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On the oscillation of certain functional differential equations via comparison methods

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Abstract

Several new criteria for the oscillation of the functional differential equations of the form

$$\frac{d}{dt}\left(\left[\frac{1}{a_{n-1}(t)}\frac{d}{dt}\frac{1}{a_{n-2}(t)}\frac{d}{dt}\cdots\frac{1}{a_1(t)}\frac{d}{dt}x(t)\right]^{\alpha}\right) \pm q(t)f\left(x\left[g(t)\right]\right)=0$$

are established in this paper.

Keywords: Oscillation; Nonoscillation; Functional; Nonlinear; Comparison

1. Introduction

Consider the functional differential equation

$$L_n x(t) + \delta q(t) f\left(x\left[g(t)\right]\right) = 0, \tag{1.1; δ}$$

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where $n \geqslant 3$, $\delta = \pm 1$,

$$\begin{cases} L_0 x(t) = x(t), \\ L_k x(t) = \frac{1}{a_k(t)} \frac{d}{dt}(L_{k-1} x(t)), \quad k = 1, 2, \ldots, n-1, \\ L_n x(t) = \frac{d}{dt}([L_{n-1} x(t)])^\alpha. \end{cases} \tag{1.2}$$

In what follows we shall assume that

(i) $a_i(t) \in C([t_0, \infty), \mathbb{R}^+ = (0, \infty))$,

$$\int^{\infty} a_i(s)\, ds = \infty, \quad i = 1, 2, \ldots, n-1, \tag{1.3}$$

(ii) $q(t) \in C([t_0, \infty), \mathbb{R}^+)$,
(iii) $g(t) \in C([t_0, \infty), \mathbb{R} = (-\infty, \infty))$, $g'(t) \geqslant 0$ for $t \geqslant t_0$ and $\lim_{t\to\infty} g(t) = \infty$,
(iv) $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ and $f'(x) \geqslant 0$ for $x \neq 0$,
(v) α is a quotient of positive odd integers.

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $x : [T_x, \infty) \to \mathbb{R}$ such that $L_j x(t)$, $j = 0, 1, \ldots, n$, exist and are continuous on $[T_x, \infty)$, $T_x \geqslant t_0$. Our attention is restricted to these solutions $x \in \mathcal{D}(L_n)$ of Eq. (1.1; δ) which satisfy $\sup\{|x(t)|:\ t \geqslant T\} > 0$ for every $T \geqslant T_x$. We make the standing hypothesis that Eq. (1.1; δ) does possess such solutions. A solution of Eq. (1.1; δ) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1.1; δ) is said to be oscillatory if all its solutions are oscillatory.

Recently, the present authors [1–3,5–7] and others in [9] have established many interesting oscillation criteria for some special cases of Eq. (1.1; δ). The obtained results extend and improve many well-known oscillation results which have appeared in the literature. The purpose of this paper is to proceed further in this direction and establish some new criteria for the oscillation of Eq. (1.1; δ).

2. Preliminaries

To formulate our results we shall use the following notation: For $p_i(t) \in C([t_0, \infty), \mathbb{R})$, $i = 1, 2, \ldots,$ we define $I_0 = 1$,

$$I_i(t, s; p_i, p_{i-1}, \ldots, p_1) = \int_s^t p_i(u) I_{i-1}(u, s; p_{i-1}, \ldots, p_1)\, du, \quad i = 1, 2, \ldots.$$

It is easy to verify from the definition of I_i that

$$I_i(t, s; p_1, \ldots, p_i) = (-1)^i I_i(s, t; p_i, \ldots, p_1)$$

and

$$I_i(t, s; p_1, \ldots, p_i) = \int_s^t p_i(u) I_{i-1}(t, u; p_1, \ldots, p_{i-1})\, du.$$

We shall need the following two lemmas.

Lemma 2.1. *If $x \in \mathcal{D}(\bar{L}_n)$, where $\bar{L}_n$ is L_n defined by* (1.2) *with $\alpha = 1$, then the following formulas hold for $0 \leqslant i \leqslant k \leqslant n-1$ and $t, s \in [t_0, \infty)$:*

$$L_i x(t) = \sum_{j=i}^{k-1} I_{j-i}(t, s; a_{i+1}, \dots, a_j) L_j x(s) + \int_s^t I_{k-i-1}(t, u; a_{i+1}, \dots, a_{k-1}) a_k(u) L_k x(u)\, du \tag{2.1}$$

and

$$L_i x(t) = \sum_{j=i}^{k-1} (-1)^{j-i} I_{j-i}(s, t; a_j, \dots, a_{i+1}) L_j x(s) + (-1)^{k-i} \int_t^s I_{k-i-1}(u, t; a_{k-1}, \dots, a_{i+1}) a_k(u) L_k x(u)\, du. \tag{2.2}$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

Lemma 2.2. *Suppose condition* (1.3) *holds. If $x \in \mathcal{D}(\bar{L}_n)$, where $\bar{L}_n$ is as in Lemma* 2.1 *is eventually of one sign, then there exist $t_x \geqslant t_0 \geqslant 0$ and an integer ℓ, $0 \leqslant \ell \leqslant n$, with $n + \ell$ even for $x(t)\bar{L}_n x(t)$ nonnegative, or $n + \ell$ odd for $x(t)\bar{L}_n x(t)$ nonpositive and such that for every $t \geqslant t_x$,*

$$\begin{cases} \ell > 0 \quad \text{implies} \quad x(t)\bar{L}_k x(t) > 0, \quad k = 0, 1, \dots, \ell, \\ \ell \leqslant n-1 \quad \text{implies} \quad (-1)^{\ell-k} x(t)\bar{L}_k x(t) > 0, \quad k = \ell, \ell+1, \dots, n. \end{cases} \tag{2.3}$$

This lemma generalizes a well-known lemma of Kiguradze and can be proved similarly.

For a function $g \in C([t_0, \infty), \mathbb{R})$ we put

$$\sigma(t) = \min\{g(t), t\},$$
$$\mathcal{A}_g = \{t \in [t_0, \infty):\ g(t) > t\}, \qquad \mathcal{R}_g = \{t \in [t_0, \infty):\ g(t) < t\}.$$

For simplicity, we put

$$J_k(t, s) = I_k(t, s; a_1, \dots, a_k), \qquad K_k(t, s) = a_{n-1}(t) I_k(t, s; a_{n-2}, \dots, a_{n-k}),$$

$k = 1, 2, \dots, n-2$ and $t, s \in [T, \infty)$ for some $T \geqslant t_0 \geqslant 0$.

3. Main results

In this section, we shall study the conditions under which the set $\mathcal{N}$ of all nonoscillatory solutions of Eq. (1.1; δ) become empty. By Lemma 2.2 and because of conditions (i)–(v), we shall consider the decomposition of $\mathcal{N}$ based on the integer ℓ in Lemma 2.2. So, we shall consider the following three cases:

(1) $1 \leqslant \ell \leqslant n-1$,
(2) $\ell = 0$,
(3) $\ell = n$.

3.1. The case $1 \leqslant \ell \leqslant n-1$

We denote $\mathcal{N}_\ell$ the subset of $\mathcal{N}$ consisting of all x satisfying (2.3). We shall provide the conditions under which $\mathcal{N}_\ell = \emptyset$, $1 \leqslant \ell \leqslant n-1$. We shall assume that

$$-f(-xy) \geqslant f(xy) \geqslant f(x)f(y) \quad \text{for } xy > 0. \tag{3.1}$$

Theorem 3.1. *Let conditions* (i)–(v) *and* (3.1) *hold and let* $1 \leqslant \ell \leqslant n-1$, $(-1)^{n-\ell}\delta = -1$. *If, for every* $T \geqslant t_0$ *and* $\sigma(t) \geqslant T$,

$$\int^{\infty} K_{n-\ell-2}(t,T)\left(\int_t^{\infty} q(u)\,du\right)^{1/\alpha} f^{1/\alpha}\bigl(J_{\ell-1}\bigl(\sigma(t),T\bigr)\bigr)\,dt = \infty, \quad \ell = 1,2,\ldots,n-2, \tag{3.2; ℓ}$$

$$\int^{\infty} q(t) f\bigl(J_{n-2}\bigl(\sigma(t),T\bigr)\bigr)\,dt = \infty, \tag{3.2; $n-1$}$$

then $\mathcal{N}_\ell = \emptyset$.

