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On the Decomposition of Brauer's Centralizer Algebras

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1. INTRODUCTION

In this paper we analyze representations of two finite-dimensional real algebras introduced by Richard Brauer [2]. The algebras were introduced in connection with the centralizer algebras of the Lie groups $O(n, \mathbb{R})$ and $Sp(2n, \mathbb{R})$ acting on $\mathrm{End}(T^f(V))$, where V is the underlying n- or $2n$-dimensional real vector space. These algebras are analogues of the group algebra of the symmetric group which plays the same role for $Sl(n, \mathbb{R})$. Brauer describes two algebras $\mathfrak{A}_f^{(n)}$ and $\mathfrak{B}_f^{(n)}$ and homomorphisms into $\mathrm{End}(T^f(V))$. The respective images are the complete centralizer algebras for the action on $\mathrm{End}(T^f(V))$. Brown [3, 4] discusses the algebra $\mathfrak{A}_f^{(n)}$. He shows in [4] that it is semisimple if and only if $n \geqslant f-1$. Weyl [14] had shown it was semisimple if $n \geqslant 2f$. Brown's methods show $\mathfrak{B}_f^{(n)}$ is semisimple if and only if $n \geqslant f-1$. No further information about the radical when it is nonzero was known. This was the starting point of our work. As the image in the centralizer algebra is semisimple, we know that the radical of each algebra must be in the kernel. So in each case, the above homomorphism is defined on the quotient of the algebra by its radical. This quotient algebra is isomorphic to a direct sum of matrix rings. Each of these matrix rings must be in the kernel or else must intersect the kernel trivially. Our aim is first to find the radicals of $\mathfrak{A}_f^{(n)}$ and $\mathfrak{B}_f^{(n)}$ and then to describe the Wedderburn decomposition of the quotients by the radicals.

In this paper we reduce questions about the algebras $\mathfrak{A}_f^{(n)}$ and $\mathfrak{B}_f^{(n)}$ to the determination of the eigenvalues and eigenspaces of certain symmetric matrices $T_{m,k}(x)$. Brown [3, 4] used similar ideas. For some values of m and k we can determine these eigenvalues and eigenspaces using the

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representation theory of the symmetric groups. The algebras are defined in Section 2 and some basic properties are determined. The algebras $\mathfrak{A}_f^{(n)}$ and $\mathfrak{B}_f^{(n)}$ are given in terms of the parameter n. We show that $\mathfrak{A}_f^{(n)}$ is isomorphic to $\mathfrak{B}_f^{(-n)}$ by a natural isomorphism. In this way, results about the representations of one give results about the representations of the other. In Section 3 we completely determine the eigenvalues of the matrix $T_{0,k}$ in terms of representations of the symmetric group. In Section 4 we look at the general case and in Section 5 we conclude with some general conjectures about the eigenvalues of the matrices $T_{m,k}(x)$. In addition, we include an appendix in which we prove a conjecture of Richard Stanley which generalizes the main result of Section 3.

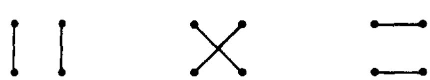
In a subsequent paper, we discuss conjectures of a combinatorial nature about the eigenvalues and eigenspaces of the matrices $T_{m,k}(x)$. These amazing conjectures, if true, help to determine the structure of $\mathfrak{A}_f^{(n)}$ and $\mathfrak{B}_f^{(n)}$ and of the appropriate centralizer algebras.

2. Brauer's Braid Algebras

In this section we describe a pair of $\mathbb{R}$-algebras defined by Richard Brauer in his 1937 paper "On Algebras which Are Connected with the Semisimple Continuous Groups" (see [2]). These algebras, which share a common basis, are parametrized by two numbers f and x, where f is a positive integer and x is a real number. We begin by describing this basis and by describing the multiplication of basis elements. We show later that replacing x by $-x$ in one of the algebras gives the other by a natural isomorphism.

Definition 2.1. A *1-factor* on $2f$ points is a graph with $2f$ points and f 2-element lines having the property that each point is incident to exactly one line. The 2-element lines are called edges.

Let F_f denote the set of 1-factors on $2f$ points and let V_f be the real vector space with basis F_f. We will often draw the 1-factors δ in F_f as having two rows, the points $1, 2, ..., f$ in a top row, denoted $t(\delta)$, and the points $f+1, ..., 2f$ in a bottom row denoted $b(\delta)$. With this convention, the three 1-factors in F_2 appear as

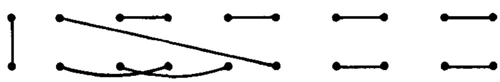


Let δ be a 1-factor in F_f. An edge of δ is called *vertical* if it joins a point in $t(\delta)$ to a point in $b(\delta)$. If the edge joins two points in $t(\delta)$ or two points in $b(\delta)$, it is called *horizontal*. We let $v(\delta)$ and $h(\delta)$ be the number of vertical and horizontal edges of δ. The number of horizontal edges joining points in $t(\delta)$ equals the number of horizontal edges joining points in $b(\delta)$ equals $\frac{1}{2}(f-v(\delta))$. Clearly $h(\delta)$ is always even. We let $V_f(2k)$ denote the subspace of V_f spanned by all δ in F_f with $h(\delta) \geqslant 2k$.

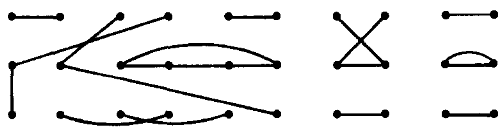
Let δ_1 and δ_2 be 1-factors in F_f. We will be interested in the graph $U(\delta_1, \delta_2)$ with $3f$ points obtained by identifying the bottom row of δ_1 with the top row of δ_2. For example, if

$\delta_1=$

and $\delta_2=$



then $U(\delta_1, \delta_2) =$



It is easy to check, for any δ_1 and δ_2, that $U(\delta_1, \delta_2)$ consists of exactly f paths $P_1, ..., P_f$ and some number $\gamma(U(\delta_1, \delta_2))$ of cycles $C_1, ..., C_{\gamma(U(\delta_1, \delta_2))}$ satisfying:

(1) The endpoints of the paths P_i lie in the set $t(\delta_1) \cup b(\delta_2)$.

(2) Each cycle C_i is of even length and consists entirely of points in the set $b(\delta_1) = t(\delta_2)$.

Let P_i be a path in $U(\delta_1, \delta_2)$ joining u to v. We say u and v are the *initial* and *terminal* points of P_i, respectively, if either u and v are in the same row with u to the left of v or u is in $t(\delta_1)$ and v is in $b(\delta_2)$. Define the *sign of* P_i, denoted $\operatorname{sgn}(P_i)$, to be -1 if the number of edges of P_i in $b(\delta_1) = t(\delta_2)$ traversed from right to left when the P_i is traversed from its initial to its terminal endpoint is odd and 1 otherwise. For C_i a cycle of $U(\delta_1, \delta_2)$ define the *sign of* C_i, denoted $\operatorname{sgn}(C_i)$, to be -1 if the number of edges traversed from right to left when the cycle C_i is traversed in either direction is odd and 1 otherwise (this notion of sign is independent of direction since the cycle C_i has even length). Last, define the *sign of* δ_1 *over* δ_2, denoted $\operatorname{sgn}(\delta_1, \delta_2)$, to be the product of the signs of all paths and cycles of $U(\delta_1, \delta_2)$.

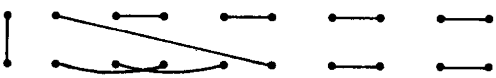
DEFINITION 2.2. Let δ_1 and δ_2 be 1-factors in F_f. Define the *braid of* δ_1

over δ_2, denoted $\beta(\delta_1, \delta_2)$, to be the 1-factor with top row $t(\delta_1)$ and bottom row $b(\delta_2)$ and with points u and v adjacent if and only if there is a path in $U(\delta_1, \delta_2)$ joining u to v. Define algebras $\mathfrak{A}_f^{(x)} = (V_f, \circ)$ and $\mathfrak{B}_f^{(x)} = (V_f, *)$ to be the $\mathbb{R}$-algebras with vector space bases V_f and multiplication of 1-factors given by

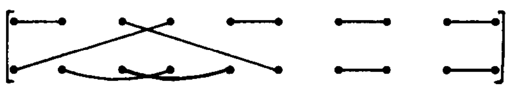
$$(\text{in } \mathfrak{A}_f^{(x)}) \qquad \delta_1 \circ \delta_2 = x^{\gamma(U(\delta_1, \delta_2))} \beta(\delta_1, \delta_2)$$
$$(\text{in } \mathfrak{B}_f^{(x)}) \qquad \delta_1 * \delta_2 = \operatorname{sgn}(\delta_1, \delta_2)\, x^{\gamma(U(\delta_1, \delta_2))} \beta(\delta_1, \delta_2).$$

As an example of these multiplications, let δ_1 and δ_2 be

$\delta_1 =$ 

$\delta_2 =$ 

Then in $\mathfrak{A}_{10}^{(x)}$ we have

$\delta_1 \circ \delta_2 = x^2$ 

whereas in $\mathfrak{B}_{10}^{(x)}$ we have

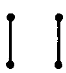
$$\delta_1 * \delta_2 = -\delta_1 \circ \delta_2,$$

the extra factor of -1 coming because $U(\delta_1, \delta_2)$ has one path and two cycles of sign -1.

Our motivation for this work is the following problem (∗):

(∗) Describe the structure of the algebras $\mathfrak{A}_f^{(x)}$ and $\mathfrak{B}_f^{(x)}$, i.e., find their radicals $\mathfrak{C}$ and $\mathfrak{D}$ and decompose $\mathfrak{A}_f^{(x)}/\mathfrak{C}$ and $\mathfrak{B}_f^{(x)}/\mathfrak{D}$ as direct sums of matrix rings.

