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A Correspondence Between Bi-Ideals and Sub-Hopf Algebras in Cocommutative Hopf Algebras*

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The main result of this paper is that for a cocommutative Hopf algebra H over a field, the correspondence $\tau: G \to HG^+$ from sub-Hopf algebras of H to left bi-ideals is a bijection. ($G^+ = \{x \in G \mid \epsilon(x) = 0\}$. We call I a *left bi-ideal* if it is a left ideal and a two-sided coideal, i.e., $\Delta I \subset I \otimes H + H \otimes I$ and $\epsilon(I) = 0$.)

The inverse to τ, which we will call μ, is defined as follows:

Let I be a left bi-ideal in H. By [7, Theorem 1.4.8(a), p. 22], H/I is a coalgebra. Thus by [7, Lemma 16.1.1, p. 312], if we let

$$G = \mathscr{H}\text{-ker}(H \xrightarrow{\pi} H/I) \equiv \{x \in H \mid (I_H \otimes \pi)\, \Delta x = x \otimes 1\},$$

the cocommutivity of H implies that G is a subcoalgebra of H. It is easy to see that, since I is a left ideal, that G is closed under multiplication. Thus G is a bialgebra. In addition, we will show in Corollaries 3.4 and 3.5 that G has an antipode, i.e., that G is a Hopf algebra. Consequently, $\mathscr{H}$-ker gives a correspondence from left bi-ideals to sub-Hopf algebras.

To show that μ and τ are inverse maps, we will show that, if G is a sub-Hopf algebra, then $\mathscr{H}\text{-ker}(H \to H/HG^+) = G$, and that, if I is a left bi-ideal, then $H/(\mathscr{H}\text{-ker}(H \to H/I))^+ = I$. We establish both these equalities, first, for irreducible Hopf algebras over perfect fields (we rely heavily on Theorem 1.3, which is a generalization of M. Sweedler's structure theorem for irreducible, cocommutative Hopf algebras [8, Theorem 3, p. 521]), and then use Kostant's structure theorem for pointed, cocommutative Hopf algebras [7, Theorem 8.1.5, p. 176] to obtain the result when the ground field is algebraically closed. A simple scalar extension completes the theorem.

As a corollary of this result, we have that left bi-ideals are left Hopf ideals if and only if they are two-sided ideals, and thus $\tau(G)$ is a Hopf ideal if and

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only if $G^+H = HG^+$. Also we show that the main result yields another proof that the category of commutative, cocommutative Hopf algebras is abelian. See also [1, Corollaire 7.4, p. 355; 4, Satz 12; 9, Corollary 4.16].

In [9], Mitsuhiro Takeuchi has a parallel result for the commutative case. He shows that for a commutative Hopf algebra H, there is a bijective correspondence between sub-Hopf algebras and Hopf ideals I with the property that if $x \in I$, $\sum_{(x)} x_{(1)}S(x_{(3)}) \otimes x_{(2)} \in H \otimes I$.

Throughout the remainder of this paper H will be a cocommutative Hopf algebra over a field K, with diagonalization Δ, augmentation ϵ, and antipode S.

Other frequently used notations will be:

(1) $P(H)$ = primitives of $H = \{x \in H \mid \Delta x = 1 \otimes x + x \otimes 1\}$.

(2) PIC-Hopf algebra (resp. PIC-coalgebra) means pointed, irreducible, cocommutative Hopf algebra (resp. coalgebra).

(3) SDP stands for sequence of divided powers, i.e., a set of elements in H: $1 = x_0, x_1, x_2, \ldots, x_n$ such that

$$\Delta x_i = \sum_{j=0}^{i} x_{i-j} \otimes x_j .$$

The definitions of the map V, coheight, and Sweedler basis can be found in [8], p. 520, p. 520, and p. 521, respectively (the latter within the hypothesis of Theorem 3), or in [5] p. 26, p. 26, and p. 30, respectively. To simplify some statements, we will also use the expression "Sweedler basis" in the char. 0 case. Here, it will mean just a basis of $P(H)$.

1. Preliminaries

Lemma 1.1. *If H is a* PIC-*Hopf algebra with bounded coheight over a perfect field K of char. $p > 0$ and, if G is a sub-Hopf algebra of H, then there exists of Sweedler basis of H which contains a Sweedler basis of G.*

Proof. Assume $V^{n+1}(H) = 1 \cdot K$, but $V^n(H) \neq 1 \cdot K$. Pick a basis of $V^n(H) \cap V^n(G) \cap P(H)$. Extend this to a basis of $V^n(H) \cap V^{n-1}(G) \cap P(H)$. Extend *this* to a basis of $V^n(H) \cap V^{n-2}(G) \cap P(H)$. Continue until we have a basis of $V^n(H) \cap P(H)$. Now extend *this* basis to one of

$$[(V^{n-1}(H) \cap V^{n-1}(G)) + V^n(H)] \cap P(H).$$

And extend this to one of

$$[(V^{n-1}(H) \cap V^{n-2}(G)) + V^n(H)] \cap P(H).$$

Continue until we have a basis for $V^{n-1}(H) \cap P(H)$. Repeat this process until we have a basis for all of $P(H)$. It's clear that this basis satisfies the statement of the theorem.

Note. If H is a finitely generated PIC-Hopf algebra over a perfect field of char. $p > 0$, then it has bounded coheight.

Proof. Each of the generators generate a finite dimensional coalgebra. The product of these coalgebras is H. Since V commutes with multiplication [3, Prop. 4.1.6(c), p. 279], the statement follows.