Proof. Assume that $\mathcal{N}_\ell$ has an element x. We may suppose that $x(t)$ is eventually positive. Since $L_n x(t)$ is eventually of one sign and because of (v) one can easily see that $L_j x(t)$ $(0 \leqslant j \leqslant n)$ are eventually of one sign. By Lemma 2.2, there exists $t_1 \geqslant t_0$ such that

$$L_i x(t) > 0 \quad (0 \leqslant i \leqslant \ell) \quad \text{and} \quad (-1)^{i-\ell} L_i x(t) > 0 \quad (\ell \leqslant i \leqslant n) \quad \text{on } [t_1,\infty). \tag{3.3}$$

Assume first that $\ell < n-1$. From (2.2) with i, k and t replaced by ℓ, $n-1$ and t_1, respectively, we obtain in view of (3.3),

$$\begin{aligned} L_\ell x(t_1) &= \sum_{j=\ell}^{n-2} (-1)^{j-\ell} I_{j-\ell}(s,t_1;a_j,\ldots,a_{\ell+1}) L_j x(s) \\ &\quad + (-1)^{n-\ell-1} \int_{t_1}^{s} I_{n-\ell-2}(u,t_1;a_{n-2},\ldots,a_{\ell+1}) a_{n-1}(u) L_{n-1} x(u)\,du \\ &\geqslant (-1)^{n-\ell-1} \int_{t_1}^{s} K_{n-\ell-2}(u,t_1) L_{n-1} x(u)\,du. \end{aligned} \tag{3.4}$$

Integrating Eq. (1.1; δ) from u to T and letting $T \to \infty$, we find

$$\delta L_{n-1}x(u) \geqslant \left(\int_u^\infty q(\tau)\,d\tau\right)^{1/\alpha} f^{1/\alpha}\big(x\big[g(u)\big]\big) \geqslant \left(\int_u^\infty q(\tau)\,d\tau\right)^{1/\alpha} f^{1/\alpha}\big(x\big[\sigma(u)\big]\big)$$
$$\text{for } t_1 \leqslant u \leqslant s. \tag{3.5}$$

Using (3.5) in (3.4) and letting $s \to \infty$, we have

$$\infty > L_\ell x(t_1) \geqslant \int_{t_1}^{s} K_{n-\ell-2}(u,t_1)\left(\int_u^\infty q(\tau)\,d\tau\right)^{1/\alpha} f^{1/\alpha}\big(x\big[\sigma(u)\big]\big)\,du. \tag{3.6}$$

On the other hand, by integrating $L_\ell x(t) > 0$ $(t \geqslant t_1)$ ℓ times, we have

$$x\big[\sigma(t)\big] \geqslant c J_{\ell-1}\big(\sigma(t),t_1\big) \quad \text{for } t \geqslant t_2 \text{ for some } t_2 \geqslant t_1, \tag{3.7}$$

where c is a positive constant. Combining (3.6) with (3.7) and using condition (3.1), we obtain

$$\int_{t_2}^\infty K_{n-\ell-2}(u,t_1)\left(\int_u^\infty q(\tau)\,d\tau\right)^{1/\alpha} f^{1/\alpha}\big(J_{\ell-1}\big(\sigma(u),t_1\big)\big)\,du < \infty,$$

which contradicts condition (3.2; ℓ).

Next, suppose $\ell = n-1$. Integrating Eq. (1.1; δ) from t_1 to u and letting $u \to \infty$ we have

$$\infty > \delta L_{n-1}^{\alpha} x(t_1) \geqslant \int_{t_1}^\infty q(u) f\big(x\big[\sigma(u)\big]\big)\,du. \tag{3.8}$$

From (3.8) and (3.7) with $\ell = n-1$, and by using (3.1), we get

$$\int_{t_2}^\infty q(u) f\big(J_{n-2}\big(\sigma(u),t_1\big)\big)\,du < \infty,$$

which again contradicts condition (3.2; $n-1$). This completes the proof. □

We note that a nontrivial solution $x(t)$ of Eq. (1.1; δ) is said to be strongly decreasing if it satisfies

$$(-1)^j x(t) L_j x(t) > 0 \quad \text{eventually for } 0 \leqslant j \leqslant n-1,$$

i.e., $x \in \mathcal{N}_0$.

Also, a nontrivial solution $x(t)$ of Eq. (1.1; 1) is said to be strongly increasing if it satisfies

$$x(t) L_j x(t) > 0 \quad \text{eventually for } 0 \leqslant j \leqslant n-1,$$

i.e., $x \in \mathcal{N}_n$.

The following corollary is immediate.

Corollary 3.1. *Let the hypotheses of Theorem* 3.1 *hold. Then,*

(a_1) *for* $\delta = 1$, n *is even and* $\ell \in \{1, 3, \ldots, n-1\}$, *Eq.* (1.1; 1) *is oscillatory,*
(a_2) *for* $\delta = 1$, n *is odd and* $\ell \in \{2, 4, \ldots, n-1\}$, *every nonoscillatory solution of Eq.* (1.1; 1) *is strongly decreasing,*
(a_3) *for* $\delta = -1$, n *is odd and* $\ell \in \{1, 3, \ldots, n-2\}$, *every nonoscillatory solution of Eq.* (1.1; −1) *is strongly increasing,*
(a_4) *for* $\delta = -1$, n *is even and* $\ell \in \{2, 4, \ldots, n-2\}$, *every nonoscillatory solution of Eq.* (1.1; −1) *is either strongly decreasing, or strongly increasing.*

In the following result, we require the following notations. For $t \geqslant T \geqslant t_0$, we put

$$Q_j(t) = a_{j+1}(t) \int_t^\infty I_{n-j-3}(u, t; a_{n-2}, \ldots, a_{j+1}) a_{n-1}(u) \times f^{1/\alpha}\big(I_{j-1}\big(g(u), \sigma(t); a_1, \ldots, a_{j-1}\big)\big) \left(\int_u^\infty q(\tau)\, d\tau \right)^{1/\alpha} du, \quad 1 \leqslant j \leqslant n-3, \tag{3.9; j}$$

$$Q_{n-2}(t) = q(t) f\big(I_{n-2}\big(\sigma(t), T, a_1, \ldots, a_{n-2}\big)\big), \quad \sigma(t) \geqslant T, \tag{3.9; $n-2$}$$

$$Q_{n-1}(t) = q(t) f\big(I_{n-2}\big(g(t), \eta(t); a_1, \ldots, a_{n-2}\big)\big) \tag{3.9; $n-1$}$$

for every $g(t) \geqslant \eta(t) \in C([t_0, \infty), \mathbb{R})$ and

$$Q^*_{n-1}(t) = q(t) f\big(I_{n-1}\big(\sigma(t), T, a_1, \ldots, a_{n-1}\big)\big), \quad \sigma(t) \geqslant T. \tag{3.9*; $n-1$}$$

Theorem 3.2. *Let conditions* (i)–(v) *and* (3.1) *hold and let* $1 \leqslant \ell \leqslant n-1$ *with* $(-1)^{n-\ell}\delta = -1$. *If* $1 \leqslant \ell \leqslant n-3$ *assume all the second order differential equations*

$$\left(\frac{1}{a_\ell(t)} y'(t) \right)' + Q_\ell(t) f^{1/\alpha}\big(y\big[\sigma(t)\big]\big) = 0, \quad 1 \leqslant \ell \leqslant n-3, \tag{3.10; ℓ}$$

with $\sigma(t) \geqslant T \geqslant t_0$ *are oscillatory. If* $\ell = n-2$ *assume all bounded solutions of the equation*

$$\left(\left(\frac{1}{a_{n-1}(t)} z'(t) \right)^\alpha \right)' - Q_{n-2}(t) f\big(z\big[\sigma(t)\big]\big) = 0 \tag{3.10; $n-2$}$$

are oscillatory. If $\ell = n-1$ *assume either* (a) *there exists* $\eta(t) \in C([t_0, \infty), \mathbb{R})$ *such that* $\eta(t) \leqslant \sigma(t)$ *for* $t \geqslant t_0$, $\lim_{t\to\infty} \eta(t) = \infty$ *and the equation*

$$\left(\left(\frac{1}{a_{n-1}(t)} w'(t) \right)^\alpha \right)' + Q_{n-1}(t) f\big(w\big[\eta(t)\big]\big) = 0 \tag{3.10; $n-1$}$$

is oscillatory, or (b) *the first order equation*

$$u'(t) + Q^*_{n-1}(t) f\big(u^{1/\alpha}\big[\sigma(t)\big]\big) = 0 \tag{3.10*; $n-1$}$$

is oscillatory. Then, $\mathcal{N}_\ell = \emptyset$.

Proof. Assume that $\mathcal{N}_\ell$ has an element x. We may suppose that $x(t)$ is eventually positive. Since $L_n x(t)$ is eventually of one sign, we see that $L_j x(t)$ $(0 \leqslant j \leqslant n-1)$ are also eventually of constant sign. Moreover,

$$L_n x(t) = \frac{d}{dt}\big(L_{n-1}x(t)\big)^\alpha = \alpha L_{n-1}^{\alpha-1}x(t)\bar{L}_n x(t), \tag{3.11}$$

where $\bar{L}_n$ is defined as in Lemma 2.1, we see that the sign of L_n and $\bar{L}_n$ are the same. As in the proof of Theorem 3.1, we obtain (3.3).