As an example, we consider the case $f = 2$. A common basis for these algebras is the set of three 1-factors


e


(1,2)

α

Note that e and $(1, 2)$ generate subalgebras of both $\mathfrak{A}_2^{(x)}$ and $\mathfrak{B}_2^{(x)}$

isomorphic to the group algebra of the symmetric group Sym(2). In the algebra $\mathfrak{A}_2^{(x)}$ we have

(1) $e \circ \alpha = \alpha \circ e = \alpha$,

(2) $(1, 2) \circ \alpha = \alpha \circ (1, 2) = \alpha$,

(3) $\alpha \circ \alpha = x\alpha$

and in $\mathfrak{B}_2^{(x)}$ we have

(1) $e * \alpha = \alpha * e = \alpha$,

(2) $(1, 2) * \alpha = \alpha * (1, 2) = -\alpha$,

(3) $\alpha * \alpha = -x\alpha$.

In both cases, the subspace $V_2(2) = \langle \alpha \rangle$ of V_2 is an ideal. Both algebras are commutative (which is not the case for $f \geqslant 3$).

For x nonzero, one can check that both $\mathfrak{A}_2^{(x)}$ and $\mathfrak{B}_2^{(x)}$ are semisimple. Each is a direct sum of three one-by-one matrix rings. The orthogonal idempotents for these decompositions are

$$\begin{aligned}\mathfrak{A}_2^{(x)} &= \langle (1/2)(e + (1, 2)) - (1/x)\alpha \rangle \\ &\quad \oplus \langle (1/2)(e - (1, 2)) \rangle \oplus \langle (1/x)\alpha \rangle \\ \mathfrak{B}_2^{(x)} &= \langle (1/2)(e + (1, 2)) \rangle \\ &\quad \oplus \langle (1/2)(e - (1, 2)) + (1/x)\alpha \rangle \oplus \langle -(1/x)\alpha \rangle.\end{aligned}$$

When $x = 0$, the radicals $\mathfrak{C}$ and $\mathfrak{D}$ are equal to $\langle \alpha \rangle$. Both $\mathfrak{A}_2^{(0)}/\mathfrak{C}$ and $\mathfrak{B}_2^{(0)}/\mathfrak{D}$ are isomorphic to $\mathbb{R}\,\mathrm{Sym}(2)$, which is a direct sum of the two one-by-one matrix rings $\langle (1/2)(e + (1, 2)) \rangle \oplus (1/2)(e - (1, 2)) \rangle$. Thus,

$$\mathfrak{A}_2^{(0)}/\mathfrak{C} = \langle (1/2)(e + (1, 2)) + \mathfrak{C} \rangle \oplus \langle (1/2)(e - (1, 2)) + \mathfrak{C} \rangle$$

and

$$\mathfrak{B}_2^{(0)}/\mathfrak{D} = \langle (1/2)(e + (1, 2)) + \mathfrak{D} \rangle \oplus \langle (1/2)(e - (1, 2)) + \mathfrak{D} \rangle.$$

Certain questions in invariant theory led Richard Brauer to define these algebras. Let W_n denote $\mathbb{R}^n$ with a nondegenerate bilinear form $\langle\ ,\ \rangle$. Let $e_1, ..., e_n$ be the standard basis of $\mathbb{R}^n$ and let $e_1^*, ..., e_n^*$ be the dual basis (dual with respect to $\langle\ ,\ \rangle$). Let G be the Lie group preserving the form $\langle\ ,\ \rangle$. G acts on $\mathbb{R}^n$ hence on $T^f\mathbb{R}^n$. Brauer was interested in a description of the commutator algebra of G in $\mathrm{End}(T^f\mathbb{R}^n)$, i.e., the set of $T \in \mathrm{End}(T^f\mathbb{R}^n)$ satisfying $TA = AT$ for all $A \in G$.

Define a linear map ϕ_n from V_f into $\mathrm{End}(T^f W_n)$ in the following way. Let $\delta \in F_f$ and let i_a and j_a $(a = 1, 2, ..., f)$ be integers in the range 1 to n.

Then the $e_{i_1} \otimes \cdots \otimes e_{i_f}$, $e_{j_1} \otimes \cdots \otimes e_{j_f}$ entry of $\phi_n(\delta)$ is obtained according to the following procedure:

Step 1: Label the points in $t(\delta)$ from left to right with $e_{i_1}, ..., e_{i_f}$ and the points in $b(\delta)$ from left to right with $e^*_{j_1}, ..., e^*_{j_f}$.

Step 2: The desired entry in $\phi_n(\delta)$ is then the product, over all edges $\mathfrak{E}$ of $\langle x, y \rangle$, where x and y are the labels on the endpoints of ε with x to the left of y if $\mathfrak{E}$ is horizontal and x in $t(\delta)$, y in $b(\delta)$ if ε is vertical.

As an example, take $\langle \, , \, \rangle$ to be the form given (with respect to the basis $\{e_1, ..., e_n\}$) by the identity matrix. If

$\delta =$ 

then the $e_{i_1} \otimes e_{i_2} \otimes e_{i_3}$, $e_{j_1} \otimes e_{j_2} \otimes e_{j_3}$ entry in $\phi_n(\delta)$ is 1 if $i_1 = i_2$, $i_3 = j_2$, and $j_1 = j_3$, but is 0 if any of those equalities fail.

One can check the following two facts:

(F1) If $\langle \, , \, \rangle$ is symmetric then ϕ_n is an algebra homomorphism from $\mathfrak{A}_f^{(n)}$ into the commutator algebra of the orthogonal group G in $\mathrm{End}(T^f W_n)$.

(F2) If $\langle \, , \, \rangle$ is skew-symmetric then ϕ_n is an algebra homomorphism from $\mathfrak{B}_f^{(n)}$ into the commutator algebra of the symplectic group G in $\mathrm{End}(T^f W_n)$. Here n is even.

One of the main results in Brauer's 1937 paper is the following theorem.

THEOREM. *Let f be a fixed positive integer.*

(a) *If $\langle \, , \, \rangle$ is symmetric of dimension n, then ϕ_n maps $\mathfrak{A}_f^{(n)}$ onto the commutator algebra of the orthogonal group in* $\mathrm{End}(T^f W_n)$.

(b) *If $\langle \, , \, \rangle$ is skew-symmetric. ϕ_n maps $\mathfrak{B}_f^{(n)}$ onto the commutator algebra of the symplectic group in* $\mathrm{End}(T^f W_n)$.

Brown showed in [4] that the map ϕ_n in (a) is an isomorphism if and only if $n \geqslant f-1$. His methods show the same for the map in (b). Weyl [14] had shown that ϕ_n in (a) was an isomorphism if $n \geqslant 2f$.

Our motivation for this work was a need to know more about the image of ϕ_n when $n \leqslant f-2$. The commutator algebra in $\mathrm{End}(T^f W_n)$ of either the orthogonal or symplectic groups is semisimple so the radical of $\mathfrak{A}_f^{(n)}$ and $\mathfrak{B}_f^{(n)}$ is contained in the kernel of ϕ_n. Each of the matrix algebras in $\mathfrak{A}_f^{(n)}/\mathfrak{C}$ and $\mathfrak{B}_f^{(n)}/\mathfrak{D}$ must either be in the kernel of ϕ_n or must be mapped isomorphically by ϕ_n. So a complete answer to the question will give a great deal of insight into the exact structure of the algebras $\phi_n(\mathfrak{A}_f^{(n)})$ and

$\phi_n(\mathfrak{B}_f^{(n)})$. For an alternative approach to the study of these centralizer algebras, see the recent work of Berele [1].

Let δ_1 and δ_2 be 1-factors in F_f. If e is a horizontal edge of δ_1 joining two points in $t(\delta_1)$ then e remains a horizontal edge in $\delta_1 \circ \delta_2$ and $\delta_1 * \delta_2$ joining points in $t(\delta_1 \circ \delta_2)$ and $t(\delta_1 * \delta_2)$. Thus $h(\delta_1 \circ \delta_2) \geqslant h(\delta_1)$ and $h(\delta_1 * \delta_2) \geqslant h(\delta_1)$. By a similar argument, $h(\delta_1 \circ \delta_2) \geqslant h(\delta_2)$ and $h(\delta_1 * \delta_2) \geqslant h(\delta_2)$. Thus $V_f(2k)$ is an ideal in both $\mathfrak{A}_f^{(x)}$ and $\mathfrak{B}_f^{(x)}$. We denote these ideals by $\mathfrak{A}_f^{(x)}(2k)$ and $\mathfrak{B}_f^{(x)}(2k)$.

By general results $\mathfrak{C} \cap \mathfrak{A}_f^{(x)}(2k)$ (which we denote $\mathfrak{C}(2k)$) and $\mathfrak{D} \cap \mathfrak{B}_f^{(x)}$ (which we denote $\mathfrak{D}(2k)$) are the radicals of $\mathfrak{A}_f^{(x)}(2k)$ and $\mathfrak{B}_f^{(x)}(2k)$. One approach to finding $\mathfrak{C}$ and $\mathfrak{D}$ is to analyze the radicals of the quotients $\mathfrak{A}_f^{(x)}(2k)/\mathfrak{A}_f^{(x)}(2k+2)$ and $\mathfrak{B}_f^{(x)}(2k)/\mathfrak{B}_f^{(x)}(2k+2)$.

In the extreme case $k = f/2$ for even f this approach is highly successful. We will end this section by analyzing that case. The more general situation is considered in Section 4.

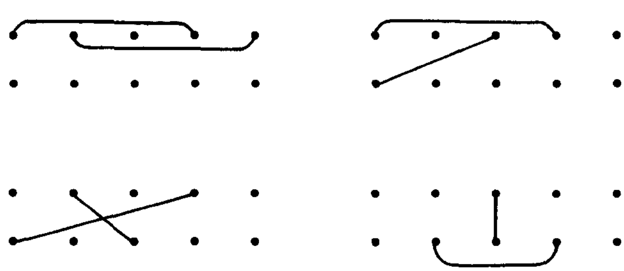
Before doing this, however, we prove that

$$\mathfrak{A}_f^{(x)} \simeq \mathfrak{B}_f^{(-x)}$$

by a natural monomial isomorphism preserving the ideals $\mathfrak{A}_f^{(x)}(2k)$ and $\mathfrak{B}_f^{(-x)}(2k)$. Because of this, information about one of them can be translated into information about the other and so we need consider $\mathfrak{A}_f^{(x)}$ only.