DEFINITION. Let $\delta^{-1} =$ identity, $\delta^0 = I - \epsilon$, and

$$\delta^n = \underbrace{(I-\epsilon)\otimes(I-\epsilon)\otimes\cdots\otimes(I-\epsilon)}_{n+1\text{-times}} \Delta_n \qquad \text{for} \quad n > 0.$$

Remark. By [7, Prop. 11.0.5, p. 220] if we let $H_n = \ker \delta^n$, the H_n form the coradical filtration of H. Thus for $n \geqslant 0$, if $\delta^{n+1}(x) = 0$ and $\delta^n(x) \neq 0$, $\delta^n(x) \in H_1^{[n+1]}$. By [7, Prop. 10.0.1, p. 200], $H_1 = K \cdot 1 \oplus P(H)$. But by definition, all the tensorands of $\delta^n(x)$ have augmentation 0, i.e.,

$$\delta^n(x) \in [P(H)]^{[n+1]}.$$

Thus for any $x \in H$ such that $\epsilon(x) = 0$, $\exists n$ such that $\delta^n(x)$ is a symmetric element of the tensor algebra of $P(H)$.

The following theorem uses this mapping to obtain a sufficient condition for a subset of H to be a basis.

LEMMA 1.2. *Let H be a* PIC-*Hopf algebra with Sweedler basis B. Assume the ground field K is either perfect or has* char. 0. *Let S be the subset of the symmetric tensors in the tensor algebra of $P(H)$ with the property that each tensorand is in B and, if* char. $K = p > 0$ *and if the coheight of the tensorand is $n < \infty$, then it is repeated in each term less than p^{n+1} times. (If the coheight is infinite or if the* char. $K = 0$ *make no restriction on repetition.) Add the identity to S. Then if T is any set of elements of H such that for each $x \in T$ $\exists n \geqslant -1 \ni \delta^n(x) \in S$ and if this gives a bijective correspondence between T and S, then T is a basis of H.*

A basis of H satisfying these conditions for some Sweedler basis of H, will be called a *proper basis*.

Proof. Similar to the second and third paragraphs of the proofs of [5, Theorem 9, pp. 30–32].

The typical method of forming a proper basis is by taking maximal SDP's over each element of a Sweedler basis and then to form monomials from elements in these sequences, no monomial containing more than one factor from any one sequence and no combination of factors being repeated in more than one monomial [8, Theorem 3, p. 521]. For the char. $p > 0$ case, we need a generalization of this construction.

Let G be a sub-Hopf algebra of PIC-Hopf algebra H over a perfect field of char. $p > 0$. If $x \in G$, let G-coh. x = maximal coheight of x as an element of G; and let H-coh. x = maximal coheight of x as an element of $H\}$. By [8, Lemma 7, p. 522], if $x \in G$ and G-coh. $x = m < \infty$, we can construct a $p^{m+1} - 1$ SDP: $1, x = {}^1g, {}^2g, \ldots, {}^{p^{m+1}-1}g$ in G and if H-coh. $x = n < \infty$, we can construct a $p^{n+1} - 1$ SDP: $1, x = {}^1h, {}^2h, \ldots, {}^{p^{n+1}-1}h$ in H. In general, the second sequence will not be an extension of the first. (See Example 1, below.) (The infinite coheight case presents some additional complications [5, Example 1, p. 27], which, for this paper, we need not get into.) Now note that there exists a bijective correspondence between:

$$\{{}^jh\,{}^ig \mid 0 \leqslant i \leqslant p^{m+1} - 1; 0 \leqslant j \leqslant p^{n+1} - 1 \text{ and } p^{m+1} \text{ divides } j\}$$

and $\{x^{[t]} \mid 0 \leqslant t \leqslant p^{n+1} - 1\}$ using δ^{i+j-1} where

$$x^{[0]} = 1, x^{[t]} = \underbrace{x \otimes x \cdots \otimes x}_{t\text{-times}}.$$

Thus, instead of forming a proper basis from monomials of elements of SDP's, whenever the primitive is in H we can replace the SDP by elements of the above form, and still have a proper basis. Further since

$$\delta^{a+b+c+d-1}({}^ah_\alpha\,{}^bg_\alpha\,{}^ch_\beta\,{}^dg_\beta) = \delta^{a+b+c+d-1}({}^ah_\alpha\,{}^ch_\beta\,{}^bg_\alpha\,{}^dg_\beta)$$

we can insist that our proper basis consist of monomials where all the g's come after the h's. We have shown:

THEOREM 1.3. *Let H be a* PIC *Hopf algebra over a field K and let G be a sub-Hopf algebra of H. If* char. $K = p > 0$, *assume that K is perfect and that H has bounded coheight. Let $B' = \{x_\alpha\}_{\alpha \in I}$ be a Sweedler basis of G contained in $B = \{x_\alpha\}_{\alpha \in J}$ a Sweedler basis of H. ($I \subset J$).*

(a) Char. $K = p > 0$.

*If $x_\alpha \in B'$, let $m = G$-*coh. x_α *and let $n = H$-*coh. x_α *and select* SDP's:

$$1, {}^1g_\alpha = x_\alpha, {}^2g_\alpha, \ldots, {}^{p^{m+1}-1}g_\alpha \qquad \textit{in } G$$

and

$$1, {}^1h_\alpha = x_\alpha, {}^2h_\alpha, \ldots, {}^{p^{n+1}-1}h_\alpha \qquad \textit{in } H.$$

If $x_\alpha \in B - B'$ *and* H-coh. $x_\alpha = n$ *select* SDP, $1, {}^1h_\alpha = x_\alpha, {}^2h_\alpha, \ldots, {}^{p^{n+1}-1}h_\alpha$. *Order I and J independently. Then*

$$\{{}^{b_1}h_{\beta_1}\,{}^{b_2}h_{\beta_2} \cdots {}^{b_j}h_{\beta_j}\,{}^{a_1}g_{\alpha_1}\,{}^{a_2}g_{\alpha_2} \cdots {}^{a_i}g_{\alpha_i}\},$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_i$ *in* I; $\beta_1 < \beta_2 < \cdots < \beta_j$ *in* J;

$$0 \leqslant a_s < p^{m_s+1}, \qquad m_s = G\text{-coh. } x_{\alpha_s};$$

$$0 \leqslant b_r < p^{n_r+1}, \qquad m_r = H\text{-coh. } x_{\beta_r}$$

and $p^{m_r+1} \mid b_r$ *if* $x_{\beta_r} \in B'$, *is a proper basis of* H *containing a proper basis of* G. (*The latter is just those monomials with no* h's *in them.*)

(b) char. $K = 0$.