Let $\ell \leqslant n-3$. Putting $i = \ell+1$, $k = n-1$, $s \geqslant t \geqslant t_1$ in (2.2) we have

$$\begin{aligned} L_{\ell+1}x(t) = {} & \sum_{j=\ell+1}^{n-2} (-1)^{j-\ell-1} I_{j-\ell-1}(s,t;a_j,\dots,a_{\ell+2}) L_j x(s) \\ & + (-1)^{n-\ell-2} \int_t^s I_{n-\ell-3}(u,t;a_{n-2},\dots,a_{\ell+2}) a_{n-1}(u) L_{n-1}x(u)\,du. \end{aligned} \tag{3.12}$$

Using (3.3) and letting $s \to \infty$ in (3.12), we obtain

$$-L_{\ell+1}x(t) \geqslant (-1)^{n-\ell-1} \int_t^\infty I_{n-\ell-3}(u,t;a_{n-2},\dots,a_{\ell+2}) a_{n-1}(u) L_{n-1}x(u)\,du. \tag{3.13}$$

Integrating Eq. $(1.1;\delta)$ for $u \geqslant t \geqslant t_1$ to s, we find

$$-\delta L_{n-1}^\alpha x(s) + \delta L_{n-1}^\alpha x(u) = \int_u^s q(\tau) f\big(x\big[g(\tau)\big]\big)\,d\tau.$$

Letting $s \to \infty$, we have

$$\delta L_{n-1}x(u) \geqslant \left(\int_u^\infty q(\tau) f\big(x\big[g(\tau)\big]\big)\,d\tau \right)^{1/\alpha}. \tag{3.14}$$

Substituting (3.14) in (3.13), we get

$$\begin{aligned} -L_{\ell+1}x(t) \geqslant {} & \int_t^\infty I_{n-\ell-3}(u,t;a_{n-2},\dots,a_{\ell+2}) a_{n-1}(u) \\ & \times \left(\int_u^\infty q(\tau) f\big(x\big[g(\tau)\big]\big)\,d\tau \right)^{1/\alpha} du. \end{aligned}$$

Since $x(t)$ and $g(t)$ are nondecreasing for $t \geqslant t_1$, we have

$$-L_{\ell+1}x(t) \geqslant \int_t^\infty I_{n-\ell-3}(u,t;a_{n-2},\ldots,a_{\ell+2})a_{n-1}(u) \times \left(\int_u^\infty q(\tau)\,d\tau\right)^{1/\alpha} f^{1/\alpha}\bigl(x\bigl[g(u)\bigr]\bigr)\,du. \tag{3.15}$$

On the other hand, using (2.1) with i, k, t and s replaced by 0, $\ell-1$, u and t_1, respectively, and (3.3), we get

$$\begin{aligned} x(u) &\geqslant \int_{t_1}^u I_{\ell-2}(u,v;a_1,\ldots,a_{\ell-2})a_{\ell-1}(v)L_{\ell-1}x(v)\,dv \\ &\geqslant \left(\int_t^u I_{\ell-2}(u,v;a_1,\ldots,a_{\ell-2})a_{\ell-1}(v)\,dv\right)L_{\ell-1}x(t) \\ &= I_{\ell-1}(u,t;a_1,\ldots,a_{\ell-1})L_{\ell-1}x(t) \quad \text{for } u \geqslant t \geqslant t_1. \end{aligned}$$

There exists $t_2 \geqslant t_1$ such that

$$x\bigl[g(t)\bigr] \geqslant I_{\ell-1}\bigl(g(u),\sigma(t);a_1,\ldots,a_{\ell-1}\bigr)L_{\ell-1}x\bigl[\sigma(t)\bigr], \quad t \geqslant t_2. \tag{3.16}$$

Substituting (3.16) in (3.15) and using (3.1) we get

$$-L_{\ell+1}x(t) \geqslant \left[\int_t^\infty I_{n-\ell-3}(u,t;a_{n-2},\ldots,a_{\ell+2})a_{n-1}(u)\left(\int_u^\infty q(\tau)\,d\tau\right)^{1/\alpha} \times f^{1/\alpha}\bigl(I_{\ell-1}\bigl(g(u),\sigma(t);a_1,\ldots,a_{\ell-1}\bigr)\bigr)\,du\right] f^{1/\alpha}\bigl(L_{\ell-1}x\bigl[\sigma(t)\bigr]\bigr),$$

or

$$-\frac{d}{dt}\bigl(L_\ell x(t)\bigr) \geqslant Q_\ell(t) f^{1/\alpha}\bigl(L_{\ell-1}x\bigl[\sigma(t)\bigr]\bigr) \quad \text{for } t \geqslant t_2.$$

Let $y(t)$ be given by $y(t) = L_{\ell-1}x(t)$. Then $y(t) > 0$ on $[t_2,\infty)$ and $y(t)$ satisfies the differential inequality

$$\left(\frac{1}{a_\ell(t)}y'(t)\right)' + Q_\ell(t) f^{1/\alpha}\bigl(y\bigl[\sigma(t)\bigr]\bigr) \leqslant 0 \quad \text{for } t \geqslant t_2.$$

Theorem 2 in [10] now implies that the equation

$$\left(\frac{1}{a_\ell(t)}z'(t)\right)' + Q_\ell(t) f^{1/\alpha}\bigl(z\bigl[\sigma(t)\bigr]\bigr) = 0$$

has an eventually positive solution. But this contradicts our assumptions.

Let $\ell = n-2$. This occurs when $\delta = -1$. From (2.1) with i, k, t and s replaced by 0, $n-2$, t and $t_1 \geqslant t_0$, respectively, we have

$$
\begin{aligned}
x(t) &= \sum_{j=0}^{n-2} I_j(t, t_1; a_1, \ldots, a_j) L_j x(t_1) \\
&\quad + \int_{t_1}^{t} I_{n-3}(t, u; a_1, \ldots, a_{n-3}) a_{n-2}(u) L_{n-2} x(u)\, du \\
&\geqslant \int_{t_1}^{t} I_{n-3}(t, u; a_1, \ldots, a_{n-3}) a_{n-2}(u) L_{n-2} x(u)\, du. \qquad (3.17)
\end{aligned}
$$

Using the fact that $L_{n-2}x(t)$ is decreasing on $[t_1, \infty)$ there exists $t_2 \geqslant t_1$ such that

$$
x\big[g(t)\big] \geqslant x\big[\sigma(t)\big] \geqslant I_{n-2}\big(\sigma(t), t_1; a_1, \ldots, a_{n-2}\big) L_{n-2} x\big[\sigma(t)\big] \quad \text{for } t \geqslant t_2. \qquad (3.18)
$$

Using (3.18) in Eq. $(1, 1; -1)$ and condition (3.1) we get

$$
L_n x(t) = q(t) f\big(x\big[g(t)\big]\big) \geqslant q(t) f\big(I_{n-2}\big(\sigma(t), t_1; a_1, \ldots, a_{n-2}\big)\big) f\big(L_{n-2} x\big[\sigma(t)\big]\big), \quad t \geqslant t_2.
$$

Let $y(t)$ be given by $y(t) = L_{n-2}x(t)$. Then, $y(t) > 0$ on $[t_2, \infty)$ and is bounded and satisfies the inequality

$$
\left(\left(\frac{1}{a_{n-1}(t)} y'(t)\right)^{\alpha}\right)' \geqslant q(t) f\big(I_{n-2}\big(\sigma(t), t_1; a_1, \ldots, a_{n-2}\big)\big) f\big(y\big[\sigma(t)\big]\big).
$$

By applying a similar argument given in [11] (see also [4]), one can easily see that the equation

$$
\left(\left(\frac{1}{a_{n-1}(t)} w'(t)\right)^{\alpha}\right)' = q(t) f\big(I_{n-2}\big(g(t), t_1; a_1, \ldots, a_{n-2}\big)\big) f\big(w\big[\sigma(t)\big]\big)
$$

has a positive bounded solution, which contradicts the hypothesis of the theorem.

Finally, let $\ell = n-1$. This is the case when $\delta = 1$. Assume (a) in Theorem 3.2 occurs. From (3.17), there exist $t_2 \geqslant t_1$ such that

$$
x\big[g(t)\big] \geqslant I_{n-2}\big(g(t), \eta(t); a_1, \ldots, a_{n-2}\big) L_{n-2} x\big[\eta(t)\big] \quad \text{for } t \geqslant t_2. \qquad (3.19)
$$

Using (3.19) in Eq. (1.1; 1) and condition (3.1) we obtain

$$
\begin{aligned}
-L_n x(t) &= q(t) f\big(x\big[g(t)\big]\big) \\
&\geqslant q(t) f\big(I_{n-2}\big(g(t), \eta(t); a_1, \ldots, a_{n-2}\big)\big) f\big(L_{n-2} x\big[\eta(t)\big]\big) \quad \text{for } t \geqslant t_2.
\end{aligned}
$$

Let $y(t)$ be given by $y(t) = L_{n-2}x(t)$. Then $y(t) > 0$ on $[t_2, \infty)$ and satisfies the inequality

$$
\left(\left(\frac{1}{a_{n-1}(t)} y'(t)\right)^{\alpha}\right)' + q(t) f\big(I_{n-2}\big(g(t), \eta(t); a_1, \ldots, a_{n-2}\big)\big) f\big(y\big[\eta(t)\big]\big) \leqslant 0.
$$

Once again Theorem 2 in [10] implies that the equation

$$
\left(\left(\frac{1}{a_{n-1}(t)} u'(t)\right)^{\alpha}\right)' + q(t) f\big(g(t), \eta(t); a_1, \ldots, a_{n-1}\big) f\big(u\big[\eta(t)\big]\big) = 0
$$

has an eventually positive solution. But this contradicts the hypothesis of the theorem. Also, for the case $\ell = n-1$, if (b) in Theorem 3.2 occurs, we can proceed as follows: From (2.1) and (3.3) we see that there exists $t_2 \geqslant t_1$ such that

$$x\big[g(t)\big] \geqslant x\big[\sigma(t)\big] \geqslant I_{n-1}\big(\sigma(t), t_1; a_1, \ldots, a_{n-1}\big) L_{n-1}x\big[\sigma(t)\big] \quad \text{for } t \geqslant t_2. \tag{3.20}$$

Using (3.20) in Eq. (1.1; 1) and condition (3.1) we have

$$-\frac{d}{dt}\big(L_{n-1}x(t)\big)^{\alpha} \geqslant q(t) f\big(I_{n-1}\big(\sigma(t), t_1; a_1, \ldots, a_{n-1}\big)\big) f\big(L_{n-1}x\big[\sigma(t)\big]\big).$$

Let $y(t)$ be given by $y(t) = L_{n-1}^{\alpha}x(t)$. Then, $y(t) > 0$ on $[t_2, \infty)$ and satisfies the inequality

$$y'(t) + Q_{n-1}^{*}(t) f\big(y^{1/\alpha}\big[\sigma(t)\big]\big) \leqslant 0 \quad \text{for } t \geqslant t_2.$$

By a result in [11] we find that the equation

$$v'(t) + Q_{n-1}^{*}(t) f\big(v^{1/\alpha}\big[\sigma(t)\big]\big) = 0$$

has an eventually positive solution. This again contradicts our assumptions and completes the proof. □

Remark 3.1. In the case when $g(t) > \sigma(t)$ for $t \geqslant t_0$, Eq. (3.10; $n-1$) can be replaced by

$$\left(\left(\frac{1}{a_{n-1}(t)} w'(t)\right)^{\alpha}\right)' + Q_{n-1}(t) f\big(w\big[\sigma(t)\big]\big) = 0.$$

The following corollary is immediate.

