Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 216 (2008) 157 – 163
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

# Approximate analytical solution for the Zakharov–Kuznetsov equations with fully nonlinear dispersion

Khaldoun Batiha*

*Business Networking and Systems Management, Philadelphia University, Jordan*

Received 7 January 2007
Available online 7 January 2007

**Abstract**

In this paper, variational iteration method (VIM) is used to obtain numerical and analytical solutions for the Zakharov–Kuznetsov equations with fully nonlinear dispersion. Comparisons with exact solution show that the VIM is a powerful method for the solution of nonlinear equations.


*MSC:* 76M30; 78M30; 35A15

*Keywords:* Variational iteration method; Zakharov–Kuznetsov (ZK) equations; Solitary wave solution

## 1. Introduction

The investigation of the traveling wave solutions play an important role in nonlinear science. These solutions may well describe various phenomena in nature, such as vibrations, solitons and propagation with a finite speed. Recently, directly searching for exact solutions of nonlinear differential equations has become more and more attractive partly due to the availability of computer symbolic systems like Maple which allow us to perform some complicated algebraic calculation on a computer, as well as help us to find new exact solutions of these kinds of equations.

The Zakharov–Kuznetsov equations (shortly called $ZK(m,n,k)$) of the form

$$u_t + a(u^m)_x + b(u^n)_{xxx} + c(u^k)_{yyx} = 0, \quad m,n,k \neq 0, \tag{1}$$

where $a$, $b$ and $c$ are arbitrary constants and $m$, $n$ and $k$ are integers, governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [21,22]. The Zakharov–Kuznetsov equation (ZK) was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimensions [28].

Wazwaz [25] used extended tanh method for analytic treatment of the ZK equation, the modified ZK equation, and the generalized forms of these equations. Huang [15] applied the polynomial expansion method to solve the coupled ZK equations. Zhao et al. [29] obtained numbers of solitary waves, periodic waves and kink waves using the theory

* Tel.: +962 777414588.
*E-mail address:* khl_67@yahoo.com.

0377-0427/$ - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.cam.2007.04.020

of bifurcations of dynamical systems for the modified ZK equation. Inc [16] solved nonlinear dispersive ZK equations using the Adomian decomposition method (ADM).

Another powerful analytical method is called the variational iteration method (VIM), which was first envisioned by He [6] (see also [7–14]). The VIM has successfully been applied to many situations. For example, He [7] solved the classical Blasius' equation using VIM. He [8] used VIM to give approximate solutions for some well-known non-linear problems. He [9] used VIM to solve autonomous ordinary differential systems. He [10] coupled the iteration method with the perturbation method to solve the well-known Blasius' equation. He [11] solved strongly nonlinear equations using VIM. Soliman [24] applied the VIM to solve the KdV-Burger's and Lax's seventh-order KdV equations. The VIM has recently been applied for solving nonlinear coagulation problem with mass loss by Abulwafa et al. [3]. Momani et al. [20] applied VIM to Helmholtz equation. The VIM has been applied for solving nonlinear differential equations of fractional order by Odibat et al. [23]. Bildik et al. [4] used VIM for solving different types of nonlinear partial differential equations. Wazwaz [26] used VIM to determine rational solutions for the KdV, the $K(2,2)$, the Burgers, and the cubic Boussinesq equations. Wazwaz [27] presented a comparative study between the VIM and ADM. Tamer et al. [5] introduced a modification of VIM. Abbasbandy [1] solved the quadratic Riccati differential equation by He's VIM with considering Adomian's polynomials. Junfeng [18] introduced VIM to solve two-point boundary value problems.

The purpose of this paper is to obtain approximate analytical solutions of the ZK equations, and to determine the accuracy of VIM in solving these kind of problems.