DEFINITION 2.3. Let δ be a 1-factor on $2f$ points and suppose these points are totally ordered "$<$." An *inversion of* δ is a pair of edges $e = \{a, b\}$ and $f = \{c, d\}$ with $a < c < b < d$. We let $i(\delta)$ denote the number of inversions of δ.

Recall that a basis for both $\mathfrak{A}_f^{(x)}$ and $\mathfrak{B}_f^{(-x)}$ consists of the set of 1-factors on $2f$ points. By convention we draw these 1-factors δ with the points $1, 2, ..., f$ from left to right in $t(\delta)$ and the points $f+1, ..., 2f$ from left to right in $b(\delta)$. For order we take $1 < 2 < \cdots < f < 2f < 2f-1 < 2f-2 \cdots < f+1$ (i.e., points are ordered from left to right on the top and then right to left on the bottom). An inversion in δ can look like any of the following:



Note that if δ is a permutation then our definition of inversion agrees with the usual definition.

DEFINITION 2.4. Define a linear map $\Gamma\colon \mathfrak{A}_f^{(x)} \to \mathfrak{B}_f^{(-x)}$ by

$$\Gamma(\delta) = (-1)^{i(\delta)}\,\delta.$$

We will show that the map Γ is an $\mathbb{R}$-algebra isomorphism. We begin with a pair of technical lemmas.

LEMMA 2.5. *Let i be an element of $\{1, 2, ..., f-1\}$ and let δ by the permutation $(i, i+1)$ in $\mathfrak{A}_f^{(x)}$. For any other 1-factor δ_1 we have*

$$\text{(A)} \quad \Gamma(\delta \circ \delta_1) = \Gamma(\delta) * \Gamma(\delta_1)$$

and

$$\text{(B)} \quad \Gamma(\delta_1 \circ \delta) = \Gamma(\delta_1) * \Gamma(\delta).$$

Proof. We will prove (A) and leave (B) to the reader. There are two cases.

Case 1: δ_1 has an edge joining i to $i+1$.
Note that $\delta \circ \delta_1 = \delta_1$ and that $\delta * \delta_1 = -\delta_1$. So,

$$\begin{aligned} \Gamma(\delta \circ \delta_1) &= \Gamma(\delta_1) \\ &= (-1)^{i(\delta_1)}\,\delta_1 \\ &= (-\delta) * ((-1)^{i(\delta_1)}\,\delta_1) \\ &= \Gamma(\delta) * \Gamma(\delta_1). \end{aligned}$$

Case 2: In δ_1, i and $i+1$ belong to distinct edges $e = \{i, b\}$ and $f = \{i+1, d\}$.
Note that $\delta \circ \delta_1 = \delta_1'$, where δ_1' is obtained from δ_1 by replacing the edges e and f with e' and f', where $e' = \{i+1, b\}$ and $f' = \{i, d\}$. The reader can check that

(1) e and f are an inversion of δ_1 if and only if e' and f' are not an inversion of δ_1';

(2) for any edge g common to both δ_1 and δ_1', g is an inversion

with e if and only if g is an inversion with e', and g is an inversion with f if and only if g is an inversion with f'.

Thus $(-1)^{i(\delta_1)} = -(-1)^{i(\delta_1')}$.

Also it is easy to see that $\operatorname{sgn}(\delta, \delta_1) = 1$. So,

$$\begin{aligned}\Gamma(\delta \circ \delta_1) &= \Gamma(\delta_1') \\ &= (-1)^{i(\delta_1')} \\ &= -(-1)^{i(\delta_1)}\, \delta * \delta_1 \\ &= (-\delta) * ((-1)^{i(\delta_1)}\, \delta_1) \\ &= \Gamma(\delta) * \Gamma(\delta_1).\end{aligned}$$

This completes the proof of the lemma.

It is well known that the adjacent transpositions generate the entire symmetric group. This gives us the following corollary to Lemma 2.5.

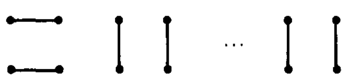
COROLLARY 2.6. *Let δ be a permutation in $\mathfrak{A}_f^{(x)}$ and let δ_1 be any other 1-factor. Then,*

$$\text{(A)} \quad \Gamma(\delta \circ \delta_1) = \Gamma(\delta) * \Gamma(\delta_1)$$

and

$$\text{(B)} \quad \Gamma(\delta_1 \circ \delta) = \Gamma(\delta_1) * \Gamma(\delta).$$

Next let π be the 1-factor with edges from 1 to 2, $f+1$ to $f+2$, and i to $f+i$ for $3 \leqslant i \leqslant f$:



LEMMA 2.7. *Let δ_1 be any 1-factor. Then*

$$\text{(A)} \quad \Gamma(\pi \circ \delta_1) = \Gamma(\pi) * \Gamma(\delta_1)$$

and

$$\text{(B)} \quad \Gamma(\delta_1 \circ \pi) = \Gamma(\delta_1) * \Gamma(\pi).$$

Proof. We prove part (A) only. There are two cases.

Case 1: δ_1 has an edge from 1 to 2.

In this case, $\pi \circ \delta_1 = x\delta_1$ and $\pi^* \delta_1 = -(-x)\,\delta_1$ (here the first minus is

$\operatorname{sgn}(\pi, \delta_1)$ and the $-x$ comes about because Γ maps $\mathfrak{A}_f^{(x)}$ to $\mathfrak{B}_f^{(-x)}$). Also $i(\pi)=0$ so

$$\begin{aligned}\Gamma(\pi \circ \delta_1) &= x\Gamma(\delta_1)\\ &= (-1)^{i(\delta_1)}\, x\delta_1\\ &= \pi * ((-1)^{i(\delta_1)}\delta_1)\\ &= \Gamma(\pi) * \Gamma(\delta_1).\end{aligned}$$

Case 2: δ_1 has distinct edges $e=\{1, b\}$ and $f=\{2, d\}$.

Note that $\pi \circ \delta_1 = \delta_1'$, where δ_1' is obtained from δ_1 by replacing e by $e'=\{1, 2\}$ and f by $f'=\{b, d\}$. We need to compare $i(\delta_1)$ and $i(\delta_1')$. Let $g=\{u, v\}$ (with $u<v$) be an edge common to both δ_1 and δ_1'. We will show that the number of inversions that g forms with e and f is congruent mod 2 to the number of inversions that g forms with e' and f'. Observe that g does not form an inversion with $e'=\{1, 2\}$ so that latter number is 1 or 0 depending on whether g does or does not form an inversion with f'. There are three subcases.

(1) If g forms an inversion with neither e nor f then v is less than both b and d or u is greater than both b and d. In this case g does not form an inversion with f'.

(2) If g forms an inversion with exactly one of e and f then either u is less than both b, d and v is between b and d or v is greater than both b, d and u is between b and d. In this case g does form an inversion with f'.

(3) If g forms an inversion with both e and f then u is less than both b, d and v is greater than both b, d. In this case g does not form an inversion with f'.

To complete the comparison of $i(\delta_1)$ and $i(\delta_1')$ we must consider e and f. Note that e and f form an inversion in δ_1 if and only if $b<d$, whereas e' and f' never form an inversion in δ_1'. So

$$(2.8)\qquad i(\delta_1') \equiv \begin{cases} i(\delta_1) & \text{if } d<b \\ i(\delta_1)+1 & \text{if } b<d \end{cases} \pmod 2.$$

Next we compute $\operatorname{sgn}(\pi, \delta_1)$. The sign of every path in $\pi \cup \delta_1$ is 1 except perhaps the path P joining b and d. If $d<b$ then there are two edges ($\{d, 2\}$ and $\{2, 1\}$) traced backwards in going from d to b. If $b<d$ then there is one edge ($\{b, 1\}$) traced backwards in going from b to d. Hence,

$$(2.9)\qquad \operatorname{sgn}(\pi, \delta_1) = \begin{cases} 1 & \text{if } d<b \\ -1 & \text{if } d>b. \end{cases}$$

Combining Eqs. (2.8) and (2.9) we have

$$\operatorname{sgn}(\pi, \delta_1)(-1)^{i(\delta_1)} = (-1)^{i(\delta_1')}.$$

So,

$$\begin{aligned}\Gamma(\pi \circ \delta_1) &= \Gamma(\delta_1') \\ &= (-1)^{i(\delta_1')}\, \delta_1' \\ &= (-1)^{i(\delta_1)}\, \pi * \delta_1 \\ &= \Gamma(\pi) * \Gamma(\delta_1).\end{aligned}$$

This completes the proof of Lemma 2.7.

The last three results lead up to the following theorem.

THEOREM 2.10. *The map Γ is an $\mathbb{R}$-algebra isomorphism of $\mathfrak{A}_f^{(x)}$ onto $\mathfrak{B}_f^{(-x)}$.*

Proof. Let δ be any 1-factor. It is easy to see that δ can be written in $\mathfrak{A}_f^{(x)}$ as

$$\delta = \sigma_1 \circ \pi \circ \sigma_2 \circ \cdots \circ \pi \circ \sigma_a,$$

where $\sigma_1, ..., \sigma_a$ are suitably chosen permutations. Let δ_1 be any other 1-factor. Then Corollary 2.6 and Lemma 2.7 give

$$\begin{aligned}\Gamma(\delta \circ \delta_1) &= \Gamma(\sigma_1) * \Gamma(\pi) * \cdots * \Gamma(\sigma_a) * \Gamma(\delta_1) \\ &= \Gamma(\sigma_1 \circ \pi \circ \cdots \circ \sigma_a) * \Gamma(\delta_1) \\ &= \Gamma(\delta) * \Gamma(\delta_1)\end{aligned}$$

and similarly $\Gamma(\delta_1 \circ \delta) = \Gamma(\delta_1) * \Gamma(\delta)$. This completes the proof of the theorem.