For each $x_\alpha \in B'$ *pick* SDP: $1, {}^1g_\alpha = x_\alpha, {}^2g_\alpha, \ldots$ *in* G *and for each* $x_\alpha \in B - B'$ *pick* SDP: $1, {}^1h_\alpha = x_\alpha, {}^2h_\alpha, \ldots$ *in* H. *Order I and J independently. Then*

$$\{{}^{b_1}h_{\beta_1}\,{}^{b_2}h_{\beta_2} \cdots {}^{b_j}h_{\beta_j}\,{}^{a_1}g_{\alpha_1}\,{}^{a_2}g_{\alpha_2} \cdots {}^{a_i}g_{\alpha_i}\},$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_i$ *in* I; $\beta_1 < \beta_2 < \cdots < \beta_j$ *in* J; *and* $0 \leqslant a_s, b_r$, *is a proper basis of* H *containing a proper basis of* G. (*The latter is just those monomials with no* h's *in them.*)

Notation. We will frequently abbreviate an element in the above proper basis $\prod {}^b h \cdot \prod {}^a g$ and differentiate among them by subscripts on the h and g.

COROLLARY 1.4. *With the hypothesis as in Theorem* 1.3, H *is a free right* G-*module with basis* $\{\prod {}^b h\}$.

EXAMPLE 1. We give an example of a PIC-Hopf algebra H and sub-Hopf algebra G, where no maximal SDP over a primitive in G can be extended to a maximal SDP in H over that primitive.

Let K be the field of 2 elements and let $H = K[X, Y, Z, W]/(X^2, Y^2, Z^2, W^2)$ with $\Delta X = 1 \otimes X + X \otimes 1$, $\Delta Y = 1 \otimes Y + X \otimes X + Y \otimes 1$, $\Delta Z = 1 \otimes Z + X \otimes XY + Y \otimes Y + XY \otimes X + Z \otimes 1$, and $\Delta W = 1 \otimes W + X \otimes X + W \otimes 1$. Let $G = K[X, W]/(X^2, W^2)$. Note that $V(Z) = Y$ and $V(Y) = V(W) = X$. Then $V^2(Z) = X$, so H-coh. $X = 2$ and since $V(W) = X$, G-coh. $X = 1$. Thus a maximal SDP over X in H has length 7 and a maximal SDP over X in G has length 3. But $(V^{-1}(X) \cap G^+) \cap V(H) = 0$, i.e., every 2nd divided power of X in G has coheight 0 in H and no maximal SDP over X in G can be extended in H.

COROLLARY 1.5. *G and H as in Theorem* 1.3. *The monomials of the form* $\prod {}^b h \cdot \prod {}^a g$ *with* $\prod {}^a g \neq 1$ *form a basis of* HG^+.

Proof. Clearly any such monomial is in HG^+. Conversely, let $x \in HG^+$. Then $x = \sum_\alpha y_\alpha z_\alpha$ with $y_\alpha \in H$ and $z_\alpha \in G^+$, and we can write

$$y_\alpha = \sum_\beta c_{\alpha,\beta} \prod {}^b h_{\alpha,\beta} \prod {}^a g_{\alpha,\beta}, \qquad \text{and} \qquad z_\alpha = \sum_\gamma c_{\alpha,\gamma} \prod {}^a g_{\alpha,\gamma}$$

with no $\prod {}^a g_{\alpha,\gamma} = 1$ since $z_\alpha \in G^+$ ($c_{\alpha,\beta}, c_{\alpha,\gamma} \in K$). Now when we multiply the expressions for y_α and z_α, each time we multiply a $\prod {}^a g_{\alpha,\beta}$ and a $\prod {}^a g_{\alpha,\gamma}$ we can replace the product by a $\sum_\psi c_\psi \prod {}^a g_\psi$ with $c_\psi \in K$. Since

$$\epsilon \left(\prod {}^a g_{\alpha,\beta} \prod {}^a g_{\alpha,\gamma} \right) = 0, \qquad \prod {}^a g_\psi \neq 1$$

for any ψ. Thus we can write x as a linear combination of basis elements with nontrivial G factors.

COROLLARY 1.6. *G and H as in Theorem* 1.3. *The* $\prod {}^b h$ *form a vector space compliment to* HG^+, *i.e.*, $\prod {}^b \bar{h}$ *form a basis of* H/HG^+.

COROLLARY 1.7. *G and H as in Theorem* 1.3. *Assume for all* α *such that* $x_\alpha \in B'$, *that* ${}^i h_\alpha \in HG^+$ *if* $p^{m+1} \nmid i$, $m = G\text{-coh.}\, x_\alpha$. *Then* $\pi: H \to H/HG^+$ *has a right* H/HG^+ *comodule splitting* $\rho: H/HG^+ \to H$ *via* $\prod {}^b \bar{h} \to \prod {}^b h$. (*H is a right* H/HG^+ *comodule via* $(1 \otimes \pi)\Delta$).

Proof. We wish to show that for $\prod {}^b \bar{h} \in H/HG^+$,

$$(\rho \otimes 1)\, \Delta_{H/HG^+} \left(\prod {}^b \bar{h} \right) = (1 \otimes \pi)\, \Delta \left(\prod {}^b h \right).$$

Assume first that $\prod {}^b h$ is a divided power, i.e., is of the form ${}^i h_\alpha$ with $p^{m+1} \mid i$ and $m = G\text{-coh.}\, x_\alpha$. Then for $0 \leqslant j \leqslant i$, either p^{m+1} divides both j and $i - j$ or neither. Consequently (using the hypothesis)

$$(1 \otimes \pi)\, \Delta {}^i h_\alpha = \sum {}^j h_\alpha \otimes {}^{i-j} \bar{h}_\alpha$$

summing over j such that $p^{m+1} \mid j$. On the other hand,

$$\Delta_{H/HG^+} {}^i \bar{h}_\alpha = (\pi \otimes \pi)\, \Delta {}^i h_\alpha = \sum \pi({}^j h_\alpha) \otimes \pi({}^{i-j} h_\alpha)$$

again summing over j such that $p^{m+1} \mid j$. The desired equality follows.
It is straightforward to extend this equality to arbitrary $\prod {}^b h$.