## 2. Variational iteration method

This method, which is a modified general Lagrange's multiplier method [17], has been shown to solve effectively, easily and accurately a large class of nonlinear problems [2–4,6–14,19,20,23,24]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution. To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation:

$$Lu + Nu = g(x), \tag{2}$$

where $L$ is a linear operator, $N$ is a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to the VIM [8–14], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\}\,d\tau, \tag{3}$$

where $\lambda$ is a general Lagrangian multiplier [17], which can be identified optimally via the variational theory, the subscript $n$ denotes the $n$th-order approximation, $\tilde{u}_n$ is considered as a restricted variation [8–10,14], i.e., $\delta\tilde{u}_n = 0$.

## 3. Analysis of ZK equations

In this paper, we present numerical and analytical solutions for the ZK equations:

$$u_t + a(u^m)_x + b(u^n)_{xxx} + c(u^k)_{yyx} = 0, \quad m, n, k \neq 0. \tag{4}$$

To solve Eq. (4) by means of He's VIM, we construct a correction functional,

$$u_{i+1}(x, y, t) = u_i(x, y, t) + \int_0^t \lambda(s)[(u_i)_s + a(\tilde{u}_i^m)_x + b(\tilde{u}_i^n)_{xxx} + c(\tilde{u}_i^k)_{yyx}]\,ds,$$

$$\delta u_{i+1}(x, y, t) = \delta u_i(x, y, t) + \delta\int_0^t \lambda(s)[(u_i)_s + a(\tilde{u}_i^m)_x + b(\tilde{u}_i^n)_{xxx} + c(\tilde{u}_i^k)_{yyx}]\,ds,$$

$$\delta u_{i+1}(x,y,t)=\delta u_i(x,y,t)+\delta\int_0^t \lambda(s)(u_i)_s\,\mathrm{d}s,$$
$$\delta u_{i+1}(x,y,t)=\delta u_i(x,y,t)+\lambda(s)\delta u_i(x,y,s)-\int_0^t \delta u_i(x,y,s)\lambda'(s)\,\mathrm{d}s,$$

where $\tilde{u}_i$ is considered as restricted variations, which mean $\delta\tilde{u}_i=0$. Its stationary conditions can be obtained as follows:

$$\lambda'(s)=0,\quad 1+\lambda(s)|_{s=t}=0. \tag{5}$$

The Lagrange multipliers, therefore, can be identified as $\lambda=-1$, and the following variational iteration formula is obtained:

$$u_{i+1}(x,y,t)=u_i(x,y,t)-\int_0^t[(u_i)_s+a(u_i^m)_x+b(u_i^n)_{xxx}+c(u_i^k)_{yyx}]\,\mathrm{d}s. \tag{6}$$

For simplicity, we can take an initial approximation $u_0=u(x,y,0)$. The next iterates are easily obtained from (6) and are given by

$$u_1(x,y,t)=u_0(x,y,t)-\int_0^t[(u_0)_s+a(u_0^m)_x+b(u_0^n)_{xxx}+c(u_0^k)_{yyx}]\,\mathrm{d}s,$$
$$u_2(x,y,t)=u_1(x,y,t)-\int_0^t[(u_1)_s+a(u_1^m)_x+b(u_1^n)_{xxx}+c(u_1^k)_{yyx}]\,\mathrm{d}s.$$
$$\vdots$$

## 4. Applications

We will choose two special equations, namely $ZK(2,2,2)$ and $ZK(3,3,3)$ with specific initial conditions to illustrate the concrete scheme.

### 4.1. Example 1

First we consider the following $ZK(2,2,2)$ equation:

$$u_t+(u^2)_x+\tfrac{1}{8}(u^2)_{xxx}+\tfrac{1}{8}(u^2)_{yyx}=0, \tag{7}$$

the exact solution to Eq. (7) subject to the initial condition

$$u(x,y,0)=\tfrac{4}{3}\lambda\sinh^2(x+y), \tag{8}$$

where $\lambda$ is an arbitrary constant, was derived by Inc [16] using ADM and is given by

$$u(x,y,t)=\tfrac{4}{3}\lambda\sinh^2(x+y-\lambda t). \tag{9}$$

To solve Eq. (7) by means of He's VIM, we construct a correction functional (see (6)),

$$u_{i+1}(x,y,t)=u_i(x,y,t)-\int_0^t\left[(u_i)_s+(u_i^2)_x+\frac{1}{8}(u_i^2)_{xxx}+\frac{1}{8}(u_i^2)_{yyx}\right]\mathrm{d}s. \tag{10}$$

