free when the $(k + s_k) \times q$ matrix

$$D_2(f) = \begin{pmatrix} \xi_a f^i \\ \{\xi_a, \xi_b\} f^i \end{pmatrix}$$

is full-rank at every point of every local trivialization (the definition of $\mathcal{H}$-free map was first introduced by Gromov in [4]). Clearly TM-immersions are the usual immersions and TM-free maps are the usual free maps.

We denote by $\mathrm{Imm}_{\mathcal{H}}(M, \mathbb{R}^q)$ and $\mathrm{Free}_{\mathcal{H}}(M, \mathbb{R}^q)$ the sets of $\mathcal{H}$-immersions and $\mathcal{H}$-free maps $M \to \mathbb{E}^q$ and endow $C^\infty(M, \mathbb{R}^q)$ with the strong Whitney topology. Observe that, since the matrix $D_1(f)$ is equal to the first k lines of $D_2(f)$, every $\mathcal{H}$-free map is also an $\mathcal{H}$-immersion, namely $\mathrm{Free}_{\mathcal{H}}(M, \mathbb{R}^q) \subset \mathrm{Imm}_{\mathcal{H}}(M, \mathbb{R}^q)$.

Both $\mathrm{Imm}_{\mathcal{H}}(M, \mathbb{R}^q)$ and $\mathrm{Free}_{\mathcal{H}}(M, \mathbb{R}^q)$ are open subsets of $C^\infty(M, \mathbb{R}^q)$ and are clearly empty for, respectively, $q < k$ and $q < k + s_k$ (we say that k and $k + s_k$ are *critical dimensions* for, respectively, $\mathcal{H}$-immersions and $\mathcal{H}$-free maps). Next theorem shows that, independently on the topology of $\mathcal{H}$ and of M, both sets are non-empty if q is big enough:

Theorem 1. *(See [2].) The sets* $\mathrm{Imm}_{\mathcal{H}}(M, \mathbb{R}^q)$ *and* $\mathrm{Free}_{\mathcal{H}}(M, \mathbb{R}^q)$ *are dense in* $C^\infty(M, \mathbb{R}^q)$ *for, respectively,* $q \geqslant m + k$ *and* $q \geqslant m + k + s_k$.

What happens in general in the range $k \leqslant q < m + k$ for $\mathcal{H}$-immersions and $k + s_k \leqslant q < m + k + s_k$ for $\mathcal{H}$-free maps is still an open question. When $\mathcal{H} = TM$ it is known that the h-principle holds for immersions (resp. free maps) for $q \geqslant n$ (resp. $q \geqslant m + s_m$) when M is open and for $q > m$ (resp. $q > m + s_m$) when M contains a closed component (see [3] and [5]). This means that, under those conditions, free maps arise whenever the appropriate topological obstructions vanish.

Example 1. The set $\mathrm{Free}(\mathbb{R}^m, \mathbb{R}^{m+s_m})$ is non-empty. A concrete element of that set is the polynomial map

$$F_m(x^1, \dots, x^m) = (x^1, \dots, x^m, (x^1)^2, x^1 x^2, \dots, (x^m)^2)$$

of all possible monic monomials of first and second degree in the coordinates.

The critical dimension case for immersions is trivial since no compact m-manifold can be immersed into $\mathbb{R}^m$. The question of the existence of free maps in critical dimension on compact sets is instead of particular interest and still open. For example, it is still unknown whether the tori $\mathbb{T}^m$, $m > 1$, admit free maps in critical dimension (see [5], Section 1.1.4).

When $\mathcal{H} \neq TM$ still no $\mathcal{H}$-immersion can arise if M is compact but interesting cases arise even when M is topologically trivial, as the next example shows:

Example 2. Let ξ be a vector field without zeros on a Riemannian manifold (M, g) and $\mathcal{H} = \mathrm{span}\{\xi\}$. Assume that the 1-form $\xi^\flat$, obtained by "raising the index" of ξ, is *intrinsically exact*, namely that $\xi^\flat = \lambda df$ for some smooth functions f and $\lambda > 0$. Then $\xi f = \|\xi\|_g^2/\lambda > 0$, so that $f \in \mathrm{Imm}_{\mathcal{H}}(M, \mathbb{R})$. For example consider

$$\xi(x, y) = y(1 - y^2)\partial_x + (1 - 3y^2)\partial_y$$

in $\mathbb{R}^2$. Then $\xi^\flat = e^{-x} d(y(1 - y^2)e^x)$ and therefore

$$\xi(y(1 - y^2)e^x) = y^2(1 - y^2)^2 + (1 - 3y^2)^2 > 0,$$

namely $y(1 - y^2)e^x \in \mathrm{Imm}_{\mathcal{H}}(\mathbb{R}^2, \mathbb{R})$. Observe that ξ is not topologically conjugate to a constant vector field so that, in principle, the solvability of $\xi f > 0$ is not a trivial matter.