DEFINITION. If G is a sub-Hopf algebra of a Hopf algebra H, we say G is *normal* if $HG^+ = G^+H$.

Remark 1.8. G and H as in Theorem 1.3 with G normal in H. Assume $1, {}^1h_\alpha, {}^2h_\alpha, \ldots, {}^{p^{n+1}-1}h_\alpha$ is an extension of $1, {}^1g_\alpha, {}^2g_\alpha, \ldots, {}^{p^{n+1}-1}g_\alpha$. Then the ${}^ih_\alpha$ could have been chosen to be in HG^+ whenever $p^{m+1} \nmid i$, i.e., we can choose *SDP*'s which satisfy the hypothesis of Corollary 1.7.

Proof. By [6, 1.24, p. 6], $V^t({}^ih_\alpha) = {}^{i/p^t}h_\alpha$ if $p^t \mid i$. Thus H-coh. ${}^ig_\alpha = n - \| i \|$, where $p^{\|i\|} \leqslant i < p^{\|i\|+1}$. By [8, Lemma 7, p. 522] and [6, Definition 2.6, p.10] we can inductively construct for $p^{m+1} < i < p^{n+1}$ and $p^{m+1} \nmid i$ a ${}^ih_\alpha$ which is a polynomial in the lower ${}^jh_\alpha$. Clearly we can insist that the sum of the superscripts on the ${}^jh_\alpha$ in each term of this polynomial be equal to i. Thus, each term contains a ${}^jh_\alpha$ with $p^{m+1} \nmid j$ and, using normality, we can inductively conclude that ${}^ih_\alpha \in HG^+$.

Definition. We define the norm of

$$M = {}^{b_1}h_{\beta_1}\, {}^{b_2}h_{\beta_2} \cdots {}^{b_j}h_{\beta_j}\, {}^{a_1}g_{\alpha_1}\, {}^{a_2}g_{\alpha_2} \cdots {}^{a_i}g_{\alpha_i}$$

to be $\sum_{k=1}^{j} b_k + \sum_{k=1}^{i} a_k$. Thus the norm is the smallest $n \ni \delta^n(M) = 0$.
We write norm $M = | M |$.

Remark 1.9. In the char. $p > 0$ case, if ${}^bh_\alpha$ is any bth divided power of ${}^1h_\alpha \in B$, then in terms of the basis of Theorem 1.3: ${}^bh_\alpha = {}^{tp^{m+1}}h_\alpha\, {}^ug_\alpha +$ (terms of norm $< b$) where $m = G$-coh. h_α (if $h_\alpha \in G$, otherwise $m = -1$), t is maximal such that $tp^{m+1} \leqslant b$ and $u = b - tp^{m+1}$. This statement follows from applying δ^{b-1} to both sides and noting that the set $\{\delta^n(M) \mid M \in$ proper basis, $n \geqslant 0\}$ is independent in the tensor algebra of H.

Thus in either the char. 0 or the char. p case, if $\Delta\, {}^bh_\alpha$ is written as a tensor product of basis elements, the sum of the norms in each term of the tensor product will be $\leqslant b$. Consequently, if M is any basis element, and if ΔM is written in terms of the basis, then the sum of the norms in each term of the tensor product will be $\leqslant | M |$.

2. τ is Injective

Note. Since any PIC-Hopf algebra over a field of char. 0 is the universal enveloping algebra of the Lie algebra of its primitives [7, Theorem 13.0.1, p. 274], the char. 0 case of those theorems in Sections 2 and 3 dealing with PIC-Hopf algebra are actually well-known Lie algebra theory statements. However, since it does not entail any additional work, I have included them here for completeness.

Theorem 2.1. *Let H be a* PIC-*Hopf algebra over a field K and let G be a* *sub-Hopf algebra of H. If* char. $K = p > 0$, *assume K is perfect and H has bounded coheight. Then any coalgebra contained in* $HG^+ \oplus 1 \cdot K$ *is actually in G.*

Proof. Pick a Sweedler basis B of H containing a Sweedler basis B' of G, and form a proper basis of H containing a proper basis of G as per Theorem 1.3.

Pick $x \in HG^+ - G^+$. We will show that the coalgebra generated by x is not contained in $HG^+ \oplus 1 \cdot K$. Write $x = \sum_\alpha c_\alpha \prod {}^b h_\alpha \prod {}^a g_\alpha$, $c_\alpha \in K$. Among those terms where $\prod {}^b h_\alpha \neq 1$ (there must be at least one since $x \notin G$) let $\prod {}^b h_t \prod {}^a g_t$ be one of maximal norm. Now define a functional f on H via $f(\prod {}^a g_t) = 1$ and f is zero on all other basis elements.

Define $f \cdot y = \sum_{(y)} y_{(1)} f(y_{(2)})$ where $y \in H$ and $\Delta y = \sum_{(y)} y_{(1)} \otimes y_{(2)}$. It follows from Remark 1.9 that in terms of our basis:

$$f \cdot x = \sum c_\alpha \prod {}^b h_\alpha + (\text{terms in } G^+) + (\text{terms of lower norm}),$$

where the first sum is over $\alpha \ni |\prod {}^b h_\alpha| = |\prod {}^b h_t|$, and $\prod {}^a g_\alpha = \prod {}^a g_t$. Since the first term is not empty (it contains in particular $c_t \prod {}^b h_t$) we have by Corollary 1.5 that $f \cdot x \notin HG^+$. But by [7, Prop. 2.1.1, p. 34] $f \cdot x$ is in the coalgebra generated by x.