For simplicity, we can take an initial approximation $u_0=u(x,y,0)$ as given by (8). The next iterate is easily obtained from (10) and is given by

$$u_1(x,y,t)=\tfrac{4}{3}\lambda\sinh^2(x+y)-\tfrac{224}{9}\lambda^2\sinh^3(x+y)\cosh(x+y)t$$
$$-\tfrac{32}{3}\lambda^2\sinh(x+y)\cosh^3(x+y)t.$$

In the same manner, the rest of the components of the iteration formulae (10) can be obtained using the Maple Package. In Table 1 we present the absolute error between the 3-iterate of VIM and the exact solution. Fig. 1 shows the comparison between the 3-iterate of VIM and the exact solution.

Table 1
Absolute errors between the 3-iterate of VIM, and the exact solutions for Examples 1–4

| $x$ | $y$ | $t$ | Example 1 | Example 2 | Example 3 | Example 4 |
|---|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.2 | 3.85217E − 07 | 1.20265E − 06 | 4.99520E − 08 | 1.65433E − 09 |
| | | 0.3 | 5.75911E − 07 | 1.81281E − 06 | 7.49279E − 08 | 2.48087E − 09 |
| | | 0.4 | 7.65350E − 07 | 2.42896E − 06 | 9.99039E − 08 | 3.30699E − 09 |
| 0.6 | 0.6 | 0.2 | 4.66337E − 05 | 6.21444E − 05 | 5.08988E − 08 | 9.99177E − 09 |
| | | 0.3 | 6.86056E − 05 | 9.55665E − 05 | 7.63480E − 08 | 1.49870E − 08 |
| | | 0.4 | 8.98243E − 05 | 1.30793E − 04 | 1.01797E − 07 | 1.99818E − 08 |
| 0.9 | 0.9 | 0.2 | 5.12131E − 04 | 7.28468E − 04 | 5.21228E − 08 | 1.51072E − 08 |
| | | 0.3 | 7.38186E − 04 | 1.21342E − 03 | 7.81841E − 08 | 2.26602E − 08 |
| | | 0.4 | 9.57942E − 04 | 1.81646E − 03 | 1.04245E − 07 | 3.02127E − 08 |

Where $\lambda = 0.001$.

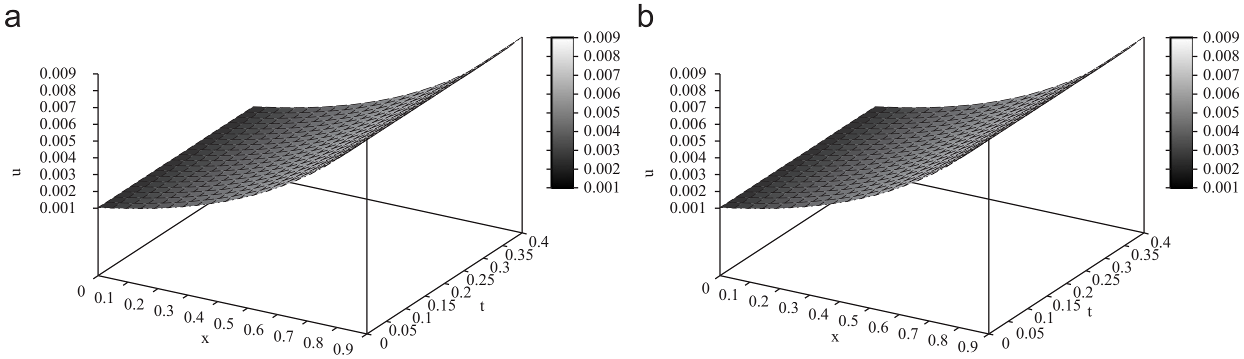


Fig. 1. Comparison between the 3-iterate of VIM and the exact solutions for Example 1. Where $\lambda = 0.001$ and $y = 0.9$.