Moreover, the next theorem shows that $\mathcal{H}$-immersions can be used to build $\mathcal{H}$-free maps:

Theorem 2. *Let $F \in \mathrm{Free}(\mathbb{R}^q, \mathbb{R}^{q'})$ and $f \in \mathrm{Imm}_{\mathcal{H}}(M, \mathbb{R}^q)$. Then $F \circ f \in \mathrm{Free}_{\mathcal{H}}(M, \mathbb{R}^{q'})$. In particular, if $\mathcal{H}$ admits $\mathcal{H}$-immersions in critical dimension, then $\mathcal{H}$ admits $\mathcal{H}$-free maps in critical dimension.*

Proof. We can prove the claim without loss of generality in the critical dimension case, namely when $q = k$ and $q' = k + s_k$. Let $f = (f^a) : M \to \mathbb{R}^k$ an $\mathcal{H}$-immersion and $F = (F^i) : \mathbb{R}^k \to \mathbb{R}^{k+s_k}$ a free map (see Example 2).

Now observe that

$$\begin{cases} \xi_a F^i(f^1, \dots, f^k) = \xi_a f^c \partial_c F, \\ \xi_a(\xi_a F^i)(f^1, \dots, f^k) = \xi_a f^c \xi_a f^d \partial^2_{cd} F + \xi_a \xi_a f^c \partial_c F, \\ \{\xi_a, \xi_b\} F^i(f^1, \dots, f^k) = 2\xi_a f^c \xi_b f^d \partial^2_{cd} F + \{\xi_a, \xi_b\} f^c \partial_c F. \end{cases}$$

This shows that

$$D_2(F \circ f) = \left(\begin{array}{c|c} D_1(f) & \mathbb{O}_{k,s_k} \\ \hline C & D \end{array} \right) D_2(F),$$

where $\mathbb{O}_{k,s_k}$ is the $k \times s_k$ zero matrix,

$$C = \begin{pmatrix} 2\xi_1(\xi_1 f^1) & \dots & 2\xi_1(\xi_1 f^k) \\ \{\xi_1, \xi_2\} f^1 & \dots & \{\xi_1, \xi_2\} f^k \\ \vdots & \vdots & \vdots \\ 2\xi_k(\xi_k f^1) & \dots & 2\xi_k(\xi_k f^k) \end{pmatrix}$$

and

$$D = \begin{pmatrix} (\xi_1 f^1)^2 & 2\xi_1 f^1 \xi_1 f^2 & \dots & (\xi_1 f^k)^2 \\ \xi_1 f^1 \xi_2 f^1 & \xi_1 f^1 \xi_2 f^1 + \xi_2 f^1 \xi_1 f^1 & \dots & \xi_1 f^k \xi_2 f^k \\ \vdots & \vdots & \vdots & \vdots \\ (\xi_k f^1)^2 & 2\xi_1 f^1 \xi_1 f^2 & \dots & (\xi_1 f^k)^2 \end{pmatrix}.$$

Clearly then $\det D_2(F \circ f) = \det D_1(f) \det D \det D_2(F)$. It is easy to check that the matrix D can be written as $\rho(D_1(f))$, where $\rho : GL_k(\mathbb{R}) \to GL_{s_k}(\mathbb{R})$ is a linear representation of $GL_k(\mathbb{R})$ over $\mathbb{R}^{s_k}$, and therefore $\det D_1(f) \neq 0$ implies $\det D \neq 0$. In particular it is easy to check (e.g. it is enough to consider the case of diagonal matrices) that $\det D = (\det D_1(f))^{k+1}$, so that

$$\det D_2(F \circ f) = (\det D_1(f))^{k+2} \det D_2(F).$$

Hence, if f is an $\mathcal{H}$-immersion and F is free, the map $F \circ f$ is $\mathcal{H}$-free. □

3. $\mathcal{H}$-immersions in critical dimension

Thanks to Theorem 2, we can now reformulate Theorems 1.1–1.3 of [2] so that it is clear that they all depend on the existence of an $\mathcal{H}$-immersion.