Remark. The referee has pointed out that Corollary 1.4 yields an alternative proof of this theorem. Since H is a free right G-module there exists a G-module splitting of $G \subset H$. Then the proof of [7, Theorem 16.0.3, p. 309] gives another demonstration of the theorem.

COROLLARY 2.2. *If H is any* PIC-*Hopf algebra over a field K of* char. 0 *or a perfect field of* char. $p > 0$ *and if G is a sub-Hopf algebra of H, then any sub-coalgebra in* $H \cdot G^+ \oplus 1 \cdot K$ *is actually in G.*

Proof. The char. 0 case was shown in the theorem, so assume K is perfect of char. $p > 0$. Now assume $\exists x \in HG^+ - G^+$ which generates a coalgebra in $HG^+ \oplus 1 \cdot K$. We want a contradiction. Write $x = \sum y_\alpha z_\alpha$ with $y_\alpha \in H$ and $z_\alpha \in G^+$. Let H' be the Hopf algebra generated by the y_α's and the z_α's and let $G' = H' \cap G$. Then since H' is finitely generated it has bounded coheight. Thus Theorem 2.1 applies to H' and G' and we have the desired contradiction.

COROLLARY 2.3. *Let H be a* PIC-*Hopf algebra over a field K. Assume that either* char. $K = 0$ *or that K is perfect. Then* τ *the correspondence between sub-Hopf algebras of H and left bi-ideals is injective.*

Proof. We show that τ followed by μ is the identity, i.e., if G is a sub-Hopf algebra of H then $\mathscr{H}\text{-ker}(H \to H/HG^+) = G$. It follows from the definition of $\mathscr{H}$-ker that $\mathscr{H}\text{-ker}(H \to H/HG^+) \subset 1 \cdot K \oplus HG^+$. But $\mathscr{H}$-ker $(H \to H/HG^+)$ is a coalgebra, so by Corollary 2.2, $\mathscr{H}\text{-ker}(H \to H/HG^+) \subset G$. The opposite inclusion is clear.

Corollary 2.4. *If H is a cocommutative Hopf algebra over an algebraically closed field K, then τ is injective.*

Proof. Again we want to show that if G is a sub-Hopf algebra of H, then $\mathscr{H}\text{-ker}(H \to H/HG^+) = G$. By [7, Lemma 8.0.1(c), p. 158] since K is algebraically closed, H is a pointed Hopf algebra. Therefore by [7, Theorem 8.1.5, p. 176] we can write H as $H' \# K\mathfrak{H}$ where H' is a PIC-Hopf algebra and $\mathfrak{H}$ is a group, $H' \# K\mathfrak{H} \approx H' \otimes K\mathfrak{H}$ as coalgebras and if $a_1, a_2 \in \mathfrak{H}$ and $h_1, h_2 \in H'$ then $(h_1 \# a_1)(h_2 \# a_2) = h_1 a_1 h_2 a_1^{-1} \# a_1 a_2$.

Similarly, write G as $G' \# K\mathfrak{G}$.

Note that $H'(G')^+ = HG^+ \cap H'$. We show this by observing that since

$$G^+ = ((G')^+ \# K\mathfrak{G}) + (H' \# (K\mathfrak{G})^+),$$
$$HG^+ = (H'(G')^+ \# K\mathfrak{H}) + (H' \# K\mathfrak{H}(K\mathfrak{G})^+).$$

Thus if $z \in HG^+$, we can write $z = \sum x_i \# a_i + \sum y_i \# b_i$ with $x_i \in H'(G')^+$, $a_i \in K\mathfrak{H}$, $y_i \in H'$ and $b_i \in K\mathfrak{H}(K\mathfrak{G})^+$. Since $\epsilon(b_i) = 0$ each b_i must be the nontrivial linear combination of more than one grouplike. Consequently, if we assume $z \in H'$ (i.e., assume that z as an element of $H' \# K\mathfrak{H}$ has its right hand term equal to 1) and if we take the y_i's to be linearly independent, then each y_i must be in the span of the x_i's,

$$\Rightarrow y_i \in H'(G')^+ \Rightarrow z \in (H'(G')^+ \# K\mathfrak{H}) \cap (H' \# 1 \cdot K) = H'(G')^+.$$

That $H'(G')^+ \subset HG^+ \cap H'$ is clear.

Thus $\mathscr{H}\text{-ker}(H' \hookrightarrow H \to H/HG^+) = \mathscr{H}\text{-ker}(H' \to H'/H'(G')^+)$. And by Corollary 2.3, the latter equals G', i.e., the irreducible component of the identity of $\mathscr{H}\text{-ker}(H \to H/HG^+)$ is G'.

Finally, j a grouplike in H is in

$$\mathscr{H}\text{-ker}(H \xrightarrow{\pi} H/HG^+) \Leftrightarrow (I \otimes \pi)\, \Delta j = j \otimes 1 \Leftrightarrow 1 - j \in HG^+$$
$$\Leftrightarrow 1 - j \in K\mathfrak{H}(K\mathfrak{G})^+ \Leftrightarrow j \in \mathfrak{G}.$$

Thus, $\mathfrak{G} =$ the set of grouplikes in $\mathscr{H}\text{-ker}(H \to H/HG^+)$, and we can conclude ([7, Theorem 8.1.5, p. 176] and [7, Lemma 8.0.1 (c), p. 158]) that $\mathscr{H}\text{-ker}(H \to H/HG^+) = G' \# K\mathfrak{G} = G$.

Corollary 2.5. *If H be a cocommutative Hopf algebra over a field K, then τ is injective.*

Proof. Let $\bar{K}$ = algebraic closure of K. Clearly

$$\mathscr{H}\text{-ker}(H \to H/HG^+) \otimes \bar{K} = \mathscr{H}\text{-ker}(H \otimes \bar{K} \to H \otimes \bar{K}/HG^+ \otimes \bar{K}).$$

Thus Corollary 2.5 follows from Corollary 2.4.