### 4.2. Example 2

Now we consider the $ZK(2, 2, 2)$ equation:

$$u_t + (u^2)_x + \tfrac{1}{8}(u^2)_{xxx} + \tfrac{1}{8}(u^2)_{yyx} = 0, \tag{11}$$

the exact solution to Eq. (11) subject to the initial condition

$$u(x, y, 0) = -\tfrac{4}{3}\lambda \cosh^2(x + y), \tag{12}$$

where $\lambda$ is an arbitrary constant, was derived by Inc [16] using ADM and is given by

$$u(x, y, t) = -\tfrac{4}{3}\lambda \cosh^2(x + y - \lambda t). \tag{13}$$

In VIM we construct a correction functional for Eq. (11) as follows (see (6)):

$$u_{i+1}(x, y, t) = u_i(x, y, t) - \int_0^t \left[ (u_i)_s + (u_i^2)_x + \frac{1}{8}(u_i^2)_{xxx} + \frac{1}{8}(u_i^2)_{yyx} \right] \mathrm{d}s. \tag{14}$$

Again, we can take an initial approximation $u_0 = u(x, y, 0)$ as given by (12). The next iterate is easily obtained from (14) and is given by

$$\begin{aligned} u_1(x, y, t) = & -\tfrac{4}{3}\lambda \cosh^2(x + y) - \tfrac{224}{9}\lambda^2 \cosh^3(x + y)\sinh(x + y)t \\ & - \tfrac{32}{3}\lambda^2 \cosh(x + y)\sinh^3(x + y)t. \end{aligned}$$

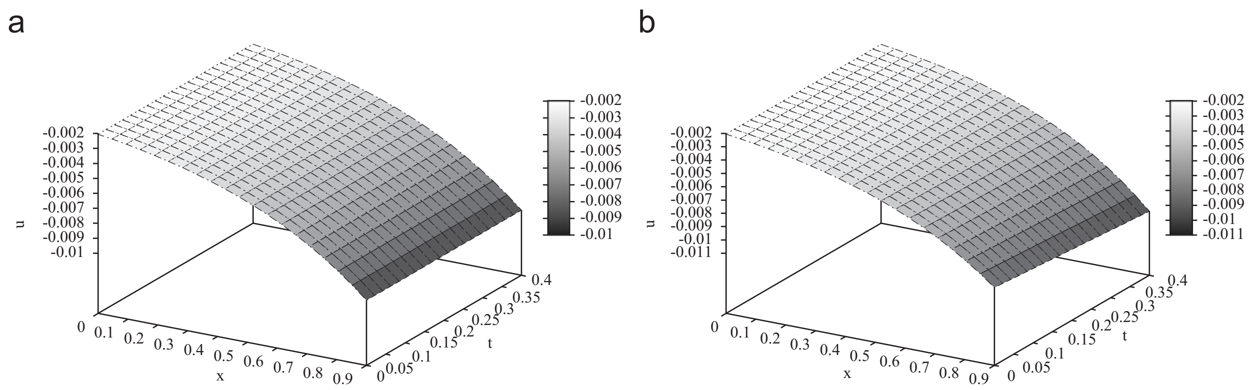


Fig. 2. Comparison between the 3-iterate of VIM and the exact solutions for Example 2. Where $\lambda = 0.001$ and $y = 0.9$.

Again, the rest of the components of the iteration formulae (14) can be obtained using the Maple Package. In Table 1 we present the absolute error between the 3-iterate of VIM and the exact solution. Fig. 2 shows the comparison between the 3-iterate of VIM and the exact solution.