Consider first the case of 1-distributions $\mathcal{H}$ in the plane (see Section 3.1 in [2]). Kaplan proved [6] that all 1-distributions in the plane are orientable, so that there exists a vector field everywhere non-zero such that $\mathcal{H} = \operatorname{span}\xi$. We say that $\mathcal{H}$ is *Hamiltonian* when it is tangent to the level sets of a regular[1] function f, i.e. $\mathcal{H} = \ker df$. Let now $\mathcal{F}$ be the integral foliation of $\mathcal{H}$. Two leaves are said *separatrices* when they cannot be separated in the quotient topology on $\mathcal{F}$. We say that $\mathcal{H}$ is of *finite type* when the set of the separatrices of $\mathcal{F}$ is closed and every separatrix is inseparable from just a finite number of other leaves; for example, if $\mathcal{H}$ is the span of a polynomial vector field then it is of finite type [7].

Theorem 3. *Let $\mathcal{H}$ be a planar 1-distribution which is either Hamiltonian or of finite type. Then $\mathcal{H}$ admits an $\mathcal{H}$-immersion in critical dimension.*

Proof. An $\mathcal{H}$-immersion $f : \mathbb{R}^2 \to \mathbb{R}$ is a function f such that either $\xi f > 0$ or $\xi f < 0$ at every point. When $\mathcal{H}$ is Hamiltonian, the existence of such a function was proved by Weiner in its lemma in [10]. When $\mathcal{H}$ is of finite type, it was proved in Lemma 3.1 of [2]. $\square$

Example 3. Consider the distribution $\mathcal{H}_\xi = \operatorname{span}\xi \subset T\mathbb{R}^2$, with $\xi = 2y\partial_x + (1-y^2)\partial_y$. Since the components of ξ depend only on y, only vertical straight lines can be separatrices for $\mathcal{H}_\xi$; in particular only $y = \pm 1$ are separatrix leaves for $\mathcal{H}_\xi$. A direct calculation shows that $\ker\xi$ is functionally generated by the regular smooth function $f(x, y) = (1-y^2)e^x$, namely $\mathcal{H}_\xi = \ker df$ is a Hamiltonian distribution. Now let $g(x, y) = ye^x$. It is easy to check that $\xi g(x, y) = (1+y^2)e^x > 0$, so that $g \in \mathrm{Imm}_{\mathcal{H}_\xi}(\mathbb{R}^2, \mathbb{R})$ and, for example, $F = (g, g^2) \in \mathrm{Free}_{\mathcal{H}_\xi}(\mathbb{R}^2, \mathbb{R}^2)$.

Consider now the distribution $\mathcal{H}_\eta = \operatorname{span}\eta$, with $\eta = (3y-1)\partial_x + (1-y^2)\partial_y$. Considerations similar to the ones made above show that $y = \pm 1$ are the only separatrices for $\mathcal{H}_\eta$. A functional generator for $\ker\eta$ is given by $f'(x, y) = (1-y)(1+y)^2 e^x$, whose gradient is null on the separatrix $y = -1$, so that $\mathcal{H}_\eta$ is not Hamiltonian; nevertheless $\mathcal{H}_\eta$ is of finite type since η is polynomial. A direct calculation shows that $\eta g(x, y) = (2y^2 - y + 1)e^x > 0$, so that $g \in \mathrm{Imm}_{\mathcal{H}_\eta}(\mathbb{R}^2, \mathbb{R})$ and, for example, $F = (g, g^2) \in \mathrm{Free}_{\mathcal{H}_\eta}(\mathbb{R}^2, \mathbb{R}^2)$.

Consider now the case of completely integral systems (Section 3.2 in [2]). Recall that, on a $2n$-dimensional symplectic manifold (M, ω), a completely integrable system is a collection of n functions $\{h_1, \ldots, h_n\}$ in involution, i.e. such that the Poisson bracket of every pair of h_i is identically zero.

Theorem 4. *Let $\mathcal{H} = \bigcap_{i=1}^n \ker dI_i$ be the Lagrangian n-distribution of a completely integrable system $\{h_1, \ldots, h_n\}$ on a symplectic manifold (M^{2n}, ω) such that:*

1. *the Hamiltonian vector fields ξ_i of the h_i are all complete;*
2. *the (Lagrangian) leaves of $\mathcal{H}$ are all diffeomorphic to $\mathbb{R}^n$.*

Then $\mathcal{H}$ admits an $\mathcal{H}$-immersion $f : M^{2n} \to \mathbb{R}^n$.