Corollary 3.2. *Let Theorem* 3.1 *in Corollary* 3.1 *be replaced by Theorem* 3.2. *Then the conclusion of Corollary* 3.1 *holds.*

We introduce the notation

$$\tau(t) = \max\big\{\min\big\{s, g(s)\big\}\colon\ 0 \leqslant s \leqslant t\big\} \quad \text{and} \quad \rho(t) = \min\big\{\max\big\{s, g(s)\big\}\colon\ s \geqslant t\big\}.$$

Note that the following inequalities hold:

$$g(s) \leqslant \tau(t) \quad \text{for } \tau(t) < s < t \quad \text{and} \quad g(s) \geqslant \rho(t) \quad \text{for } t < s < \rho(t).$$

In Theorem 3.2 for the case $\ell = n-1$ one can replace Eq. (3.10*; $n-1$) is oscillatory by the first order advanced equation

$$v'(t) - Q_{n-1}^{**}(t) f^{1/\alpha}\big(v\big[\rho(t)\big]\big) = 0 \tag{3.21}$$

is oscillatory, where

$$Q_{n-1}^{**}(t) = a_{n-1}(t)\left[\int_t^{\rho(t)} q(s) f\big(I_{n-2}\big(g(s), \rho(t); a_1, \ldots, a_{n-2}\big)\big)\, ds\right]^{1/\alpha}.$$

To show this, we proceed as in Theorem 3.2, case $\ell = n-1$, and obtain (3.17),

$$x\big[g(s)\big] \geqslant I_{n-2}\big(g(s), \rho(t); a_1, \ldots, a_{n-2}\big) L_{n-2} x\big[\rho(t)\big] \quad \text{for } g(s) \geqslant \rho(t) \geqslant t_1. \tag{3.22}$$

Substituting (3.22) in Eq. (1.1; 1), we have

$$\begin{aligned} -L_n x(s) &= q(s) f\big(x\big[g(s)\big]\big) \\ &\geqslant q(s) f\big(I_{n-2}\big(g(s), \rho(t); a_1, \ldots, a_{n-2}\big)\big) f\big(L_{n-2} x\big[\rho(t)\big]\big). \end{aligned}$$

Integrating this inequality from t to $\rho(t)$ we have

$$\begin{aligned} &\big(L_{n-1} x(t)\big)^\alpha - \big(L_{n-1} x\big[\rho(t)\big]\big)^\alpha \\ &\quad \geqslant \int_t^{\rho(t)} q(s) f\big(I_{n-2}\big(g(s), \rho(t); a_1, \ldots, a_{n-2}\big)\big)\, ds\, f\big(L_{n-2} x\big[\rho(t)\big]\big). \end{aligned}$$

Thus,

$$L_{n-1} x(t) \geqslant \left[\int_t^{\rho(t)} q(s) f\big(I_{n-2}\big(g(s), \rho(s); a_1, \ldots, a_{n-2}\big)\big)\, ds\right]^{1/\alpha} f^{1/\alpha}\big(L_{n-2} x\big[\rho(t)\big]\big).$$

Let $y(t) = L_{n-2} x(t)$. Then $y(t) > 0$ for $t \geqslant t_1$ and satisfies the inequality

$$y'(t) \geqslant Q_{n-1}^{**}(t) f^{1/\alpha}\big(y\big[\rho(t)\big]\big) \quad \text{for } t \geqslant t_1.$$

By a similar result in [11], we see that the equation

$$z'(t) - Q^{**}(t) f^{1/\alpha}\big(z\big[\rho(t)\big]\big) = 0$$

has an eventually positive solution, a contradiction.

3.2. *The case $\ell = 0$*

Here we present some sufficient conditions which ensure that $\mathcal{N}_0 = \emptyset$. It is easy to check that the case when $\ell = 0$ occurs for Eq. (1.1; 1) with n odd and Eq. (1.1; −1) with n even. Now, we present the following results.

Theorem 3.3. *Let $(-1)^n \delta = -1$ and*

$$f(x) \geqslant x^\beta \quad \textit{for } x \neq 0, \tag{3.23}$$

where β is a quotient of positive odd integers. If

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(s) I_{n-1}^{\alpha}\big(\tau(t), g(s); a_{n-1}, \ldots, a_1\big)\, ds > 1 \quad \textit{if } \alpha = \beta, \tag{3.24}$$

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(s) I_{n-1}^{\beta}\big(\tau(t), g(s); a_{n-1}, \ldots, a_1\big)\, ds > 0 \quad \textit{if } \alpha > \beta, \tag{3.25}$$

then $\mathcal{N}_0 = \emptyset$.

Proof. Assume that $x \in \mathcal{N}_0$ and suppose that $x(t) > 0$ for $t \geqslant t_0$. Then there exists $t_1 \geqslant t_0$ such that $\inf\{g(t)\colon t \geqslant t_1\} > t_0$ and

$$(-1)^i L_i x(t) > 0 \quad (0 \leqslant i \leqslant n) \text{ on } [t_1, \infty). \tag{3.26}$$

Replacing i, k, t and s by $0, n-1, \sigma$ and τ, respectively, in (2.2), we get

$$x(\sigma) = \sum_{j=0}^{n-2} (-1)^j I_j(\tau, \sigma; a_j, \ldots, a_1) L_j x(\tau) + (-1)^{n-1} \int_\sigma^\tau I_{n-2}(u, \sigma; a_{n-2}, \ldots, a_1) a_{n-1}(u) L_{n-1} x(u)\, du.$$

Using (3.26) and the fact that $(-1)^{n-1} L_{n-1} x(t)$ is decreasing on $[t_1, \infty)$, we obtain

$$\begin{aligned} x(\sigma) &\geqslant (-1)^{n-1} \int_\sigma^\tau I_{n-2}(u, \sigma; a_{n-2}, \ldots, a_1) a_{n-1}(u)\, du\, L_{n-1} x(\tau) \\ &= I_{n-1}(\tau, \sigma; a_{n-1}, \ldots, a_1)\big((-1)^{n-1} L_{n-1} x(\tau)\big), \quad \tau \geqslant \sigma \geqslant t_1. \end{aligned} \tag{3.27}$$

Substituting $g(s)$ and $\tau(t)$ for σ and τ, respectively, in (3.27), we have

$$x\big[g(s)\big] \geqslant I_{n-1}\big(\tau(t), g(s); a_{n-1}, \ldots, a_1\big)\big((-1)^{n-1} L_{n-1} x\big[\tau(t)\big]\big) \quad \text{for } \tau(t) < s < t. \tag{3.28}$$

Using (3.23) and (3.28) in Eq. $(1.1;\delta)$, for $t \geqslant s \geqslant t_1$, we get

$$\begin{aligned} -\frac{d}{ds}\big((-1)^{n-1} L_{n-1} x(s)\big)^\alpha &= q(s) f\big(x\big[g(s)\big]\big) \geqslant q(s) x^\beta\big[g(s)\big] \\ &\geqslant q(s) I_{n-1}^\beta\big(\tau(t), g(s); a_{n-1}, \ldots, a_1\big)\big((-1)^{n-1} L_{n-1} x\big[\tau(t)\big]\big)^\beta. \end{aligned} \tag{3.29}$$

Integrating both sides of inequality (3.29) from $\tau(t)$ to t, we obtain

$$\begin{aligned} &-\big((-1)^{n-1} L_{n-1} x(t)\big)^\alpha + \big((-1)^{n-1} L_{n-1} x\big[\tau(t)\big]\big)^\alpha \\ &\quad \geqslant \big((-1)^{n-1} L_{n-1} x\big[\tau(t)\big]\big)^\beta \int_{\tau(t)}^t q(s) I_{n-1}^\beta\big(\tau(t), g(s); a_{n-1}, \ldots, a_1\big)\, ds. \end{aligned} \tag{3.30}$$

Now, we consider two cases.

Case 1: $\alpha = \beta$. In this case (3.30) is reduced to

$$\big((-1)^{n-1} L_{n-1} x\big[\tau(t)\big]\big)^\alpha \left[\int_{\tau(t)}^t q(s) I_{n-1}^\alpha\big(\tau(t), g(s); a_{n-1}, \ldots, a_1\big)\, ds - 1\right] \leqslant 0 \quad \text{for } t \geqslant t_1.$$

But this is inconsistent with (3.24).