### *4.3. Example 3*

Next we exam the following $ZK(3,3,3)$ equation:

$$u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx} = 0, \tag{15}$$

subject to the initial condition

$$u(x,y,0) = \tfrac{3}{2}\lambda \sinh[\tfrac{1}{6}(x+y)], \tag{16}$$

where $\lambda$ is an arbitrary constant, the exact solution to Eq. (15) was derived by Inc [16] using ADM and is given by

$$u(x,y,t) = \tfrac{3}{2}\lambda \sinh[\tfrac{1}{6}(x+y-\lambda t)]. \tag{17}$$

By using the VIM, the correction functionals for Eq. (15) is (see (6)),

$$u_{i+1}(x,y,t) = u_i(x,y,t) - \int_0^t [(u_i)_s + (u_i^3)_x + 2(u_i^3)_{xxx} + 2(u_i^3)_{yyx}]\,\mathrm{d}s. \tag{18}$$

As before, we can take an initial approximation $u_0 = u(x,y,0)$ as given by (16). The next iterate is easily obtained from (18) and is given as follows:

$$\begin{aligned} u_1(x,y,t) = {} & \tfrac{3}{2}\lambda \sinh[\tfrac{1}{6}(x+y)] - 3\lambda^3 \sinh^2[\tfrac{1}{6}(x+y)]\cosh[\tfrac{1}{6}(x+y)]t \\ & - \tfrac{3}{8}\lambda^3 \cosh^3[\tfrac{1}{6}(x+y)]t. \end{aligned}$$

As before, the rest of the components of the iteration formulae (18) can be obtained using the Maple Package. In Table 1 we present the absolute error between the 3-iterate of VIM and the exact solution. Fig. 3 shows the comparison between the 3-iterate of VIM and the exact solution.

### *4.4. Example 4*

Finally, we exam the following $ZK(3,3,3)$ equation:

$$u_t + (u^3)_x + \tfrac{1}{8}(u^3)_{xxx} + \tfrac{1}{8}(u^3)_{yyx} = 0, \tag{19}$$

with initial condition

$$u(x,y,0) = \tfrac{3}{2}\lambda \cosh[\tfrac{1}{6}(x+y)], \tag{20}$$

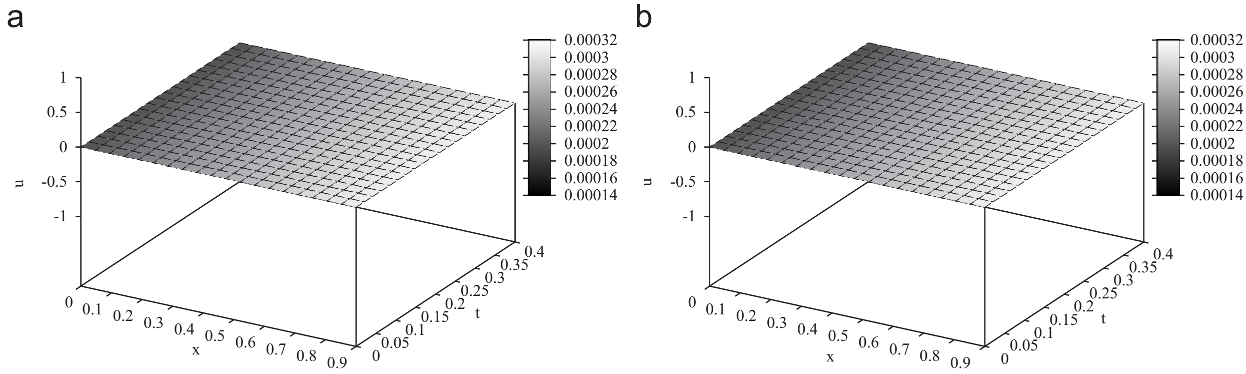


Fig. 3. Comparison between the 3-iterate of VIM and the exact solutions for Example 3. Where $\lambda = 0.001$ and $y = 0.9$.