For the rest of this section assume f is even. We will examine the ideal $\mathfrak{A}_f^{(x)}(f)$ which has a basis $V_f(f)$. Here $V_f(f)$ is the span of all 1-factors on $2f$ points with f horizontal edges and no vertical edges. In other words, the standard basis of $V_f(f)$ consists of all 1-factors δ which are split as a 1-factor δ_t of f on $t(\delta)$ and a 1-factor δ_b of f on $b(\delta)$. The map $\delta \to (\delta_t, \delta_b)$ is a bijection between the standard basis for $V_f(f)$ and the direct product $F_{f/2} \times F_{f/2}$. Thus,

(2.11) $$V_f(f) \simeq V_{f/2} \otimes V_{f/2}.$$

Let $r=f/2$ and define a matrix $T_r(x)$ whose rows and columns are indexed by F_r as

$$(T_r(x))_{\delta_1,\delta_2}=x^{\gamma(\delta_1,\delta_2)},$$

where $\gamma(\delta_1,\delta_2)$ is the number of connected components of $\delta_1\cup\delta_2$. Our rules for multiplication in $\mathfrak{A}_f^{(x)}$ stated in terms of the isomorphism (2.11) are

$$(a\otimes b)\circ(c\otimes d)=\{b^tT_f(x)c\}(a\otimes d). \tag{2.12}$$

For any square complex matrix M let $\mathcal{N}(M)$ denote the nullspace of M and let $\mathcal{R}(M)$ denote the sum of all eigenspaces of M corresponding to nonzero eigenvalues.

THEOREM 2.13. *Fix a real number x. Then in terms of the isomorphism* $V_f(f)=V_r\otimes V_r$ *we have*

$$\mathfrak{C}\cap\mathfrak{A}_f^{(x)}(f)\simeq\mathcal{N}(T_r(x))\otimes V_r+V_r\otimes\mathcal{N}(T_r(x))$$

and

$$\mathfrak{A}_f^{(x)}(f)/\mathfrak{C}(f)=\mathrm{End}(\mathcal{R}(T_r(x))).$$

In particular, $\mathfrak{A}_f^{(x)}(f)$ *modulo its radical is a complete matrix ring.*

Proof. Let $Y=\mathcal{N}(T_r(x))\otimes V_r+V_r\otimes\mathcal{N}(T_f(x))$.

If $c\in\mathcal{N}(T_r(x))$ then $(a\otimes b)\circ(c\otimes d)=0$ for all a, b, and d by (2.12). So $\mathcal{N}(T_f(x))\otimes V_r$ is a nilpotent two-sided ideal of $\mathfrak{A}_f^{(x)}(f)$. Similarly, $V_r\otimes\mathcal{N}(T_r(x))$ is a nilpotent two-sided ideal so Y is contained in $\mathfrak{C}(f)$. Recall that $T_r(x)$ is a symmetric real matrix.

We consider the structure of $\mathfrak{A}_f^{(x)}(f)/Y$. As a vector space,

$$\mathfrak{A}_f^{(x)}(f)/Y\simeq\mathcal{R}(T_r(x))\otimes\mathcal{R}(T_r(x)). \tag{2.14}$$

Note also that $\mathcal{R}(T_r(x))\otimes\mathcal{R}(T_r(x))$ is a subalgebra of $\mathfrak{A}_f(f)$. So (2.14) is an isomorphism of algebras. The matrix $T_r(x)$ is symmetric hence diagonalizable. Let $u_1, ..., u_s$ be an orthonormal basis of eigenvectors for $\mathcal{R}(T_r(x))$ with corresponding eigenvalues $\lambda_1, ..., \lambda_s$. Let $z_{i,j}$ be the s-by-s matrix with a 1 in the i, j entry and zeros elsewhere. By (2.12), the map

$$z_{i,j}\to(1/\lambda_i\lambda_j)(u_i\otimes u_j)$$

is an algebra isomorphism from $\mathrm{End}(\mathcal{R}(T_r(x)))$ onto $\mathcal{R}(T_r(x))\otimes\mathcal{R}(T_r(x))$. This shows that $\mathfrak{A}_f^{(x)}(f)/Y$ is simple so Y equals the radical of $\mathfrak{A}_f^{(x)}(f)$, which completes the proof.

There are connections between these results and representations of $\mathfrak{A}_f^{(x)}(f)$. In particular Theorem 2.13 shows that $\mathfrak{A}_f^{(x)}(f)$ has exactly one irreducible representation. Let $V_f(d)$ be the span of all $a \otimes d$ in $F_r \otimes F_r$. Let $W_f(d)$ be $\mathcal{N}(T_r(x)) \otimes d$. Then $V_f(d)/W_f(d)$ affords the irreducible representation. These connections will be explored more fully in Section 4.

3. Eigenvalues of $T_r(x)$

In this section we assume f is even and set $r = f/2$. Let Ω be a set with f elements and F_r the set of all 1-factors on Ω. As in Section 2, let $T_r(x)$ be the $F_r \times F_r$ matrix whose (δ_i, δ_j) entry is $x^{\gamma(\delta_i, \delta_j)}$. Again, $\gamma(\delta_1, \delta_2)$ is the number of connected components of $\delta_1 \cup \delta_2$. If $r = 2$,

$$T_2(x) = \begin{pmatrix} x^2 & x & x \\ x & x^2 & x \\ x & x & x^2 \end{pmatrix}$$

In this section we will determine the eigenvalues of $T_r(x)$ in terms of representations of $S_f = S_\Omega$, the symmetric group on f points.

A permutation σ in S_f induces a permutation of F_r by permuting the lines of the 1-factors. If p and q, elements of Ω, are joined in δ, then $\sigma(p)$ and $\sigma(q)$ are joined in $\sigma(\delta)$. The permutation action of S_f on F_r is transitive and equivalent to the action on the conjugate class of involutions fixing no points.

Suppose δ_1 and δ_2 are 1-factors on Ω and Ω_1 is a connected component of $\delta_1 \cup \delta_2$. Using the definition of $\sigma(\delta_1)$ and $\sigma(\delta_2)$, it is clear that $\sigma(\Omega_1)$ is a connected component of $\sigma(\delta_1) \cup \sigma(\delta_2)$. In particular, the number and sizes of the connected components of $\delta_1 \cup \delta_2$ and of $\sigma(\delta_1) \cup \sigma(\delta_2)$ are the same. This means

$$\gamma(\delta_1, \delta_2) = \gamma(\sigma(\delta_1), \sigma(\delta_2)). \tag{3.1}$$

Let $V = V_r$ be the real vector space with basis F_r. For σ in S_Ω, let P_σ be the permutation matrix corresponding to the permutation of F_r induced by σ. In particular, if $\sigma\delta_2 = \delta_1$, P_σ has a 1 in the $P_{\delta_1 \delta_2}$ entry and 0's elsewhere in the δ_1 row and δ_2 column. As a consequence of (3.1), P_σ and $T_r(x)$ commute to give

$$P_\sigma T_r(x) = T_r(x) P_\sigma. \tag{3.2}$$

This follows as $(P_\sigma T_r(x))_{\delta_i \delta_j} = x^{\gamma(\sigma^{-1}\delta_i, \delta_j)}$ and $(T_r(x) P_\sigma)_{\delta_i \delta_j} = x^{\gamma(\delta_i, \sigma\delta_j)}$.

Equation (3.2) states that $T_r(x)$ commutes with the action of S_f on F_r. This permutation module has a decomposition as an S_f module into irreducible subspaces corresponding to irreducible representations of S_f. The irreducibles of S_f are indexed by partitions of f (see [12, I.7]). In this notation, the partition (f) corresponds to the trivial character and the partition (1^f) corresponds to the sgn character. It follows from [12, Ex. 5, p. 45] that the irreducibles which occur as constituents of this permutation module are indexed by partitions of f in which all of the parts are even. That is, if $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ is the partition corresponding to an irreducible occurring in the permutation module, all λ_i are even. Furthermore, the multiplicity of each representation is 1. This means

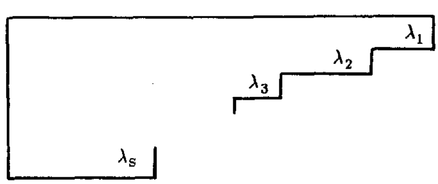
$$V = V_1 + \cdots + V_n,$$

where $V_1, ..., V_n$ are invariant subspaces for P_σ for all σ in S_f and n is the number of even partitions of f. The action of S_f on V_i corresponds to the action of the irreducible representation of S_f indexed by the ith even partition of f. As the irreducibles are distinct, $T_r(x)\, V_i \subseteq V_i$. As each V_i is irreducible, $T_r(x)$ restricted to V_i is a scalar, denoted $h_i(x) I$. Here $h_i(x)$ must be taken from a suitable extension field [5]. In order to find the eigenvalues of $T_r(x)$, it is only necessary to determine the scalars for $T_r(x)$ restricted to V_i. The multiplicity will be dim V_i. These dimensions are known according to various formulas [9, 12].