3. τ is Surjective

Theorem 3.1. *Let H be a* PIC-*Hopf algebra over a field K and let I be a left bi-ideal in H. If* char. $K = p > 0$, *assume that K is perfect and that H has bounded coheight. Then $G = \mathscr{H}\text{-ker}(H \xrightarrow{\varphi} H/I)$ is a Hopf algebra and $HG^+ = I$.*

Proof. By the introductory remarks we know that G is a bialgebra; but any PIC-bialgebra is a Hopf algebra. [7, Theorem 9.2.2 (3), p. 193].

Now clearly $G^+ \subset I$ so we have a surjection of PIC-coalgebras

$$\psi\colon H/HG^+ \to H/I.$$

$$\begin{array}{ccccc} & & & & H/HG^+ \\ & & & \overset{\rho}{\nearrow} & \downarrow \psi \\ G & \xrightarrow{\iota} & H & \xrightarrow{\varphi} & H/I \end{array}$$

We want to show that ψ is an isomorphism. By [7, Lemma 11.0.1, p. 217] we need only show that ψ is injective on primitives.

So pick $z \in P(H/HG^+) \ni \psi(z) = 0$. We will show that $z = 0$.

If char. $K = p > 0$ we make an initial assumption (*) that (considering G as a sub-Hopf algebra of H) there exists a Sweedler basis B of H containing a Sweedler basis B' of G, such that $\forall b \in B' \exists$ a maximal S.D.P. over b in G which extends to a maximal S.D.P. over b in H. (If char. $K > 0$, this assumption is of course always fulfilled, trivially.) Thus if we pick a proper basis of H in the manner of Theorem 1.3, we can assume that for fixed α,

$$1, {}^1h_\alpha, {}^2h_\alpha, \ldots \text{ is just an extension of}$$
$$1, {}^1g_\alpha, {}^2g_\alpha, \ldots .$$

Now, by Corollary 1.5 if $\rho\colon H \to H/HG^+$ is the canonical map, then $\{\rho(\prod {}^bh_\beta)\}$ form a basis of H/HG^+. Thus we can select $y \in H \ni \rho(y) = z$ and $\ni y$ is in the vector space spanned by the $\{\prod {}^bh_\beta\}$.

Let $y = \sum c_\beta \prod {}^bh_\beta$ and let $M = {}^{b_1}h_{\beta_1} {}^{b_2}h_{\beta_2} \cdots {}^{b_n}h_{\beta_n}$ be a term of maximal norm in the sum.

If $n \geqslant 2$ then

$$N = {}^{b_1}h_{\beta_1}\, {}^{b_2}h_{\beta_2} \cdots {}^{b_{n-1}}h_{\beta_{n-1}} \otimes {}^{b_n}h_{\beta_n}$$

will occur in ΔM. Since (by Remark 1.9), for any β every term in $\Delta \prod {}^{b}h_\beta$ has total norm less than or equal to $| \prod {}^{b}h_\beta |$ and, since N cannot occur in the diagonalization of any $\prod {}^{b}h_\beta$ of norm $| M |$ (other than M), we can conclude that N occurs in Δy with nonzero coefficient. But then Corollary 1.5 implies that $\rho({}^{b_1}h_{\beta_1}\, {}^{b_2}h_{\beta_2} \cdots {}^{b_{n-1}}h_{\beta_{n-1}}) \otimes \rho({}^{b_n}h_{\beta_n})$ occurs in $\Delta(\rho(y)) = \Delta z$. This contradicts $z \in P(H/HG^+)$. Thus $n \geqslant 2$ is impossible.

If $n = 1$ and if char. $K = p > 0$, then M, a term of maximal norm, will be of the form: ${}^{t(p^{m_\beta+1})}h_\beta$, where $m_\beta = G$-coh x_β. (Remember that all the ${}^{b}h_\beta$ used in the proper basis have b as a multiple of $p^{m_\beta+1}$). Now if $t > 1$, then using the same reasoning as above $\rho({}^{p^{m_\beta+1}}h_\beta) \otimes \rho({}^{(t-1)p^{m_\beta+1}}h_\beta)$ will appear in Δz which is again a contradiction. Thus we conclude that any term of maximal norm is of the form ${}^{p^{m_\beta+1}}h_\beta$ with $m_\beta = G$-coh. x_β. But by assumption (*), ${}^{p^{m_\beta+1}}h_\beta$ extends a S.D.P. in $G \Rightarrow \rho({}^{p^{m_\beta+1}}h_\beta) \in P(H/HG^+)$.

If char. $K = 0$, repeat the above argument with $p = 1$. Thus, in either case, if we delete all terms of maximal norm from $\sum c_\beta \prod {}^{b}h_\beta$ the new sum is still a primitive under ρ. Therefore, by induction, we can conclude that $\sum c_\beta \prod {}^{b}h_\beta$ is of the form $\sum c_\beta {}^{b_\beta}h_\beta$ with $b_\beta = 1$ if char. $K = 0$ and $b_\beta = p^{m_\beta+1}$ if char. $K = p$. But, since $\psi(z) = 0$, and since $\rho({}^{b_\beta}h_\beta) \in P(H/HG^+)$, we have

$$((\rho \circ \psi) \otimes I) \circ \Delta \left(\sum c_\beta\, {}^{b_\beta}h_\beta\right) = \sum c_\beta(1 \otimes {}^{b_\beta}h_\beta),$$

or $\sum c_\beta\, {}^{b_\beta}h_\beta \in \mathscr{H}\text{-ker}(\rho \circ \psi) = \mathscr{H}\text{-ker}\, \varphi = G$, i.e., $z = 0$.