Case 2: $\alpha > \beta$. It follows from (3.30) that

$$\big((-1)^{n-1}L_{n-1}x\big[\tau(t)\big]\big)^{\alpha-\beta} \geqslant \int_{\tau(t)}^{t} q(s)I_{n-1}^{\beta}\big(\tau(t), g(s); a_{n-1}, \dots, a_1\big)\, ds. \tag{3.31}$$

Taking lim sup of both sides of (3.31) as $t \to \infty$, we see that the left-hand side approaches zero, which contradicts (3.25). This completes the proof. □

The following corollary is immediate.

Corollary 3.3. *Let $g(t) < t$ for $t \geqslant t_0$, $(-1)^n\delta = -1$ and condition* (3.23) *hold. If*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} q(s)I_{n-1}^{\alpha}\big(g(t), g(s); a_{n-1}, \dots, a_1\big)\, ds > 1 \quad \textit{when } \alpha = \beta, \tag{3.32}$$

$$\limsup_{t\to\infty} \int_{g(t)}^{t} q(s)I_{n-1}^{\beta}\big(g(t), g(s); a_{n-1}, \dots, a_1\big)\, ds > 0 \quad \textit{when } \alpha > \beta, \tag{3.33}$$

then $\mathcal{N}_0 = \emptyset$.

Theorem 3.4. *Let $(-1)^n\delta = -1$, condition* (3.1) *hold and*

$$\int_{-0} \frac{du}{f(u^{1/\alpha})} < \infty \quad \textit{and} \quad \int_{+0} \frac{du}{f(u^{1/\alpha})} < \infty. \tag{3.34}$$

If

$$\int_{\mathcal{R}_g} q(s) f\big(I_{n-1}\big(s, g(s); a_{n-1}, \dots, a_1\big)\big)\, ds = \infty, \tag{3.35}$$

then $\mathcal{N}_0 = \emptyset$.

Proof. Let $x \in \mathcal{N}_0$ and assume that $x(t) > 0$ for $t \geqslant t_0$. Then there exists $t_1 \geqslant t_0$ such that (3.26) holds for $t \geqslant t_1$. Choose T_1 so large that $T_1 \geqslant t_1$ and $\inf\{g(t)\colon\ t \geqslant T_1\} > t_1$. As in the proof of Theorem 3.3, we obtain (3.27) for $\tau \geqslant \sigma \geqslant T_1$. Substituting $g(t)$ and t for σ and τ, respectively, in inequality (3.27) we get

$$x\big[g(t)\big] \geqslant I_{n-1}\big(t, g(t); a_{n-1}, \dots, a_1\big)\big((-1)^{n-1}L_{n-1}x(t)\big) \quad \text{for } t \in \mathcal{R}_g \cap [T_1, \infty). \tag{3.36}$$

Using (3.1) and (3.36) in Eq. (1.1; δ) and letting $y(t) = ((-1)^{n-1}L_{n-1}x(t))^{\alpha} > 0$ on $\mathcal{R}_g \cap [T_1, \infty)$, we have

$$-y'(t) \geqslant q(t) f\big(I_{n-1}\big(t, g(t); a_{n-1}, \dots, a_1\big)\big) f\big(y^{1/\alpha}(t)\big), \quad t \in \mathcal{R}_g \cap [T_1, \infty).$$

Choose T_2 ($\geqslant T_1$) arbitrarily. Dividing both sides of the above inequality by $f(y^{1/\alpha}(t))$ and integrating the result on $\mathcal{R}_g \cap [T_1, T_2]$, we find

$$\int_{\mathcal{R}_g\cap[T_1,T_2]} q(s) f\big(I_{n-1}\big(s, g(s); a_{n-1}, \dots, a_1\big)\big)\, ds \leqslant \int_{T_2}^{T_1} \frac{y'(s)}{f(y^{1/\alpha}(s))}\, ds = \int_{y(T_2)}^{y(T_1)} \frac{du}{f(u^{1/\alpha})}.$$

Letting $T_2 \to \infty$, we conclude that

$$\int_{\mathcal{R}_g\cap[T_1,\infty)} q(s) f\big(I_{n-1}\big(s, g(s); a_{n-1}, \dots, a_1\big)\big)\, ds \leqslant \int_0^{y(T_1)} \frac{du}{f(u^{1/\alpha})} < \infty,$$

which contradicts condition (3.35). This completes the proof. □

Remark 3.2. It is easy to see that the conclusion '$\mathcal{N}_0 = \emptyset$' can be replaced by 'all bounded solutions of Eq. $(1.1; \delta)$ are oscillatory.'

3.3. The case $\ell = n$

Here, we shall present some sufficient conditions which ensure that $\mathcal{N}_n = \emptyset$. It is easy to check that the case when $\ell = n$ occurs for Eq. $(1,1;-1)$. Now, we present the following results.

Theorem 3.5. *Let $\delta = -1$ and condition* (3.23) *hold. If*

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) I_{n-1}^{\alpha}\big(g(s), \rho(t); a_1, \dots, a_{n-1}\big)\, ds > 1 \quad \text{when } \alpha = \beta, \tag{3.37}$$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) I_{n-1}^{\beta}\big(g(s), \rho(t); a_1, \dots, a_{n-1}\big)\, ds > 0 \quad \text{when } \alpha < \beta, \tag{3.38}$$

then $\mathcal{N}_n = \emptyset$.

Proof. Assume that $x \in \mathcal{N}_n$ and assume that $x(t) > 0$ for $t \geqslant t_0$. Then there exists $t_1 \geqslant t_0$ such that

$$L_i x(t) > 0 \quad (0 \leqslant i \leqslant n) \text{ on } [t_1, \infty). \tag{3.39}$$

From (2.1) with i, k, t and s replaced by $0, n-1, g(s)$ and $\rho(t)$, respectively,

$$x\big[g(s)\big] = \sum_{j=0}^{n-2} I_j\big(g(s), \rho(t); a_1, \dots, a_j\big) L_j x\big[\rho(t)\big] + \int_{\rho(t)}^{g(s)} I_{n-2}\big(g(s), u; a_1, \dots, a_{n-2}\big) a_{n-1}(u) L_{n-1} x(u)\, du.$$

Using (3.39) and noting that $L_{n-1}x$ is increasing, we easily get

$$x[g(s)] \geqslant I_{n-1}(g(s), \rho(t); a_1, \ldots, a_{n-1}) L_{n-1}x[\rho(t)] \quad \text{for } t < s < \rho(t). \tag{3.40}$$

Using (3.40) and (3.23) in Eq. (1.1; -1), we have for $t_1 \leqslant t < s < \rho(t)$,

$$\begin{aligned}\frac{d}{ds}\left(L_{n-1}^{\alpha}x(s)\right) &= q(s) f\left(x[g(s)]\right) \geqslant q(s)x^{\beta}[g(s)] \\ &\geqslant q(s) I_{n-1}^{\beta}\left(g(s), \rho(t); a_1, \ldots, a_{n-1}\right) L_{n-1}^{\beta}x[\rho(t)].\end{aligned} \tag{3.41}$$

Integrating both sides of inequality (3.41) from t to $\rho(t)$ we obtain

$$\begin{aligned}&L_{n-1}^{\alpha}x[\rho(t)] - L_{n-1}^{\alpha}x(t) \\ &\quad \geqslant \int_t^{\rho(t)} q(s) I_{n-1}^{\beta}\left(g(s), \rho(t); a_1, \ldots, a_{n-1}\right) ds\, L_{n-1}^{\beta}x[\rho(t)].\end{aligned} \tag{3.42}$$

Now, we consider two cases.

Case 1: $\alpha = \beta$. In this case (3.42) becomes

$$L_{n-1}^{\alpha}x[\rho(t)]\left[\int_t^{\rho(t)} q(s) I_{n-1}^{\alpha}\left(g(s), \rho(t); a_1, \ldots, a_{n-1}\right) ds - 1\right] \leqslant 0.$$

But this is inconsistent with (3.37).

Case 2: $\alpha < \beta$. It follows from (3.42) that

$$L_{n-1}^{\alpha-\beta}x[\rho(t)] \geqslant \int_t^{\rho(t)} q(s) I_{n-1}^{\beta}\left(g(s), \rho(t); a_1, \ldots, a_{n-1}\right) ds. \tag{3.43}$$

Taking lim sup of both sides of inequality (3.43) as $t \to \infty$, one can easily see that the left-hand side approaches zero, which contradicts (3.38). This completes the proof. □

The following corollary is immediate.

Corollary 3.4. *Let $g(t) > t$ for $t \geqslant t_0$, $\delta = -1$ and condition* (3.23) *hold. If*

$$\limsup_{t\to\infty} \int_t^{g(t)} q(s) I_{n-1}^{\alpha}\left(g(s), g(t); a_1, \ldots, a_{n-1}\right) ds > 1 \quad \textit{when } \alpha = \beta, \tag{3.44}$$

$$\limsup_{t\to\infty} \int_t^{g(t)} q(s) I_{n-1}^{\beta}\left(g(s), g(t); a_1, \ldots, a_{n-1}\right) ds > 0 \quad \textit{when } \alpha < \beta, \tag{3.45}$$

then $\mathcal{N}_n = \emptyset$.

Theorem 3.6. *Let $\delta = -1$, condition* (3.1) *hold and*

$$\int^{-\infty} \frac{du}{f(u^{1/\alpha})} < \infty \quad \textit{and} \quad \int^{+\infty} \frac{du}{f(u^{1/\alpha})} < \infty. \tag{3.46}$$

If

$$\int_{\mathcal{A}_g} q(s) f\big(I_{n-1}\big(g(s), s; a_1, \ldots, a_n\big)\big)\, ds = \infty, \tag{3.47}$$

then $\mathcal{N}_n = \emptyset$.