Proof. In Lemma 3.3 of [2] we proved the existence, under the same assumptions of this theorem, of n functions f^i such that $\{h_i, f^j\} = 0$ for $i \neq j$ and $\{h_i, f^i\} > 0$, $i = 1, \ldots, n$. Since $\mathcal{H}$ is spanned by the Hamiltonian pairwise commuting vector fields ξ_i and $\xi_i f^j = \{h_i, f^j\}$, this is enough to grant that the map $f = (f^1, \ldots, f^n) : M \to \mathbb{R}^n$ is an $\mathcal{H}$-immersion. $\square$

Example 4. Consider the symplectic manifold $T^*\mathbb{T}^n$ with canonical coordinates $(\varphi^\alpha, p_\alpha)$, so that the symplectic form is equal to $\omega = d\varphi^\alpha \wedge dp_\alpha$. The system $\{I_\alpha = e^{p_\alpha}\cos\varphi^\alpha\}_{\alpha=1,\ldots,n}$ is completely integrable on $T^*\mathbb{T}^n$, e.g. because each I_α depends only on the two coordinates with index α.

The corresponding Lagrangian distribution $\mathcal{H} = \bigcap_{\alpha=1}^n \ker dI_\alpha$ is generated by the pairwise commuting (Hamiltonian) vector fields $\xi_\alpha = e^{p_\alpha}(\sin\varphi^\alpha, \cos\varphi^\alpha)$. This system is clearly the direct product of n independent systems on the cylinders $(\varphi^\alpha, p_\alpha)$, $\alpha = 1, \ldots, n$, in such a way that the α-th system admits, as partial immersion, the function $g_\alpha = e^{p_\alpha}\sin\varphi^\alpha$; indeed $\xi_\alpha g_\alpha = e^{2p_\alpha} > 0$. Hence the map

$$G = (g_1, \ldots, g_n) : T^*\mathbb{T}^n \to \mathbb{R}^n$$

is an $\mathcal{H}$-immersion and, consequently, the map

$$F_n \circ G = \left(g_1, \ldots, g_n, g_1^2, g_1 g_2, \ldots, g_n^2\right) : T^*\mathbb{T}^n \to \mathbb{R}^{n+s_n}$$

(see Example 1) is an $\mathcal{H}$-free map.

[1] We say that a smooth function is *regular* when $df \neq 0$ at every point. Analogously, we say that a vector field is regular when it has no zeros.

Finally, consider the case of Riemann–Poisson manifolds. These are Riemannian manifolds (M, g) on which it is defined the Poisson structure

$$\{f, g\}_H \stackrel{\text{def}}{=} *[dh_1 \wedge \cdots \wedge dh_{m-2} \wedge df \wedge dg],$$

where the $H = \{h_1, \ldots, h_{m-2}\}$ are fixed smooth (possibly multivalued) functions on M.

Example 5. Consider the flat torus $\mathbb{T}^3$ with angular coordinates $(\theta^1, \theta^2, \theta^3)$ and $H = \{h(\theta^i) = B_i\theta^i\}$ for some constant 1-form $B = B_i\, d\theta^i$. Then the Riemann–Poisson bracket is given by

$$\{f, g\}_H = \epsilon^{ijk}\partial_i f\,\partial_i g B_k,$$

where ϵ^{ijk} is the totally antisymmetric Levi-Civita tensor. This bracket was introduced by S.P. Novikov as an application of his generalization of Morse theory to multivalued functions [9]. An example of the rich topological structure hidden in this Riemann–Poisson bracket can be found in [1].