Now we complete the proof by showing that in the char. p case we can drop assumption (*). Let $x_\beta \in B'$ with max. S. D. P. 1, ${}^{1}g_\beta = x_\beta$, ${}^{2}g_\beta$,... in G. Assume G-coh. $x_\beta = m_\beta$ and H-coh. $x_\beta = n_\beta$. Then adjoin to H the variables $Z_{\beta,1}, Z_{\beta,2}, \ldots, Z_{\beta,n_\beta - m_\beta}$. Define $\Delta Z_{\beta,1}$, so that it is a $p^{m_\beta+1}$ divided power in the sequence $1, {}^{1}g_\beta, {}^{2}g_\beta, \ldots$; define $\Delta Z_{\beta,2}$ as a $p^{m_\beta+2}$ divided power in this sequence, etc. (This is possible by [8, Lemma 7, p. 522]. See [6, Definition 2.6, p. 10] for more details.) Define multiplication so that the $Z_{\beta,j}$ have no relations with elements in H or among themselves. Repeat this procedure for each $x_\beta \in B'$ and let $\bar{H} = H[Z_{\beta,j}]_{\beta,j}$ and let $\bar{I} = \bar{H}I$.

Note that $G = \mathscr{H}\text{-ker}(\bar{H} \xrightarrow{\bar{\rho}} \bar{H}/\bar{I})$. For let P be an element of

$$\mathscr{H}\text{-ker}(\bar{H} \to \bar{H}/\bar{I})$$

and think of P as a polynomial in the $Z_{\beta,j}$'s with coefficients in H. Then if we let norm $Z_{\beta,j} = p^{m_\beta+j}$, and extend norm to monomials by multiplication, find

M a monomial of highest norm in P. Let h = coefficient of M. Then ΔP contains the term $h \otimes M$ which means that $h \otimes \bar{\rho}(M)$ appears in $(1 \otimes \bar{\rho}) \Delta P$, i.e., $(1 \otimes \rho) \Delta P \neq P \otimes 1$ or $P \notin \mathscr{H}\text{-ker}(\bar{H} \to \bar{H}/\bar{I})$ unless $P \in H$ (in which case, of course, $M = 1$). Put clearly

$$H \cap \mathscr{H}\text{-ker}(\bar{H} \to \bar{H}/\bar{I}) = \mathscr{H}\text{-ker}(H \to H/I) = G.$$

Therefore $P \in G$.

Now by our construction, we have assured that for each $b \in B'$, there exists a maximal S.D.P. in G over b extendable to a maximal S.D.P. in $\bar{H}$ over b. (The variables have added primitives to H, but have not increased the coheight of any primitive in G.) Thus assumption (*) is fulfilled and $\bar{H}/\bar{H}G^+ \to \bar{H}/\bar{I}$ is an isomorphism. Therefore, since $H/HG^+ \to \bar{H}/\bar{H}G^+$ is injective, $z = 0$. Q.E.D.

Remark. The referee has pointed out that if G is normal in H one can give an alternative demonstration of the part of Theorem 1.3 where we assumed (*). By Remark 1.8 and Corollary 1.7, under assumption (*) there exists a H/HG^+-comodule splitting ρ: $H/HG^+ \to H$. The proof of [7, Lemma 16.0.2, p. 306] now applies and gives the desired result.

COROLLARY 3.2. *Let H be a* PIC-*Hopf algebra with left bi-ideal I. If the ground field has* char. $p > 0$, *assume it is perfect. Then $G = \mathscr{H}\text{-ker}(H \to H/I)$ is a Hopf algebra and $HG^+ = I$.*

Proof. As in Theorem 3.1, G is a Hopf algebra since it is a PIC-bialgebra. Since Theorem 3.1 covers the char. 0 case, we assume the ground field is perfect. We again wish to show that ψ: $H/HG^+ \to H/I$ is injective when restricted to $P(H/HG^+)$. So pick $z \in P(H/HG^+) \ni \psi(z) = 0$, and pick $y \in H \ni \rho(y) = z$. Since z is a primitive we can write

$$\Delta y = 1 \otimes y + y \otimes 1 + \sum_i y_{i,1} \otimes y_{i,2},$$

with $\sum_i y_{i,1} \otimes y_{i,2} \in HG^+ \otimes H + H \otimes HG^+$. Write

$$\sum_i y_{i,1} \otimes y_{i,2} = \sum_\alpha h_\alpha g_\alpha \otimes h_\alpha' + \sum_\beta h_\beta' \otimes h_\beta g_\beta$$

with h_α, h_α', h_β, $h_\beta' \in H$ and g_α, $g_\beta \in G^+$. Let H' = sub-Hopf of H generated by $\{h_\alpha, h_\alpha', h_\beta, h_\beta', g_\alpha, g_\beta\}_{\alpha,\beta}$ and let $I' = I \cap H'$. It follows from the constructive definition of $\mathscr{H}$-ker that $\mathscr{H}\text{-ker}(H' \to H'/I') = G \cap H' \equiv G'$. Thus, if we restrict ourselves to H', I', and G', Theorem 3.1 applies (since H'

is finitely generated and therefore has bounded coheight). Consequently, $z = 0$. Q.E.D.

COROLLARY 3.3. *Let H be a cocommutative Hopf algebra over an algebraically closed field K and let I be a left bi-ideal of H. Then, if*

$$G = \mathscr{H}\text{-ker}(H \to H/I),$$

G is a Hopf algebra and $HG^+ = I$.

Proof. Since K is algebraically closed by [7, Lemma 8.0.1 (c), p. 158] G is pointed. Thus by [7, Theorem 9.2.5, p. 196] G is a Hopf algebra iff the grouplikes of G form a group. Since G is a bialgebra, the grouplikes form a monoid, so we need only show the existence of inverses. As in Corollary 2.4, a grouplike in H is in G iff $1 - g \in I$. But, since H is a Hopf algebra, g^{-1} exists in $H \Rightarrow -g^{-1}(1 - g) = 1 - g^{-1} \in I \Rightarrow g^{-1} \in G$. Thus G is a Hopf algebra.

We again want to show that the surjection ψ: $H/HG^+ \to H/I$ is a bijection. By [7, Corollary 8.0.7, p. 167], since H/HG^+ and H/I are pointed cocommutative coalgebras, each is isomorphic to the direct sum of its respective pointed irreducible components. Thus we need only show that ψ restricted to each pointed irreducible component of H/HG^+ is injective.