In Theorem 3.1 below we determine these scalars $h_i(x)$ in terms of the partition associated with the representation and the location of certain integers on a grid. Let Δ be a grid with locations as in a matrix. The first row will have locations (1, 1); (1, 2);... . The second row will have locations (2, 1); (2, 2);... . The ith column of Δ will be the locations (1, 1); (2, i); (3, i);... . Place numbers in Δ in the even columns only according to the following rule. Place in the $(i, 2j)$ position $(2j - i - 1) = c_{i,2j}$.

$$\begin{array}{lccccccccc}
\text{Column number:} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \;\ldots \\
 & & 0 & & 2 & & 4 & & 6 & & 8 \\
 & & -1 & & 1 & & 3 & & 5 & & 7 \\
\Delta = & & -2 & & 0 & & 2 & & 4 \\
 & & -3 & & -1 & & 1 & & 3 \\
 & & -4 & & -2 & & 0 & & 2
\end{array}$$

Let λ be a partition of f into even parts. Let $\lambda = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_s\}$. It is natural to place the diagram corresponding to λ on Δ. Here d is the diagram of shape λ.

d =


There will be exactly r of the integers in Δ contained inside the boundary of d. Recall that each λ_i is even. The r integers inside the boundary of d are said to be in d. We are now ready to state Theorem 3.1.

THEOREM 3.1. *Let $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ be a partition of f with all λ_j even. Denote by V_λ the subspace V_i associated to the partition λ and let $h_\lambda(x) = h_i(x)$. Then*

$$h_\lambda(x) = \Pi(x + a_{ij}),$$

where the a_{ij} are in the diagram, d, of shape λ.

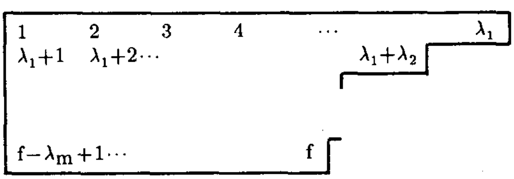
EXAMPLES. (a) If $\lambda = \{6, 4, 4\}$, $h_\lambda(x) = x^2(x+4)(x+2)(x+1)(x-1)(x-2)$.

(b) If $\lambda = \{2r\}$, $h_\lambda = x(x+2)\cdots(x+2r-2)$.

(c) If $\lambda = \{2, 2, ..., 2\}$, $h_\lambda(x) = x(x-1)(x-2)\cdots(x-r+1)$.

(d) If $\lambda = \{8, 6, 4, 2, 2\}$, $h_\lambda(x) = (x+6)(x+4)(x+3)(x+2)(x+1)xx(x-1)(x-2)(x-3)(x-4)$.

Proof. This is proved by induction on r. Let $d_\lambda(x) = \Pi(x + a_{ij})$ with a_{ij} in the diagram d of shape λ. We must show $h_\lambda(x) = d_\lambda(x)$. If r is 1, there is only one possible partition, $\{2\}$. Here $T_1(x) = h_2(x) = x$, $d_2(x) = x$ and the theorem is correct for $r = 1$. We suppose the theorem is true for all partitions of size smaller than $2r$ and let $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ be a partition of $f = 2r$ with all λ_i even. Let d be a diagram of shape λ and let t be the Standard filling of d with the integers increasing consecutively in each row.

t =


The right-hand node of the jth row has the entry $\sum \lambda_i$ for $i = 1$ to j; the left-hand node has the entry $1 + \sum \lambda_i$ for $i = 1$ to $j - 1$. There are two subgroups of S_f associated with t; the row stabilizer R_t and the column stabilizer C_t. The row stabilizer contains the elements of S_f permuting entries in the same row of t and the column stabilizer contains the elements permuting entries in the same column. Let

$$e_t = \sum \varepsilon(\sigma)\,\sigma\tau \qquad \text{for } \sigma \text{ in } C_t \text{ and } \tau \text{ in } R_t.$$

It is shown in [9, 12] that $e_t v$ is in V_λ for any v in V. Furthermore, e_t

is a multiple of a primitive idempotent affording the representation corresponding to λ.

Let δ_0 be the 1-factor of $\{1, 2, ..., f\}$ whose lines join $2i-1$ to $2i$ for $i = 1, 2, ..., r$. We will show that $e_t\delta_0$ has a nonzero δ_0 coefficient u and that the δ_0 coefficient of $T_r(x)\, e_t\delta_0$ is $p_\lambda(x)u$. As $T_r(x)$ acts as a scalar on V_λ and $e_t\delta_0$ is in V_λ, $p_\lambda(x) = h_\lambda(x)$. The theorem will be proved when we show that $p_\lambda(x)$ is the product given in the statement of the theorem.

If δ_i is a 1-factor in F_r, $T_r(x)\,\delta_i = \sum x^{\gamma(\delta_j, \delta_i)}\delta_j$, the sum being over all j for which δ_j is in F_r. The δ_0 component of $T_r(x)\,\sigma\tau\delta_0$ is therefore $x^{\gamma(\sigma\tau\delta_0, \delta_0)}$. Let $up_\lambda(x)$ be the δ_0 component of $T_r(x)\, e_t\delta_0$. Then

$$up_\lambda(x) = \sum \varepsilon(\sigma)\, x^{\gamma(\sigma\tau\delta_0, \delta_0)} \qquad \text{for } \sigma \text{ in } C_t \text{ and } \tau \text{ in } R_t. \tag{3.3}$$

Some of the terms in (3.3) give the same expressions. In particular, let R_{t0} be the subgroup of R_t which fixes δ_0 and let r_0 be its order. If τ_1 is in R_{t0}, $\tau_1\delta_0 = \delta_0$. Now for τ in R_t, $\gamma(\sigma\tau\tau_1\delta_0, \delta_0) = \gamma(\sigma\tau\delta_0, \delta_0)$. Let R_t/R_{t0} be a set of right coset representation of R_{t0} in R_t. Then (3.3) becomes

$$up_\lambda(x) = \left(\sum \varepsilon(\sigma)\, x^{\gamma(\sigma\tau\delta_0, \delta_0)}\right) r_0 \qquad \text{for } \sigma \text{ in } C_t \text{ and } \tau \text{ in } R_t/R_{t0}. \tag{3.4}$$

Let C_{t0} be the stabilizer of δ_0 in C_t and let c_0 be its order. If σ_1 is in C_{t0}, $\gamma(\sigma_1\sigma\tau\delta_0, \delta_0) = \gamma(\sigma\tau\delta_0, \sigma_1^{-1}\delta_0) = \gamma(\sigma\tau\delta_0, \delta_0)$. Furthermore, all σ in C_{t0} are even as the permutation in each odd column is identical to the permutation in the column to its immediate right as the line joining $2i-1$ to $2i$ is preserved. Let $C_{t0}\backslash C_t$ be left coset representatives of C_{t0} in C_t. Now (3.4) becomes

$$up_\lambda(x) = \left(\sum \varepsilon(\sigma)\, x^{\gamma(\sigma\tau\delta_0, \delta_0)}\right) r_0 c_0 \quad \text{for } \sigma \text{ in } C_{t0}\backslash C_t \text{ and } \tau \text{ in } R_t\backslash R_{t0}. \tag{3.5}$$

Note that for σ in C_{t0} and τ in R_{t0}, $\sigma\tau\delta_0 = \delta_0$. Suppose that $\sigma\tau\delta_0 = \delta_0$. If τ is not in R_{t0}, $\tau\delta_0$ has an entry in some row which is not joined to an adjacent entry. This is also true in $\sigma\tau\delta_0$ and so $\sigma\tau\delta_0$ is not δ_0. If τ is in R_{t0}, $\sigma\tau\delta_0 = \delta_0 = \sigma\delta_0$ and so σ is in C_{t0}. As $\varepsilon(\sigma) = 1$ for σ in C_{t0}, the coefficient u of δ_0 in $e_t\delta_0$ is r_0c_0.

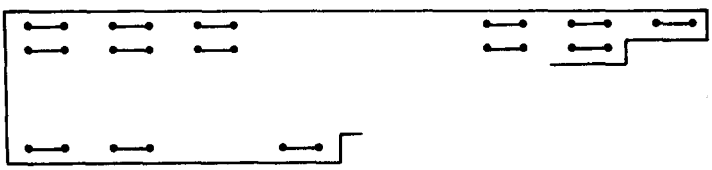
The arguments above show that $T_r(x)$ restricted to V_λ is the scalar $p_\lambda(x)$. We need only show $p_\lambda(x) = d_\lambda(x)$. We may also assume that for diagrams λ' of smaller size we have $d_{\lambda'}(x) = h_{\lambda'}(x) = p_{\lambda'}(x)$. Using (3.5) we see

$$p_\lambda(x) = \sum \varepsilon(\sigma)\, x^{\gamma(\sigma\tau\delta_0, \delta_0)} \qquad \text{for } \sigma \text{ in } C_{t0}\backslash C_t \text{ and } \tau \text{ in } R_t/R_{t0}. \tag{3.6}$$

We know that any choice of coset representatives gives the same polynomial $p_\lambda(x)$. Some arguments become clearer if we choose them in a specific way. It is clear from the definitions that C_{t0} consists of all permutations in C_t which permute the $2i-1$ column in the same way as the $2i$ column for $i=1, 2, ..., \lambda_1/2$. Coset representatives may be chosen which fix the odd-numbered columns pointwise and permute elements in the even-numbered columns. There are of course many other possibilities. We will assume for the remainder of this proof that the coset representatives in $C_{t0}\backslash C_t$ are precisely the permutations acting on even-numbered columns. This is a full set as any element of C_t is a product of a permutation in C_{t0} followed by a permutation moving only elements in even-numbered columns. They are in different cosets.

The choices for coset representatives of R_t/R_{t0} are not as natural. The group R_t is a direct product of the groups R_i, where R_i permutes only the elements of row i and fixes all other elements. Also R_{t0} is a direct product of the groups R_{0i}, where $R_{0i}=R_i \cap R_{t0}$. Coset representatives of R_t/R_{t0} may be chosen as products $r_1 r_2 \cdots r_m$, where the r_i are coset representatives in R_i/R_{0i}.