Theorem 5. *Let $(M, g, \{,\}_H)$ be a Riemann–Poisson manifold such that the $m-2$ functions in H are functionally independent at every point and let $\mathcal{H}$ be a Hamiltonian 1-distribution on it. Then $\mathcal{H}$ admits an $\mathcal{H}$-immersion $f : M \to \mathbb{R}$.*

Proof. Let h be a Hamiltonian for $\mathcal{H}$. An $\mathcal{H}$-immersion $f : M \to \mathbb{R}$ is a function f such that $\{h, f\}_H > 0$ (or $\{h, f\}_H < 0$). The existence of such a function was proven in Lemma 3.4 of [2]. $\square$

Example 6. Consider the case of $\mathbb{E}^3$ with the Riemann–Poisson structure induced by the singlet $H = \{(1 - y^2)e^x\}$, so that

$$\{f, g\}_H = e^x\big[(1 - y^2)(\partial_y f\,\partial_z g - \partial_z f\,\partial_y g) - 2y(\partial_x f\,\partial_z g - \partial_z f\,\partial_x g)\big].$$

Take a Hamiltonian of the form $h(x, y, z) = \lambda(x, y)z + \mu(x, y)$, where λ is strictly positive and μ is arbitrary. The Hamiltonian 1-dimensional distribution $\mathcal{H}$ corresponding to h is the span of the regular vector field $\xi_h = \{h, \cdot\}_H$ which, for our particular choice of h, writes as

$$\xi_h = e^x\big[\big(1 - 2y - y^2\big)\big(z\partial_y\lambda(x, y) + \partial_y\mu(x, y)\big)\partial_z - \lambda(x, y)\big(\big(1 - y^2\big)\partial_y - 2y\partial_x\big)\big].$$

Example 3 shows that $f(x, y, z) = ye^x$ solves the partial differential inequality $\{h, f\}_H > 0$. Indeed

$$\{h, f\}_H = \xi_h f = e^x\lambda(x, y)\big[\big(1 - y^2\big)e^x + 2y^2e^x\big] = \big(1 + y^2\big)\lambda(x, y)e^{2x} > 0,$$

so that $f \in \mathrm{Imm}_{\mathcal{H}}(\mathbb{R}^3, \mathbb{R})$ and $(f, f^2) \in \mathrm{Free}_{\mathcal{H}}(\mathbb{R}^3, \mathbb{R}^2)$.

We add now a fourth case where it is possible to find an $\mathcal{H}$-immersion in critical dimension. Recall that a contact structure on a $(2n+1)$-dimensional manifold M is a completely non-integrable codimension-1 distribution $\mathcal{H} \subset TM$. Locally $\mathcal{H} = \ker\theta$ for some 1-form θ, so that the non-integrability condition translates into $\theta \wedge (d\theta)^n \neq 0$.

Example 7. Consider the bundle $J^1(N, \mathbb{R}) \simeq T^*N \times \mathbb{R}$ of all 1-jets of maps $N \to \mathbb{R}$, where N is an n-dimensional manifold. This bundle has a canonical contact structure induced by the tautological 1-form θ, defined as the unique (modulo strictly positive or negative smooth functions) 1-form such that a section $\sigma : N \to J^1(N, \mathbb{R})$ is holonomic (namely is the 1-jet of a map $M \to \mathbb{R}$) iff $\sigma^*\theta = 0$. In canonical coordinates (x^α, p_α, t) a canonical contact form writes as $\theta = \lambda(x, p, t)(dt - p_\alpha\, dx^\alpha)$, where $\lambda(x, p, t)$ is never zero. For $N = \mathbb{R}^n$ this gives exactly the canonical contact structure on $\mathbb{R}^{2n+1} \simeq J^1(\mathbb{R}^n, \mathbb{R})$.

Theorem 6. *Let $\mathcal{H}$ be the canonical contact structure on $\mathbb{R}^{2n+1}$. Then $\mathcal{H}$ admits an $\mathcal{H}$-immersion in critical dimension.*

Proof. A trivialization for $\mathcal{H}$ is given by the $2n$ vectors

$$\xi_1 = \partial_{x^1} - p_1\partial_t, \quad \ldots, \quad \xi_n = \partial_{x^n} - p_n\partial_t, \quad \xi_{n+1} = \partial_{p_1}, \quad \ldots, \quad \xi_{2n} = \partial_{p_n}.$$

Hence the projection on the first $2n$ components

$$\pi\big(x^1, p_1, \ldots, x^n, p_n, z\big) = \big(x^1, p_1, \ldots, x^n, p_n\big)$$

belongs to $\mathrm{Imm}_{\mathcal{H}}(\mathbb{R}^{2n+1}, \mathbb{R}^{2n})$ and $F_{2n} \circ \pi$ belongs to $\mathrm{Free}_{\mathcal{H}}(\mathbb{R}^{2n+1}, \mathbb{R}^{2n+s_{2n}})$. $\square$

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