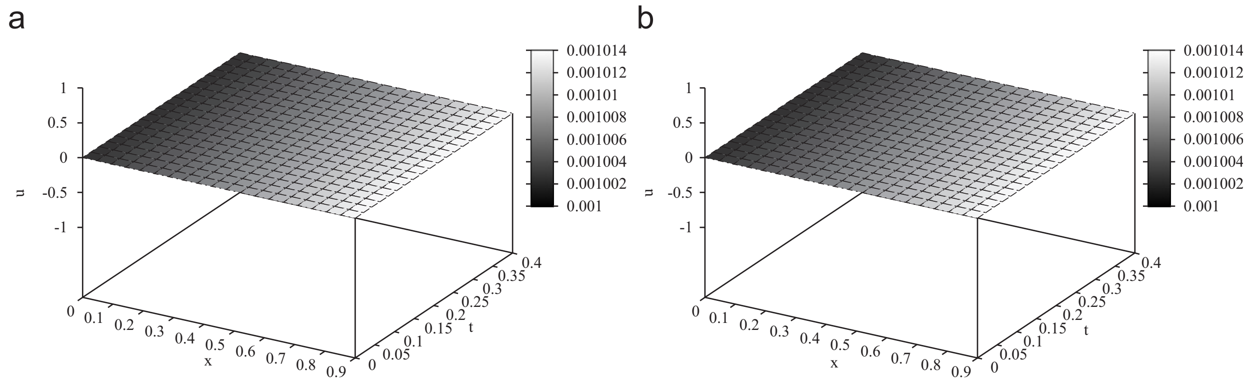


Fig. 4. Comparison between the 3-iterate of VIM and the exact solutions for Example 4. Where $\lambda = 0.001$ and $y = 0.9$.

and exact solution [17]

$$u(x, y, t) = \frac{3}{2}\lambda \cosh[\tfrac{1}{6}(x + y - \lambda t)]. \tag{21}$$

Proceeding as before, we can obtain the iterate of VIM as follows:

$$u_1(x, y, t) = \frac{3}{2}\lambda \cosh[\tfrac{1}{6}(x + y)] - \frac{453}{256}\lambda^3 \cosh^2[\tfrac{1}{6}(x + y)] \sinh[\tfrac{1}{6}(x + y)]t - \frac{3}{128}\lambda^3 \sinh^3[\tfrac{1}{6}(x + y)]t.$$

And the rest of the components of the iteration can be obtained using the Maple Package. In Table 1 we present the absolute error between the 3-iterate of VIM and the exact solution. Fig. 4 shows the comparison between the 3-iterate of VIM and the exact solution.

## 5. Conclusions

In this paper, variational iteration method (VIM) has been successfully applied to finde approximate solution of the ZK equations. The method was used in a direct way without using linearization or perturbation . It provides more realistic series solutions that converge very rapidly in real physical problems. It may be concluded that VIM is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of nonlinear differential equations.

## References


[1] S. Abbasbandy, A new application of He's variational iteration method for quadratic Riccati differential equation by using adomian's polynomials, J. Comput. Appl. Math., in press, doi:10.1016/j.cam.2006.07.012.

[2] M.A. Abdou, A.A. Soliman, Variational iteration method for solving Burger's and coupled Burger's equations, J. Comput. Appl. Math. 181 (2005) 245–251.

[3] E.M. Abulwafa, M.A. Abdou, A.A. Mahmoud, The solution of nonlinear coagulation problem with mass loss, Chaos Solitons Fractals 29 (2) (2006) 313–330.

[4] N. Bildik, A. Konuralp, The use of variational iteration method, differential transform method and adomian decomposition method for solving different types of nonlinear partial differential equations, Internat. J. Nonlinear Sci. Numer. Simulation 7 (1) (2006) 65–70.

[5] H. El Zoheiry, T.A. Abassy, M.A. El-Tawil, Toward a modified variational iteration method, J. Comput. Appl. Math., in press, doi:10.1016/j.cam.2006.07.019.