As in Corollary 2.4, if we let $G = G' \# K\mathfrak{G}$ and $H = H' \# K\mathfrak{H}$ then $H'(G')^+ = HG^+ \cap H'$. So by Corollary 3.2, ψ restricted to $H'/(HG^+ \cap H')$ is injective.

Let $x \in$ irreducible component of H/HG^+ whose grouplike is h. Pick grouplike $g \in H \ni \rho(g) = h$ and pick y in the irreducible component of H containing g such that $\rho(y) = x$. (This is possible since if C and D are two irreducible components of H, either $\rho(C) = \rho(D)$ or $\rho(C) \cap \rho(D) = 0$.) (ρ: $H \to H/HG^+$.) So if $\psi(x) = 0$, then $y \in I \Rightarrow g^{-1}y \in I$. But $g^{-1}y \in H'$ so since ψ restricted to $H'/(HG^+ \cap H')$ is injective, $g^{-1}y \in HG^+ \Rightarrow g(g^{-1}y) = y \in HG^+ \Rightarrow x = 0$.

Thus ψ restricted to any irreducible component of H/HG^+ is injective and we are done.

COROLLARY 3.4. *Let H be a cocommutative Hopf algebra over an arbitrary field K, and let I be a left bi-ideal of H. Then if $G = \mathscr{H}\text{-ker}(H \to H/I)$, G is a Hopf algebra and $HG^+ = I$.*

Proof. Let $\bar{K} =$ algebraic closure of K.

Since $\mathscr{H}\text{-ker}(H \to H/I) \otimes \bar{K} = \mathscr{H}\text{-ker}(H \otimes \bar{K} \to H \otimes \bar{K}/I \otimes \bar{K})$, and since $G \otimes \bar{K}$ is a Hopf algebra (by Corollary 3.3), G is a Hopf algebra.

Finally, since (by Corollary 3.3) $HG^+ \otimes \bar{K} = I \otimes \bar{K}$, and since $HG^+ \subset I$, $HG^+ = I$.

4. Conclusions

THEOREM 4.1. *If H is a Hopf algebra, there is a bijective correspondence* τ_L (resp. τ_R) *between sub-Hopf algebras of H and left* (resp. *right*) *bi-ideals of H. τ_L* (resp. τ_R) *of a sub-Hopf algebra G is the left* (resp. *right*) *bi-ideal*: HG^+ (resp. G^+H). τ_L^{-1} (resp. τ_R^{-1}) *of a left* (resp. *right*) *bi-ideal I is the sub-Hopf algebra*: $\mathscr{H}$-ker$(H \to H/I)$.

Proof. Corollary 2.5 shows that τ_L is injective. Corollary 3.4 shows that τ_L is surjective. Symmetrical demonstrations would yield the same result for τ_R.

LEMMA 4.2. *Let H be a cocommutative Hopf algebra and let I be left* (resp. *right*) *bi-ideal of H. Then I is a left* (resp. *right*) *Hopf ideal* (i.e., $S(I) \subset I$) *iff I is a two-sided ideal.*

Proof. If I is a left bi-ideal then by Corollary 3.4, $I = HG^+$ with G a sub-Hopf algebra of H. Then, since the antipode is anti-commutative [7, Prop. 4.0.1 (1), p. 74] $S(HG^+) = S(G^+)\, S(H) \subset G^+H$. Thus, if I is a two-sided ideal, it is a Hopf ideal.

Conversely, assume $S(I) \subset I$. Since H and G are cocommutative S: $H \to H$ and S: $G \to G$ are surjective [7, Prop. 4.0.1 (6), p. 74] $\Rightarrow S(I) = S(HG^+) = G^+H$. Therefore, $IH = I$.

The proof is symmetric if I is a right bi-ideal.

THEOREM 4.3. *If H is cocommutative Hopf algebra there is a bijective order preserving correspondence between the left* (resp. *right*) *Hopf ideals of H and the normal sub-Hopf algebras of H.*

Proof. Theorem 4.1 together with Lemma 4.2.

THEOREM 4.4. *The category of commutative, cocommutative Hopf algebras over a field K is abelian.*

Proof. By [2, p. 35] a category is abelian if:

(1) It has a zero object.

(2) For every pair of objects there is a product and sum.

(3) Every map has a kernel and cokernel.

(4) (a) Every monomorphism is a kernel of a map.
(b) Every epimorphism is a cokernel of a map.

In the given category the first three conditions are easily fulfilled as:

(1) The zero object is K.

(2) Tensor product is both product and sum.

(3) If $\varphi: G \to H$ is a Hopf algebra map, $\mathscr{H}$-ker is the categorical kernel [7, Lemma 16.1.1, p. 312] and $\mathscr{H}$-coker $= H/\varphi(G^+)H$ is the categorical cokernel. [7, p. 313].

To prove (4) (a) note first that if $\varphi: G \to H$ is a monomorphism of Hopf algebras, it is injective. For let $I =$ vector space kernel. By [7, Theorem 4.3.1 (b), p. 87] I is a Hopf ideal, so by Corollary 3.2 $\exists G' \ni H(G')^+ = I$. Then $G' \hookrightarrow G \xrightarrow{\varphi} H$ is the zero map which contradicts φ monomorphic. Therefore $I = \{0\}$.

Now by Corollary 2.5, $G \approx \mathscr{H}\text{-ker}(H \to H/H\varphi(G^+))$ so every monomorphism is a kernel.

For (4) (b) note that if $\varphi: G \to H$ is an epimorphism, it is surjective; as $H/H\varphi(G^+) \approx K \Rightarrow H\varphi(G^+) = HH^+ \Rightarrow$ (by Corollary 2.5) that $\varphi(G) = H$.

Now let $I =$ vector space kernel of φ. Then by Corollary 3.4, $\exists G' \ni G(G')^+ = I$, so $\mathscr{H}\text{-coker}(G' \to G) = H$, i.e., every epimorphism is a cokernel. Q.E.D.

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