Proof. Let $x \in \mathcal{N}_n$ and assume that $x(t) > 0$ for $t \geqslant t_0$. Then there exists $t_1 \geqslant t_0$ such that (3.39) holds for $t \geqslant t_1$. Replacing i, k, t and s in (2.1) by 0, $n-1$, $g(t)$ and t, respectively, we get

$$\begin{aligned} x\big[g(t)\big] &= \sum_{j=0}^{n-2} I_j\big(g(t), t; a_1, \ldots, a_j\big) L_j x(t) \\ &\quad + \int_t^{g(t)} I_{n-2}\big(g(t), u; a_1, \ldots, a_{n-2}\big) a_{n-1}(u) L_{n-1} x(u)\, du \\ &\geqslant \int_t^{g(t)} I_{n-2}\big(g(t), u; a_1, \ldots, a_{n-2}\big) a_{n-1}(u) L_{n-1} x(u)\, du \end{aligned}$$

for $t \in \mathcal{A}_g \cap [t_1, \infty)$ which in view of the increasing nature of $L_{n-1}x$ implies

$$x\big[g(t)\big] \geqslant I_{n-1}\big(g(t), t; a_1, \ldots, a_{n-1}\big) L_{n-1} x(t), \quad t \in \mathcal{A}_g \cap [t_1, \infty).$$

Set $u(t) = L_{n-1}^{\alpha} x(t)$. Then $u(t)$ satisfies

$$\frac{du}{dt} = L_n x(t) = q(t) f\big(x\big[g(t)\big]\big) \geqslant q(t) f\big(I_{n-1}\big(g(t), t; a_1, \ldots, a_{n-1}\big)\big) f\big(u^{1/\alpha}(t)\big) \quad \text{for } t \in \mathcal{A} \cap [t_1, \infty).$$

Dividing the above inequality by $f(u^{1/\alpha})$ and integrating on $\mathcal{A}_g \cap [t_1, T_1]$, where T_1 ($\geqslant t_1$) is arbitrary, we obtain

$$\begin{aligned} &\int_{\mathcal{A}_g \cap [t_1, T_1]} q(s) f\big(I_{n-1}\big(g(s), s; a_1, \ldots, a_{n-1}\big)\big)\, ds \\ &\quad \leqslant \int_{t_1}^{T_1} \frac{u'(s)}{f(u^{1/\alpha}(s))}\, ds = \int_{u(t_1)}^{u(T_1)} \frac{du}{f(u^{1/\alpha})} < \infty. \end{aligned}$$

This contradicts (3.47). This completes the proof. □

Theorem 3.7. *Let* $g(t) > t$ *for* $t \geqslant t_0$, $\delta = -1$ *and* (3.23) *hold. If*

$$\limsup_{t\to\infty} \int_t^{g(t)} I_{n-2}\big(g(t), u; a_1, \ldots, a_{n-2}\big) a_{n-1}(u) \left(\int_t^u q(s)\, ds \right)^{1/\alpha} du > 1 \quad \text{when } \alpha = \beta, \tag{3.48}$$

$$\limsup_{t\to\infty} \int_t^{g(t)} I_{n-2}\big(g(t), u; a_1, \ldots, a_{n-2}\big) a_{n-1}(u) \left(\int_t^u q(s)\, ds \right)^{1/\alpha} du > 0$$
$$\text{when } \alpha < \beta, \tag{3.49}$$

then $\mathcal{N}_n = \emptyset$.

Proof. Assume that $x \in \mathcal{N}_n$, say $x(t) > 0$ for $t \geqslant t_0$. Then there exists $t_1 \geqslant t_0$ such that (3.39) holds on $[t_1, \infty)$. Let $T_1 > t_1$ be such that $\inf\{g(t){:}\ t \geqslant T_1\} \geqslant t_1$. As in the proof of Theorem 3.5, one can easily find

$$x\big[g(t)\big] \geqslant \int_t^{g(t)} I_{n-2}\big(g(t), u; a_1, \ldots, a_{n-2}\big) a_{n-1}(u) L_{n-1}x(u)\, du, \quad t \geqslant T_1. \tag{3.50}$$

Integrating Eq. $(1.1;\delta)$ from t to u, we have

$$L_{n-1}^{\alpha}x(u) - L_{n-1}^{\alpha}x(t) = \int_t^u q(s) f\big(x\big[g(s)\big]\big)\, ds, \quad u \geqslant t \geqslant T_1,$$

or

$$L_{n-1}x(u) \geqslant \left(\int_t^u q(s) f\big(x\big[g(s)\big]\big)\, ds \right)^{1/\alpha} \quad \text{for } u \geqslant t \geqslant T_1. \tag{3.51}$$

Substituting (3.51) in (3.50), we get

$$x\big[g(t)\big] \geqslant \int_t^{g(t)} I_{n-2}\big(g(t), u; a_1, \ldots, a_{n-2}\big) a_{n-1}(u) \left(\int_t^u q(s)\, ds \right)^{1/\alpha} \times f^{1/\alpha}\big(x\big[g(t)\big]\big)\, du.$$

The rest of the proof is similar to that of Theorem 3.5 and hence omitted. □

3.4. Oscillation criteria

In this subsection, we combine the results of Sections 3.1–3.3 and obtain the following interesting criteria for the oscillation of Eq. $(1.1;\delta)$.

Theorem 3.8. *Let* $\delta = 1$ *and* n *be even.*

(i) *Assume that conditions* (i)–(v) *and* (3.1) *hold. If for all large* T *with* $\sigma(t) \geqslant T$ *condition* $(3.2;\ell)$ $(\ell = 1, 3, \ldots, n-3)$ *and* $(3.2; n-1)$ *hold, then Eq.* $(1.1; 1)$ *is oscillatory.*

(ii) *Assume that conditions* (i)–(v) *and* (3.1) *hold. For all large* t, *suppose the second order equations* $(3.10;\ell)$ $(\ell = 1, 3, \ldots, n-3)$ *are oscillatory and either* (a) *there exists* $\eta(t) \in C([t_0, \infty), \mathbb{R})$ *such that* $\eta(t) \leqslant \sigma(t)$ *for* $t \geqslant t_0$, $\lim_{t\in\infty} \eta(t) = \infty$ *such that the second order equation* $(3.10; n-1)$ *is oscillatory, or* (b) *the first order delay* *equation* (3.10*; $n-1$) *is oscillatory, or* (c) *the first order advanced equation* (3.21) *is oscillatory. Then Eq.* (1.1; 1) *is oscillatory.*

Remark 3.3. We note that Theorem 3.8 is applicable to equations of type (1.1; δ) with $f(x)=x^{\beta}$, say, where β is the quotient of positive odd integers, $\beta>\alpha$, $\beta=\alpha$ and $\beta<\alpha$ and with general deviating argument.

Theorem 3.9. *Let $\delta=1$ and n be odd.*

(i) *Assume that conditions* (i)–(v) *and* (3.1) *hold. If for all large $T\geqslant t_0$ and $\sigma(t)\geqslant T$, condition* (3.2; ℓ) ($\ell=2,4,\dots,n-3$), (3.2; $n-1$), (3.34) *and* (3.35) *are satisfied, then Eq.* (1.1; 1) *is oscillatory.*

(ii) *Assume that conditions* (i)–(v) *and* (3.1) *hold. For all large t, suppose the second order equations* (3.10; ℓ) ($\ell=2,4,\dots,n-3$) *are oscillatory and either* (a) *there exists $\eta(t)\in C([t_0,\infty),\mathbb{R})$ such that $\eta(t)\leqslant\sigma(t)$ for $t\geqslant t_0$, $\lim_{t\to\infty}\eta(t)=\infty$ such that the second order equation* (3.10; $n-1$) *is oscillatory, or* (b) *the first order delay equation* (3.10*; $n-1$) *is oscillatory and conditions* (3.34) *and* (3.35) *hold. Then Eq.* (1.1; 1) *is oscillatory.*

Corollary 3.5. *Let $\delta=1$ and n be odd.*

(i) *Assume that conditions* (i)–(v) *and* (3.23) *hold. If, for all large $T\geqslant t_0$ with $\sigma(t)\geqslant T$,*

$$\int^{\infty} K_{n-\ell-2}(t,T)\left(\int_t^{\infty} q(u)\,du\right)^{1/\alpha} J_{\ell-1}^{\beta/\alpha}(\sigma(t),T)\,dt=\infty,$$

$$\ell\in\{2,4,\dots,n-3\}, \tag{3.52; ℓ}$$

$$\int^{\infty} q(t)J_{n-2}^{\beta}\big(\sigma(t),T\big)\,dt=\infty, \tag{3.52; $n-1$}$$

and either condition (3.24) *when $\alpha=\beta$, or condition* (3.25) *when $\alpha>\beta$ is satisfied, then Eq.* (1.1; 1) *is oscillatory.*

(ii) *Assume that conditions* (i)–(v) *and* (3.23) *hold. For all large t, suppose the second order equations*

$$\left(\frac{1}{a_\ell(t)}y'(t)\right)'+Q_\ell(t)y^{\beta/\alpha}\big[\sigma(t)\big]=0,\quad \ell\in\{2,4,\dots,n-3\}, \tag{3.53; ℓ}$$

are oscillatory and either (a) *there exists $\eta(t)\in C([t_0,\infty),\mathbb{R})$ such that $\eta(t)\leqslant\sigma(t)$ for $t\geqslant t_0$ and $\lim_{t\to\infty}\eta(t)=\infty$ and the equation*

$$\left(\left(\frac{1}{a_{n-1}(t)}w'(t)\right)^{\alpha}\right)'+Q_{n-1}(t)w^{\beta}\big[\eta(t)\big]=0 \tag{3.53; $n-1$}$$

is oscillatory, or (b) *the first order delay equation*

$$u'(t)+Q_{n-1}^{*}(t)u^{\beta/\alpha}\big[\sigma(t)\big]=0 \tag{3.53*; $n-1$}$$

is oscillatory and either condition (3.24) *when* $\alpha = \beta$, *or condition* (3.25) *when* $\alpha > \beta$ *holds. Then Eq.* (1.1; 1) *is oscillatory.*

Remark 3.4. We note that Theorem 3.9 and Corollary 3.5 are applicable to odd order equations of type (1.1; 1) with retarded as well as general deviating arguments.