In order to prove the theorem we concentrate on the $(m, \lambda_m - 1)$ and the (m, λ_m) position. For convenience call the first position a and the second one b. In order to evaluate $\gamma(\sigma\tau\delta_0, \delta_0)$, it is convenient to place the lines from δ_0 in the diagram t. This gives



Pictured this way, $\tau\delta_0$ is a diagram with lines all in the same row. The coset representatives for R_i/R_{0i} may be picked any way for $i=1, 2, ..., m-1$. We choose coset representatives for R_m by first restricting to a group R_{m*}, the subgroup of R_m fixing a and b. Let R_{m0*} be the subgroup of R_{m*} which fixes the 1-factor which is the bottom row of t above. Choose Y a set of representatives of $R_{m*}R_{m0*}$. Let τ_i be the transposition in R_m interchanging $2r-1$ and $2r-\lambda_m+i$ for $i=1, 2, ..., \lambda_m-2$. Let $\tau_0=e$, the identity. The elements $\tau_i Y$ are a full set of representatives of R_m/R_{m0}.

We also wish to choose the coset representatives appropriately for the subgroup of C_t moving elements in the λ_mth column only. Denote this subgroup by C_m. Let C_{m*} be the subgroup of C_m fixing the bottom entry b. Let σ_i be the transposition in C_m interchanging b with the entry above it in the ith row. Here, $i=1, 2, ..., m-1$. Coset representatives for $C_m^*\backslash C_m$ may be taken to be $\sigma_i C_{m-1}$ for $i=0, 1, ..., m-1$, where σ_0 is the identity.

Now let C_{t*} be the permutations in C_t fixing a and b and let R_{t*} be the permutations in R_t fixing a and b. Let C_{t0*} and R_{t0*} be the corresponding stabilizers of δ_0 fixing a and b. Choose coset representatives K for $C_{t0^*} \backslash C_{t^*}$ and L for R_{t^*}/R_{t0^*}. Again choose representatives for C_{t^*} moving only elements in even-numbered columns. Now coset representatives for $C_{t0} \backslash C_t$ can be chosen as $\sigma_i \sigma$, for σ in K and $i = 0, 1, ..., m-1$. Coset representatives for R_t/R_{t^*} can be chosen as $\tau_j \tau$, for τ in L and $j = 0, 1, ..., \lambda_m - 2$. The coset representatives appearing in (3.6) are

$$\{\sigma_i \sigma \tau_j \tau \colon \sigma \text{ in } K, \tau \text{ in } L, i = 0, 1, ..., m-1, j = 0, 1, ..., \lambda_m - 2\}.$$

Using these choices, (3.6) becomes

$$(3.7) \qquad p_\lambda(x) = \sum_{i,j} \sum \varepsilon(\sigma_i \sigma)\, x^{\gamma(\sigma_i \sigma \tau_j \tau \delta_0, \delta_0)} \qquad \text{for } \sigma \text{ in } K \text{ and } \tau \text{ in } L.$$

We concentrate on the inner sum for fixed i and j. Denote this sum by Q_{ij}. Here

$$(3.8) \qquad Q_{ij} = \sum \varepsilon(\sigma)\, x^{\gamma(\sigma_i \sigma \tau_j \tau \delta_0, \delta_0)} \qquad \text{for } \sigma \text{ in } K \text{ and } \tau \text{ in } L.$$

We begin with the expression for Q_{00}. Here σ_i and τ_j are both the identity. Each of the terms σ, τ in the sum in Q_{00} fixes a and b. Let σ', τ' be the corresponding restriction of σ, τ to S_{f-2}. Let δ_0' be δ_0 with $\{a, b\}$ omitted and let γ' be the corresponding inner product on 1-factors of size $f-2$. The connected components $\sigma\tau\delta_0 \cup \delta_0$ are exactly the orbits of $\sigma'\tau'\delta_0' \cup \delta_0'$ with $\{a, b\}$ adjoined. This means $\gamma(\sigma\tau\delta_0, \delta_0) = \gamma'(\sigma'\tau'\delta_0', \delta_0') + 1$. In particular

$$(3.9) \qquad Q_{00} = \sum \varepsilon(\sigma)\, x x^{\gamma'(\sigma'\tau'\delta_0', \delta_0')} \qquad \text{for } \sigma \text{ in } K \text{ and } \tau \text{ in } L.$$

Of course $\varepsilon(\sigma) = \varepsilon(\sigma')$. With these choices we see that

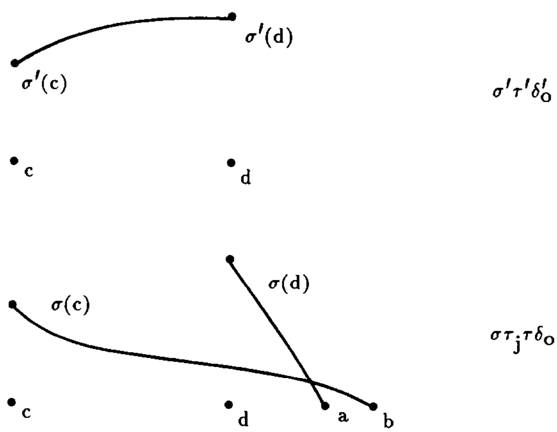
$$(3.10) \qquad Q_{00} = x p_{\lambda^*}(x),$$

where λ^* is λ with λ_m replaced by $\lambda_m - 2$. We know by induction that $p_{\lambda^*}(x) = d_{\lambda^*}(x)$ and so we get

$$(3.11) \qquad Q_{00} = x d_{\lambda^*}(x).$$

We will show that $Q_{0j} = d_{\lambda^*}(x)$ for j not 0. See the illustration below. Suppose j is fixed between 1 and $\lambda_m - 2$ and σ, τ are chosen in K, L. We wish to examine $\gamma(\sigma\tau_j\tau\delta_0, \delta_0)$. Again let $\sigma', \tau', \delta_0', \gamma'$ be the appropriate restrictions to the diagram for λ^*. We will show $\gamma(\sigma\tau_j\tau\delta_0, \delta_0) = \gamma'(\sigma'\tau'\delta_0', \delta_0')$. Let c be $f - \lambda_m + j$, the entry in position (m, j). Suppose c is joined in $\tau\delta_0$ to d. Note that $\sigma\tau_j\tau\delta_0$ is the same as $\sigma'\tau'\delta_0'$ except $\{a, b\}$ has been added and the line from $\sigma'(c)$ to $\sigma'(d)$ is replaced by two lines, one from $\sigma'(c)$ to b and one from a to $\sigma'(d)$. It is now clear that all orbits of $\sigma'\tau'(\delta_0') \cup \delta_0'$ not containing $\sigma'(c)$ and $\sigma'(d)$ are orbits of $\sigma\tau_j\tau(\delta_0) \cup \delta_0$. The orbit containing $\sigma'(c)$ and $\sigma'(d)$ together with $\{a, b\}$ is an orbit of $\sigma\tau_j\tau\delta_0 \cup \delta_0$. Recall a and b are joined in δ_0. This shows $\gamma(\sigma\tau_j\tau\delta_0, \delta_0) = \gamma'(\sigma'\tau'\delta_0', \delta_0')$. It follows from these arguments that

$$Q_{0j} = p_{\lambda^*}(x) = d_{\lambda^*}(x). \tag{3.12}$$



We now turn to Q_{i0} for $i = 1, 2, ..., m-1$ and show that each is $-d_{\lambda^*}(x)$. We need to consider orbits of $\sigma_i\sigma\tau\delta_0 \cup \delta_0$. Again we consider the restricted term $\sigma'\tau'\delta_0'$. This time let c be the entry in the (i, λ_m) position of $\sigma'\tau'\delta_0'$ and let d be the entry joined to c in $\sigma'\tau'\delta_0'$.

The lines in $\sigma_i\sigma\tau\delta_0$ are precisely the lines of $\sigma'\tau'\delta_0'$ except the line from c to d is replaced by a line from d to b and one from a to c. Again the orbits of $\sigma'\tau'\delta_0' \cup \delta_0'$ are those of $\sigma_i\sigma\tau\delta_0 \cup \delta_0$ except for this one orbit through c and d. Note $\varepsilon(\sigma_i\sigma) = -\varepsilon(\sigma) = -\varepsilon(\sigma')$. This gives

$$Q_{i0} = -d_{\lambda^*}(x). \tag{3.13}$$

We have shown

$$Q_{00} + \sum Q_{i0} + \sum Q_{0j} = (x + (\lambda_m - 2) - (m-1))\, d_{\lambda^*}(x) \quad \text{for } i \text{ and } j \text{ not } 0. \tag{3.14}$$

By using the definitions of $d_\lambda(x)$ and $d_{\lambda^*}(x)$, we see that

$$d_\lambda(x) = (x + (\lambda_m - 2) - (m-1))\, d_{\lambda^*}(x). \tag{3.15}$$

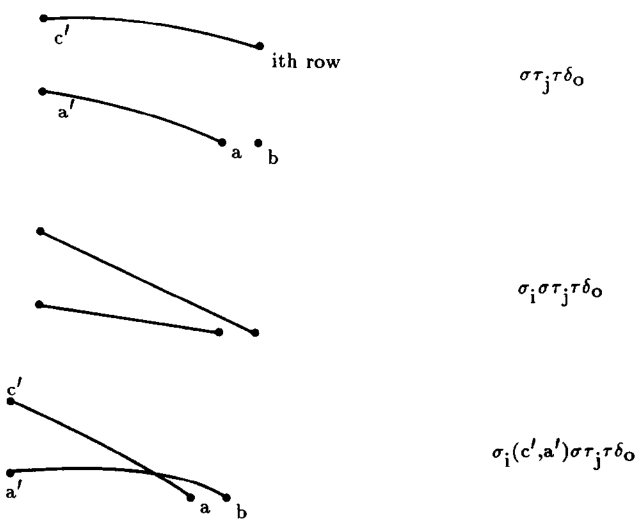
We now show that the sum over Q_{ij} with i and j both not 0 gives zero.

The contribution from a fixed i, j, σ, τ with i and j not 0 is

$\varepsilon(\sigma_i\sigma)x^{\gamma(\sigma_i\sigma\tau_j\tau\delta_0,\delta_0)}$. We will show how to combine terms for a fixed i into disjoint subsets of size two. The sum over each of these subsets will be 0 and so the sum over all Q_{ij} with i and j both not 0 will be zero.