[6] J.H. He, A new approach to nonlinear partial differential equations, Comm. Nonlinear Sci. Numer. Simulation 2 (1997) 230–235.

[7] J.H. He, Approximate analytical solution of Blasius' equation, Comm. Nonlinear Sci. Numer. Simulation 3 (1998) 260–263.

[8] J.H. He, Variational iteration method-a kind of non-linear analytical technique: some examples, Internat. J. Non-Linear Mech. 34 (1999) 699–708.

[9] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2000) 115–123.

[10] J.H. He, A simple perturbation approach to Blasius' equation, Appl. Math. Comput. 140 (2003) 217–222.

[11] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (2006) 1141–1199.

[12] J.H. He, Non-perturbative methods for strongly nonlinear problems, Dissertation, de-Verlag im Internet GmbH, Berlin, 2006.

[13] J.H. He, Variational iteration method—some recent results and new interpretations, J. Comput. Appl. Math., 2006, in press.

[14] J.H. He, Y.Q. Wan, Q. Guo, An iteration formulation for normalized diode characteristics, Internat. J. Circuit Theory Appl. 32 (2004) 629–632.

[15] W. Huang, A polynomial expansion method and its application in the coupled Zakharov–Kuznetsov equations, Chaos Solitons Fractals 29 (2006) 365–371.

[16] M. Inc, Exact solutions with solitary patterns for the Zakharov–Kuznetsov equations with fully nonlinear dispersion, Chaos Solitons Fractals 33 (15) (2007) 1783–1790.

[17] M. Inokuti, H. Sekine, T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics, in: S. Nemat-Nassed (Ed.), Variational Method in the Mechanics of Solids, Pergamon Press, Oxford, 1978, pp. 156–162.

[18] L. Junfeng, Variational iteration method for solving two-point boundary value problems, J. Comput. Appl. Math., in press, doi:10.1016/j.cam.2006.07.014.

[19] M. Moghimi, F.S.A. Hejazi, Variational iteration method for solving generalized Burger–Fisher and Burger equations, Chaos Solitons Fractals 33 (5) (2007) 1756–1761.

[20] S. Momani, S. Abuasad, Application of He's variational iteration method to Helmholtz equation, Chaos Solitons Fractals 27 (2006) 1119–1123.

[21] S. Monro, E.J. Parkes, The derivation of a modified Zakharov–Kuznetsov equation and the stability of its solutions, J. Plasma Phys. 62 (3) (1999) 305–317.

[22] S. Monro, E.J. Parkes, Stability of solitary-wave solutions to a modified Zakharov–Kuznetsov equation, J. Plasma Phys. 64 (3) (2000) 411–426.

[23] Z.M. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, Internat. J. Nonlinear Sci. Numer. Simulation 7 (1) (2006) 27–34.

[24] A.A. Soliman, A numerical simulation and explicit solutions of KdV-Burger's and Lax's seventh-order KdV equations, Chaos Solitons Fractals 29 (2) (2006) 294–302.

[25] A.M. Wazwaz, The extended tanh method for the Zakharov–Kuznetsov (ZK) equation, the modified ZK equation, and its generalized forms, Comm. Nonlinear Sci. Numer. Simulation, in press, doi:10.1016/j.cnsns.2006.10.007.

[26] A.M. Wazwaz, The variational iteration method for rational solutions for KdV, K(2,2), Burger's and cubic Boussinesq equations, J. Comput. Appl. Math., in press, doi:10.1016/j.cam.2006.07.010.

[27] A.M. Wazwaz, A comparison between the variational iteration method and adomian decomposition method, J. Comput. Appl. Math., in press, doi:10.1016/j.cam.2006.07.018.

[28] V.E. Zakharov, E.A. Kuznetsov, On three-dimensional solitons, Soviet Phys. 39 (1974) 285–288.

[29] X. Zhao, H. Zhou, Y. Tang, H. Jia, Travelling wave solutions for modified Zakharov–Kuznetsov equation, Appl. Math. Comput. 181 (2006) 634–648.