Theorem 3.10. *Let* $\delta = -1$ *and* n *be odd.*

(i) *Assume that conditions* (i)–(v) *and* (3.1) *hold. If for all large* $T \geqslant t_0$ *with* $\sigma(t) \geqslant T$ *conditions* (3.2; ℓ) ($\ell = 1, 3, \dots, n-2$), (3.46) *and* (3.47) *hold, then Eq.* (1.1; −1) *is oscillatory.*
(ii) *Assume that conditions* (i)–(v) *and* (3.1) *hold. If for all large* t *the second order equations* (3.10; ℓ) ($\ell = 1, 3, \dots, n-4$) *are oscillatory, all bounded solutions of Eq.* (3.10; $n-2$) *are oscillatory and conditions* (3.46) *and* (3.47) *hold, then Eq.* (1.1; −1) *is oscillatory.*

Corollary 3.6. *Let* $\delta = 1$ *and* n *be odd.*

(i) *Assume that conditions* (i)–(v) *and* (3.23) *hold. If for all large* $T \geqslant t_0$ *with* $\sigma(t) \geqslant T$ *conditions* (3.52; ℓ) ($\ell = 1, 3, \dots, n-2$) *and either condition* (3.37) *when* $\alpha = \beta$, *or condition* (3.38) *when* $\alpha < \beta$ *are satisfied, then Eq.* (1.1; −1) *is oscillatory.*
(ii) *Assume that conditions* (i)–(v) *and* (3.23) *hold. If for all large* t *the second order equations* (3.53; ℓ) ($\ell = 1, 3, \dots, n-3$) *are oscillatory, all bounded solutions of the equation*

$$\left(\left(\frac{1}{a_{n-1}(t)}z'(t)\right)^{\alpha}\right)' - Q_{n-2}(t)z^{\beta}\big[\sigma(t)\big] = 0 \tag{3.53; $n-2$}$$

are oscillatory and either condition (3.37) *when* $\alpha = \beta$ *or condition* (3.38) *when* $\alpha < \beta$ *is satisfied, then Eq.* (1.1; −1) *is oscillatory.*

Theorem 3.11. *Let* $\delta = -1$ *and* n *be even.*

(i) *Assume that conditions* (i)–(v) *and* (3.1) *hold. If for all large* $T \geqslant t_0$ *with* $\sigma(t) \geqslant T$ *conditions* (3.2; ℓ) ($\ell = 2, 4, \dots, n-2$),

$$\int_{\pm 0}^{\pm\infty} \frac{du}{f^{1/\alpha}(u)} < \infty, \tag{3.54}$$

and conditions (3.35) *and* (3.47) *hold, then Eq.* (1.1; −1) *is oscillatory.*
(ii) *Assume that conditions* (i)–(v) *and* (3.1) *hold. If for all large* t *Eqs.* (3.10; ℓ) ($\ell = 2, 4, \dots, n-4$) *are oscillatory, all bounded solutions of Eq.* (3.10; $n-2$) *are oscillatory and conditions* (3.54), (3.35) *and* (3.47) *hold, then Eq.* (1.1; −1) *is oscillatory.*

Corollary 3.7. *Let* $\delta = -1$ *and* n *be even.*

(i) *Assume that conditions* (i)–(v) *and* (3.23) *with* $\alpha = \beta$ *hold. If for all large* $T \geqslant t_0$ *with* $\sigma(t) \geqslant T$ *conditions* (3.52; ℓ) ($\ell = 2, 4, \ldots, n-2$), (3.24) *and* (3.37) *hold, then Eq.* (1.1; -1) *is oscillatory.*

(ii) *Assume that conditions* (i)–(v) *and* (3.23) *with* $\alpha = \beta$ *hold. Assume for all large* t *Eqs.* (3.53; ℓ) ($\ell = 2, 4, \ldots, n-4$) *are oscillatory. Also assume for large* t *all bounded solutions of Eq.* (3.53; $n-2$) *are oscillatory, and conditions* (3.24) *and* (3.37) *hold, then Eq.* (1.1; -1) *is oscillatory.*

Remark 3.5. We note that the results for the case when $\delta = -1$ and n is odd may be applied to equations of type (1.1; -1) with $g(t)$ is either advanced or of mixed type argument, while for the case when $\delta = -1$ and n is even, the obtained results can be applied to equations of type (1.1; -1) with $g(t)$ is a general argument (i.e., of mixed type, e.g., $g(t) = t + \sin t$).

3.5. *Oscillation of equation* (1.1; δ) *with* $\alpha = 1$

We shall consider Eq. (1.1; δ) with $\alpha = 1$, i.e.,

$$L_n x(t) + \delta q(t) f\big(x\big[g(t)\big]\big) = 0, \tag{3.55; δ}$$

where

$$L_0 x(t) = x(t), \quad L_n x(t) = \frac{1}{a_k(t)} \frac{d}{dt}\big(L_{k-1}x(t)\big), \quad k = 1, 2, \ldots, n,$$

and $a_n(t) = 1$, $n \geqslant 3$, $\delta = \pm 1$ and conditions (i)–(iv) hold.

Here, we shall present some criteria for the nonoscillation of Eq. (3.55; ℓ) which are different then those obtained from our earlier results by setting $\alpha = 1$.

Theorem 3.12. *Let conditions* (i)–(iv) *and* (3.1) *hold and let* $1 \leqslant \ell \leqslant n$ *with* $(-1)^{n-\ell}\delta = -1$. *If for all large* $T \geqslant t_0$ *and* $\sigma(t) \geqslant T$,

$$\int^{\infty} K_{n-\ell-1}(t, T) q(t) f\big(J_{\ell-1}\big(\sigma(t), T\big)\big)\, dt = \infty, \tag{3.56; ℓ}$$

then $\mathcal{N}_\ell = \emptyset$.

Proof. Let $x \in \mathcal{N}_\ell$ and assume that $x(t) > 0$ for $t \geqslant t_0$. There exists $t_1 \geqslant t_0$ such that (3.3) holds for $t \geqslant t_1$. Suppose $\ell \leqslant n-1$. From formula (2.2) with $i = \ell$, $k = n-1$, $t = t_1$ and $s \geqslant t_1$, it follows that

$$\begin{aligned} L_\ell x(t_1) = {} & \sum_{j=\ell}^{n-1} (-1)^{j-\ell} I_{j-\ell}(s, t_1; a_j, \ldots, a_{\ell+1}) L_j x(s) \\ & + (-1)^{n-\ell} \int_{t_1}^{s} I_{n-\ell-1}(u, t_1; a_{n-1}, \ldots, a_{\ell+1}) L_n x(u)\, du. \end{aligned} \tag{3.57}$$

Using Eq. (3.55; δ) and (3.3) in (3.57), we have

$$L_\ell x(t_1) \geqslant \int_{t_1}^{s} I_{n-\ell-1}(u, t_1; a_{n-1}, \ldots, a_{\ell+1}) q(u) f\big(x\big[g(u)\big]\big)\,du,$$

which gives in the limit as $s \to \infty$,

$$\int_{t_1}^{\infty} K_{n-\ell-1}(t, t_1) q(t) f\big(x\big[g(t)\big]\big)\,dt < \infty. \tag{3.58}$$

As in the proof of Theorem 3.1, we have (3.7) for $t \geqslant t_2 \geqslant t_1$. Combining (3.58) with (3.7) and using condition (3.1), we obtain

$$\int_{t_2}^{\infty} K_{n-\ell-1}(t, t_1) q(t) f\big(J_{\ell-1}\big(\sigma(t), t_1\big)\big)\,dt < \infty,$$

which contradicts (3.56; ℓ).

Next, suppose $\ell = n-1$. The proof of this case is similar to that of Theorem 3.1 and hence omitted. □

For simplicity, we let for all large t,

$$Q_j(t) = a_{j+1}(t) \int_{t}^{\infty} K_{n-j-2}(u, t) q(u) f\big(J_{j-1}\big(g(u), \sigma(t)\big)\big)\,du,$$
$$j = 1, 2, \ldots, n-2,$$

and

$$Q_{n-1}(t) = q(t) f\big(J_{n-2}\big(g(t), \sigma(t)\big)\big).$$

Theorem 3.13. *Let conditions* (i)–(iv) *and* (3.1) *hold and let* $1 \leqslant \ell \leqslant n-1$ *with* $(-1)^{n-\ell}\delta = -1$. *If for all large* $T \geqslant t_0$ *with* $g(t) > \sigma(t) \geqslant T$ *all of the second order equations*

$$\left(\frac{1}{a_\ell(t)} y'(t)\right)' + Q_\ell(t) f\big(y\big[\sigma(t)\big]\big) = 0 \tag{3.59; ℓ}$$

are oscillatory, then $\mathcal{N}_\ell = \emptyset$.