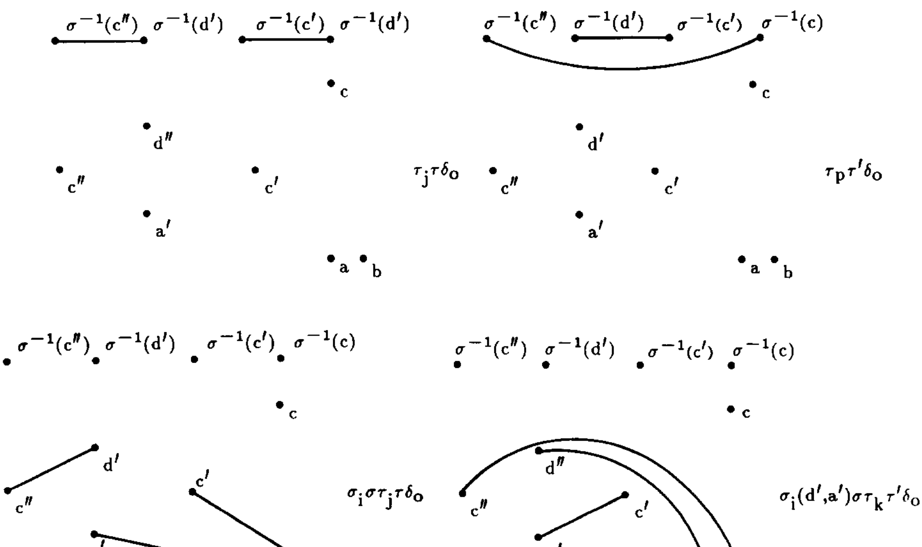
In order to choose the subsets we suppose i, j, σ, τ are chosen with i and j both not 0. We picture $\sigma_i\sigma\tau_j\tau\delta_0$ as if the lines were in the diagram t above. Important lines for us will be the line from a and the line from b. Let a' be the endpoint for the line from a and let c' be the endpoint for the line from b. The point a' is to the left of a as j is not 0.

The easiest case to handle is when a' and c' are in the same column. Suppose c' and a' are in an even-numbered column. Let σ' be $(c', a')\sigma$. As $\varepsilon(c', a') = -1$, $\varepsilon(\sigma_i\sigma') = \varepsilon(\sigma_i(c', a')\sigma) = -\varepsilon(\sigma_i\sigma)$. The orbits of $\sigma_i(c', a')\sigma\tau_j\tau\delta_0 \cup \delta_0$ are clearly the same as the orbits of $\sigma_i\sigma\tau_j\tau\delta_0 \cup \delta_0$ except for the one through a' and c'. These orbits are the same except the connections to a and b have been reversed. The lengths of the orbits are the same. Also, the number of orbits is the same and so the power of x in each term of the sum in Q_{ij} is the same. Such pairs cancel when added together because the signs are opposite. Had we started with $\sigma' = (c', a')\sigma$ instead of σ, we would have paired it with the term corresponding to σ. If c' and a' are in an odd-numbered column, the same result holds using the transposition (c^*, a^*), where c^* is to the immediate right of c' and a^* is to the immediate right of a'. The picture below illustrates these arguments.



Suppose c' and a' are in different columns. Let c be the position (i, λ_m). Note c and c' are joined in $\sigma\tau_i\tau\delta_0$. This means $\sigma^{-1}(c)$ and $\sigma^{-1}(c')$ are in the same row. Let d' be the entry such that $\sigma^{-1}(d')$ is in this row and in the same column as a'. As c' and a' are in different columns, c' is not d'.

Denote the point joined to d' by c'' and suppose c'' is in the kth column. Note $\sigma^{-1}(c'')$ is in the same row as $\sigma^{-1}(c')$, $\sigma^{-1}(c)$, and $\sigma^{-1}(d')$. Let τ' be the coset representative in L for which $\tau_k\tau'\delta_0=(\sigma^{-1}(c'), \sigma^{-1}(c''))\,\tau_j\tau\delta_0$. Then $\tau_k\tau'\delta_0$ is the same as $\tau_j\tau\delta_0$ except that $\sigma^{-1}(c)$ is joined to $\sigma^{-1}(c'')$ and $\sigma^{-1}(d')$ is joined to $\sigma^{-1}(c')$. Assume for now that c' and a' are in an even-numbered column. We examine the terms in the sum $Q_{ij}+Q_{ik}$ corresponding to $\sigma_i(d', a')\,\sigma\tau_k\tau'$ and to $\sigma_i\sigma\tau_j\tau$. Note $\varepsilon(\sigma_i(d', a')\sigma)=-\varepsilon(\sigma_i\sigma)$. We must compare the orbits of $\sigma_i(d', a')\,\sigma\tau_k\tau'\delta_0\cup\delta_0$ with the orbits of $\sigma_i\sigma\tau_j\tau\delta_0\cup\delta_0$. All orbits are the same except the ones through $\{a, b\}$, $\{a', c'\}$, and $\{d', c''\}$. In the first term a' and c' are joined directly. In the second term a' is joined to a which is joined to b in δ_0 and b is joined to c'. Consequently, this orbit has one extra pair of points in the second term. On the other hand, the orbit through d' and c'' contains $\{a, b\}$ in the first term but does not in the second term. All other points are identical. The number of orbits is the same for each and so the power of x for each term is the same. The contribution to $Q_{ij}+Q_{ik}$ for the sum of these terms is 0 as the signs are different. Furthermore, if one started with $\sigma_i(d', a)\,\sigma\tau_k\tau'\delta_0$ and proceeded as above, this term would be paired with $\sigma_i\sigma\tau_j\tau\delta_0$. If c' and a' are in an odd-numbered column, proceeding as above using the transposition (c^*, a^*) gives the same result.



This shows that the sum over all Q_{ij} with both i and j not 0 must vanish and the theorem is proved by (3.7), (3.8), (3.14), and (3.15).

4. The Structure of the Ideal $\mathfrak{A}_f^{(x)}(2k)$

A. *The Quotients* $\mathfrak{A}_f^{(x)}(2k)/\mathfrak{A}_f^{(x)}(2k+2)$

Recall that the subspace $V_f(2k)$ spanned by all 1-factors of $2f$ with at least $2k$ horizontal edges spans an ideal $\mathfrak{A}_f^{(x)}(2k)$ of $\mathfrak{A}_f^{(x)}$. Our goal in this section is to describe the structure of the quotients $\mathfrak{A}_f^{(x)}(2k)/\mathfrak{A}_f^{(x)}(2k+2)$ in terms of the eigenvalues and eigenspaces of certain matrices. In Section 2 we considered the case $f=2k$. Brown [3, 4] also studied these ideals and reduced the semisimplicity of certain algebras to nonsingularity of specific matrices. Let $\mathfrak{Q}_f^{(x)}(2k)$ denote the quotient $\mathfrak{A}_f^{(x)}(2k)/\mathfrak{A}_f^{(x)}(2k+2)$.

Definition 4.1. An m, k *partial 1-factor* is a graph with $m+2k$ points and k lines having the property that every point has degree 0 or 1. Let $P_{m,k}$ denote the set of m, k partial 1-factors, and let $V_{m,k}$ be the real vector space with basis $P_{m,k}$. Note that $V_{0,k}$ is what we called V_k in Section 2.

If $f \in P_{m,k}$ then f has exactly m points of degree 0 which are called the *free points of f*.

Let f_1 and f_2 be m, k partial 1-factors with $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ the free points of f_1 and $\beta_1 < \beta_2 < \cdots < \beta_m$ the free points of f_2. The union of f_1 and f_2 is a graph consisting of some number $\gamma(f_1, f_2)$ of disjoint cycles together with m disjoint paths $P_1, ..., P_m$ whose endpoints are in the set $\{\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_m\}$. Define an inner product $\langle f_1, f_2 \rangle$ on $V_{m,k}$ as follows. Given f_1, f_2 as above:

(1) If any path P_i joins an α_j to an α_i (or equivalently a β_j to a β_i) then

$$\langle f_1, f_2 \rangle = 0.$$

(2) If 1 is not the case then we can renumber the paths so that P_i joins β_i to $a_{\sigma i}$ for some $\sigma \in S_m$. Define

$$\langle f_1, f_2 \rangle = x^{\gamma(f_1, f_2)}\sigma.$$

Note that $\langle f_1, f_2 \rangle = \langle f_2^*, f_1^* \rangle$, where * is the anti-isomorphism defined on the algebra $\mathbb{R}S_m$ by $\sigma \mapsto \sigma^{-1}$.

Proposition 4.2. *Let* $f = m+2k$. *Then the quotient* $\mathfrak{Q}_f^{(x)}(2k)$ *is isomorphic as an algebra to* $(V_{m,k} \otimes V_{m,k} \otimes \mathbb{R}S_m, \cdot)$, *where*

$$(a \otimes b \otimes z) \cdot (c \otimes d \otimes y) = a \otimes d \otimes (z < b, c > y).$$

Proof. As a vector space $\mathfrak{Q}_f^{(x)}(2k)$ has basis the set of all $\Delta + \mathfrak{A}_f^{(x)}(2k+2)$, where Δ is a 1-factor of $2f$ points with exactly $2k$ horizontal edges.

Define the linear map ϕ,

$$\phi\colon V_{m,k}\otimes V_{m,k}\otimes \mathbb{R}S_m \to \mathfrak{D}_f^{(x)}(2k),$$

in the following way. Given $f_1, f_2 \in P_{m,k}$ with free points $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ and $\beta_1 < \beta_2 < \cdots < \beta_m$ and given $\sigma \in S_m$ define $\phi(f_1 \otimes f_2 \otimes \sigma)$ to be the 1-factor of $2f$ points with

(1) a horizontal line joining i to j in the top row if and only if i and j are adjacent in f_1;

(2) a horizontal line joining $(f+i)$ to $(f+j)$ in the bottom row if and only if i and j are adjacent in f_2;

(3) a vertical line joining $(f+\beta_i)$ to $\alpha_{\sigma i}$ for $i = 1, 2, ..., m$.