Proof. Let $x \in \mathcal{N}_\ell$ and suppose that $x(t) > 0$ for $t \geqslant t_0$. There exists $t_1 \geqslant t_0$ such that (3.3) holds for $t \geqslant t_1$. Let $\ell < n-1$. Putting $i = \ell+1$, $k = n-1$, $s \geqslant t \geqslant t_1$ in (2.2), we have

$$L_{\ell+1} x(t) = \sum_{j=\ell+1}^{n-1} (-1)^{j-\ell-1} I_{j-\ell-1}(s, t; a_j, \ldots, a_{\ell+2}) L_j x(s)$$
$$+ (-1)^{n-\ell-1} \int_{t}^{s} I_{n-\ell-2}(u, t; a_{n-1}, \ldots, a_{\ell+2}) L_n x(u)\,du.$$

Letting $s \to \infty$ in the above equality, we obtain

$$-L_{\ell+1}x(t) \geqslant \int_t^\infty I_{n-\ell-2}(u,t;a_{n-1},\dots,a_{\ell+2})q(u)f\bigl(x\bigl[g(u)\bigr]\bigr)\,du \quad \text{for } t \geqslant t_1. \tag{3.60}$$

As in the proof of Theorem 3.2, we obtain (3.16) for $t \geqslant t_2 \geqslant t_1$. Combining (3.60) with (3.16) and using condition (3.1), we have

$$-L_{\ell+1}x(t) \geqslant f\bigl(L_{\ell-1}x\bigl[\sigma(t)\bigr]\bigr)\int_t^\infty I_{n-\ell-2}(u,t;a_{n-1},\dots,a_{\ell+2})q(u) \times f\bigl(I_{\ell-1}\bigl(g(u),\sigma(t);a_1,\dots,a_{\ell-1}\bigr)\bigr)\,du.$$

The rest of the proof is similar to that of Theorem 3.2 and hence omitted.

Let $\ell = n-1$. An integration of Eq. $(3.55;\delta)$ yields

$$L_{n-1}x(t) \geqslant \int_t^\infty q(u)f\bigl(x\bigl[g(u)\bigr]\bigr)\,du \quad \text{for } t \geqslant t_1. \tag{3.61}$$

Setting $i = 0$, $k = n-3$, $t \geqslant s \geqslant t_1$ in (2.1), we have

$$x(t) = \sum_{j=0}^{n-3} I_j(t,t_1;a_1,\dots,a_j)L_jx(t_1) + \int_{t_1}^t I_{n-3}(t,u;a_1,\dots,a_{n-3})a_{n-2}(u)L_{n-2}x(u)\,du.$$

From this we easily see that

$$x(t) \geqslant \int_{t_1}^t I_{n-3}(t,u;a_1,\dots,a_{n-3})a_{n-2}(u)L_{n-2}x(u)\,du \quad \text{for } t \geqslant t_1.$$

There exists $t_2 \geqslant t_1$ such that

$$x\bigl[g(t)\bigr] \geqslant \int_{\sigma(t)}^{g(t)} I_{n-3}\bigl(g(t),u;a_1,\dots,a_{n-3}\bigr)a_{n-2}(u)\,du\,L_{n-2}x\bigl[\sigma(t)\bigr] = I_{n-2}\bigl(g(t),\sigma(t);a_1,\dots,a_{n-2}\bigr)L_{n-2}x\bigl[\sigma(t)\bigr] \quad \text{for } t \geqslant t_2. \tag{3.62}$$

Combining (3.62) with (3.61) and using (3.1), we have

$$L_{n-1}x(t) \geqslant \int_t^\infty q(u)f\bigl(I_{n-2}\bigl(g(u),\sigma(t);a_1,\dots,a_{n-2}\bigr)\bigr)f\bigl(L_{n-2}x\bigl[\sigma(u)\bigr]\bigr)\,du \quad \text{for } t \geqslant t_2.$$

Integrating this inequality from t_2 to t we see that $w(t) = L_{n-2}x(t) > 0$ satisfies

$$w(t) \geqslant w(t_2) + \int_{t_2}^{t} a_{n-1}(s) \int_{s}^{\infty} Q_{n-1}(u) f\big(w[\sigma(u)]\big)\, du\, ds \quad \text{for } t \geqslant t_2. \tag{3.63}$$

Denoting the right-hand side of (3.63) by $z(t)$, it is easy to see that

$$\left(\frac{z'(t)}{a_{n-1}(t)}\right)' + Q_{n-1}(t) f\big(z[\sigma(t)]\big) \leqslant 0 \quad \text{for } t \geqslant t_2.$$

The rest of the proof is similar to that of Theorem 3.2 and hence omitted. This completes the proof. □

Remark 3.6. In Eq. (3.55; δ) if $f(x) = x$ and $g(t) \leqslant t$ for $t \geqslant t_0$, then $Q_\ell(t)$ in Eq. (3.59; ℓ) takes the form

$$\bar{Q}_\ell(t) = a_{\ell+1}(t) \int_{t}^{\infty} K_{n-\ell-2}(u,t) q(u) J_{\ell-1}\big(g(u), \sigma(t)\big)\, du, \quad j = 1, 2, \ldots, n-2,$$

and

$$\bar{Q}_{n-1}(t) = a_{n-2}(t) \int_{\sigma(t)}^{\infty} J_{n-3}\big(g(u), t\big) q(u)\, du.$$

Now, we have the following immediate result.

Corollary 3.8. *Consider Eq.* (3.55; δ) *with* $f(x) = x$. *Let conditions* (i)–(iii) *hold,* $g(t) \leqslant t$ *for* $t \geqslant t_0$, *and let* $1 \leqslant \ell \leqslant n-1$ *with* $(-1)^{n-\ell}\delta = -1$. *If for all large* t, *the equations*

$$\left(\frac{1}{a_\ell(t)} y'(t)\right)' + \bar{Q}_\ell(t) y\big[\sigma(t)\big] = 0 \tag{3.64; ℓ}$$

are oscillatory, then $\mathcal{N}_\ell = \emptyset$.

Remark 3.7. We note that we can obtain many oscillation criteria which are similar to those given in Section 3.4 for Eq. (3.55; δ). The formulations of these results are left to the reader. As an example, we give the following oscillation criterion for Eq. (3.55; 1) when n is odd.

Corollary 3.9. *Let* $\delta = 1, n$ *be odd, conditions* (i)–(iii) *hold,* $g(t) \leqslant t$ *for* $t \geqslant t_0$ *and* $f(x) = x$. *If for all large* t, *Eqs.* (3.64; ℓ) ($\ell = 2, 4, \ldots, n-1$) *are oscillatory and condition* (3.23) *holds with* $\alpha = 1$, *then Eq.* (3.55; 1) *is oscillatory.*

Remark 3.8. In the case when condition (3.1) fails to apply to some functions f, we may employ (as an alternative) the following condition on the function $f(x)$:

$$\inf\left\{\frac{f(\eta x)}{f(\eta)}:\ \eta \neq 0\right\} > 0 \quad \text{for any } x > 0.$$

For this purpose we need the function defined by

$$w[f](x) = \begin{cases} \operatorname{sgn} x \inf\left\{\frac{f(\eta|x|)}{f(\eta)}:\ \eta x > 0\right\} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It is easy to check that $w[f]$ has the following properties: $w[f]$ is nondecreasing on $\mathbb{R}$ and $xw[f](x) > 0$ for $x \neq 0$;

$$\left|f\left(\eta|x|\right)\right| \geqslant \left|f(\eta)\right|\left|w[f](x)\right| \quad \text{for } \eta x > 0.$$

For more details of the function $w[f]$, see [8].

References

[1] R.P. Agarwal, S.R. Grace, Oscillation of certain functional differential equations, Comput. Math. Appl. 38 (1999) 143–153.
[2] R.P. Agarwal, S.R. Grace, On the oscillation of higher order differential equations with deviating arguments, Comput. Math. Appl. 38 (1999) 185–199.
[3] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic, Dordrecht, 2000.
[4] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations, Kluwer Academic, Dordrecht, 2002.
[5] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory for Second Order Dynamic Equations, Taylor & Francis, London, 2003.
[6] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation criteria for certain nth order differential equations with deviating arguments, J. Math. Anal. Appl. 262 (2001) 601–622.
[7] R.P. Agarwal, S.R. Grace, D. O'Regan, On the oscillation of certain higher order functional differential equations, Math. Comput. Modelling, submitted for publication.
[8] Y. Kitamura, Oscillation of functional differential equations with general deviating arguments, Hiroshima Math. J. 15 (1985) 445–491.
[9] T. Kusano, B.S. Lalli, On oscillation of half-linear functional differential equations with deviating arguments, Hiroshima Math. J. 24 (1994) 549–563.
[10] T. Kusano, M. Naito, Comparison theorems for functional differential equations with deviating arguments, J. Math. Soc. Japan 33 (1981) 509–532.
[11] Ch.G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, Arch. Math. 36 (1981) 168–178.