Clearly ϕ is a vector space isomorphism of $V_{m,k}\otimes V_{m,k}\otimes \mathbb{R}S_m$ onto $\mathfrak{D}_f^{(x)}(2k)$. It remains to show that ϕ is multiplicative.

Let a, b, c, d be in $P_{m,k}$ with free points $\{\alpha_i\}$, $\{\beta_i\}$, $\{\psi_i\}$, and $\{\delta_i\}$ (each set written in increasing order). Let τ and π be in S_m. We consider the product

$$x^{\gamma}\Delta' = \phi(a\otimes b\otimes \tau)\circ \phi(c\otimes d\otimes \pi).$$

The free point δ_i is adjacent to $\psi_{\pi i}$ in $\phi(c\otimes d\otimes \pi)$. There are two cases:

Case 1. Suppose $\psi_{\pi i}$ is joined by a path in $b\cup c$ to $\psi_{\pi j}$ for some i. Then we have a horizontal edge in Δ' from δ_i to δ_j. This is not a horizontal edge in d so $\Delta' \in \mathfrak{A}_f^{(x)}(2k+2)$. Also $\langle b, c\rangle = 0$ by definition so

$$\phi((a\otimes b\otimes \tau)\cdot(c\otimes d\otimes \pi)) = 0 = \phi(a\otimes b\otimes \tau)\circ\phi(c\otimes d\otimes \pi).$$

Case 2. Suppose $\psi_{\pi i}$ is joined by a path in $b\cup c$ to $\beta_{\sigma\pi i}$ for $i = 1, 2, ..., m$, and σ is a permutation in S_m. As $\beta_{\sigma\pi i}$ is adjacent to $\alpha_{\tau\sigma\pi i}$ in $\phi(a\otimes b\otimes \pi)$, δ_i is adjacent to $\alpha_{\tau\sigma\pi i}$ in Δ'. Also the number of cycles in the middle row of $U(\phi(a\otimes b\otimes \tau), \phi(c\otimes d\otimes \pi))$ is $\gamma(b, c)$. Thus

$$x^{\gamma}\Delta' = x^{\gamma(b,c)}\phi(a\otimes d\otimes \tau\sigma\pi) = \phi(a\otimes d\otimes \tau\langle b, c\rangle \pi).$$

This proves the proposition.

Remark. Brown in [3, 2.1] has a similar method for multiplying in $\mathfrak{D}_f^{(x)}(2k)$.

Our goal for the rest of this subsection is to describe the structure of the ring $\mathfrak{D}_f^{(x)}(2k)$ in terms of the eigenvalues of certain matrices. We begin by recalling several facts from the representation theory of the symmetric groups (see James and Kerber [9]). For each partition λ of m, let S^{λ}

denote the Specht module corresponding to λ and let d_λ denote the dimension of S^λ.

Fact 1. There exists in Sym(m) a unique minimal 2-sided ideal $(\mathrm{Sym}(m))^\lambda$ of dimension $(d_\lambda)^2$ which can be written as direct sums

$$(\mathrm{Sym}(m))^\lambda = I_1 \oplus \cdots \oplus I_{d_\lambda}$$
$$(\mathrm{Sym}(m))^\lambda = J_1 \oplus \cdots \oplus J_{d_\lambda},$$

where each I_i is a left ideal of Sym(m) for which multiplication on the left gives a representation isomorphic to S^λ and each J_i is a right ideal of Sym(m) for which right multiplication is isomorphic to S^λ.

Fact 2. The ideal $(\mathrm{Sym}(m))^\lambda$ considered as a vector space of linear transformations of S^λ is the full matrix algebra End(S^λ).

Fact 3. There exists a basis $A_1, ..., A_{d_\lambda}$ for S^λ with respect to which the matrices $\Psi_\lambda(\sigma)$ for $\sigma \in \mathrm{Sym}(m)$ acting on S^λ are orthogonal, i.e., $\Psi_\lambda(\sigma^{-1}) = \Psi_\lambda(\sigma)^t$.

For each $i, j \in \{1, 2, ..., d_\lambda\}$ let x_i and y_j be the elements in $(\mathrm{Sym}(m))^\lambda$ such that $\Psi_\lambda(x_i)$ is the matrix with a 1 in the i, 1 entry and zeroes elsewhere and $\Psi_\lambda(y_j)$ is the matrix with a 1 in the 1, j entry and zeroes elsewhere.

$$\Psi_\lambda(x_i) = i\begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}, \qquad \Psi_\lambda(y_j) = \begin{bmatrix} & \overset{j}{1} & \\ & & \\ & & \end{bmatrix}.$$

It is possible to choose such x_i and y_j by Fact 2 above. Note that

$$(x_i y_j)(x_{j'} y_l) = \begin{cases} x_i y_l & \text{if } j = j' \\ 0 & \text{otherwise.} \end{cases}$$

For the rest of this subsection, let p denote the size of $P_{m,k}$, i.e., the number of m, k partial 1-factors.

DEFINITION 4.3. Let $T_{m,k}(x)^\lambda$ be the (pd_λ)-by-(pd_λ) matrix which is the following p-by-p grid of d_λ-by-d_λ matrices. The matrices in the grid are indexed by pairs of m, k partial 1-factors with the matrix corresponding to the 1-factors b, c being $\Psi_\lambda(\langle b, c\rangle)$.

Let N^λ and R^λ denote the nullspace and range of $T_{m,k}(x)^\lambda$, respectively. Recall that if $\langle b, c\rangle = x^\gamma\sigma$ then $\langle c, b\rangle = x^\gamma\sigma^{-1}$. So the matrix $T_{m,k}(x)^\lambda$ is symmetric. Choose a basis $u^{(1)}, ..., u^{(n)}$ for N^λ and an orthonormal basis of eigenvectors $v^{(1)}, ..., v^{(r)}$ for the nonzero eigenvalues $\mu^{(1)}, ..., \mu^{(r)}$.

DEFINITION 4.4. For each ideal I_t and each m, k partial 1-factor d define $V_L(I_t, d)$ to be the linear span of all $c \otimes d \otimes x$, where c is arbitrary and $x \in I_t$.

Note that $V_L(I_t, d)$ is a left ideal of $\mathfrak{D}_f^{(x)}(2k)$. Define $W_L(I_t, d) \subset V_L(I_t, d)$ to be the linear span of all

$$\sum (u)_{c,i}\, c \otimes d \otimes A_{i,t},$$

where u is in N^λ and where $A_{i,t}$ is the basis element of I_t corresponding to the basis element A_i in S^λ.

PROPOSITION 4.5. *Suppose* $v = \sum (v)_{ci} c \otimes d \otimes A_{it}$ *is an element of* $V_L(I_t, d)$. *Let* a, b *be partial* 1-*factors. For any* $\sigma \in S_m$

$$(a \otimes b \otimes \sigma) v = a \otimes d \otimes \sigma \left\{ \sum \gamma_j A_{j,t} \right\},$$

where γ_j *is the* (b, j) *entry of* $T_{m,k}(x)^\lambda\,(v)$.

Proof.

$$(a \otimes b \otimes \sigma) \circ \left\{ \sum (v)_{c,i} c \otimes d \otimes A_{i,t} \right\} = a \otimes d \otimes \sigma \left\{ \sum (v)_{c,i} \langle b, c \rangle A_{i,t} \right\}$$

$$= a \otimes d \otimes \sigma \left\{ \sum \gamma_j A_{j,t} \right\},$$

where γ_j is the coefficient of A_{jt} in $\sum (v)_{c,i} \langle b, c \rangle A_{it}$. By definition of $T_{m,k}(x)^\lambda$, the coefficient of A_{jt} in $\langle b, c \rangle A_{it}$ is the $(b, j), (c, i)$ entry of $T_{m,k}(x)^\lambda$. Thus γ_j is the (b, j) entry of $(T_{m,k}(x)^\lambda) v$.

PROPOSITION 4.6. (1) $\mathfrak{D}_f^{(x)}(2k)\, W_L(I_t, d) = 0$.

(2) $\mathfrak{D}(v) = V_L(I_t, d)$ *for any* v *in* $V_L(I_t, d)$ *not in* $W_L(I_t, d)$.

(3) $V_L(I_t, d)/W_L(I_t, d)$ *is irreducible as a left* $\mathfrak{D}_f^{(x)}(2k)$ *module.*

Proof. Suppose w is a generating element of $W_L(I_t, d)$. The γ_j appearing in Proposition 4.5 are all 0 for any $a \otimes b \otimes \sigma$ by the definition of $W_L(I_t, d)$. The first equation follows. Suppose v is in $V_L(I_t, d)$ but not in $W_L(I_t, d)$. Choose a (b, j) such that $(T_{m,k}(x)^\lambda\,(v))$ is not zero. Then $a \otimes b \otimes \sigma(v)$ is not zero as γ_j is not zero. Note that a and σ were arbitrary. The images under $\sigma \in \mathrm{Sym}(m)$ of any nonzero vector in I_t generate all of I_t as I_t is an irreducible $\mathrm{Sym}(m)$ module. Hence vectors of the form

$$(a \otimes b \otimes \sigma) \circ \left(\sum (v)_{c,j} c \otimes d \otimes A_{j,t} \right)$$

generate all of $V_{\mathrm{L}}(I_t, d)$. This proves the second inequality. The third follows immediately from the first two.

Let $W_{\mathrm{L}}^{\lambda} = \bigoplus W_{\mathrm{L}}(I_t, d)$. By Proposition 4.5, W_{L}^{λ} is a nilpotent left ideal of $\mathfrak{D}_f^{(x)}(2k)$. Recall that $\mathrm{Sym}(m)^{\lambda}$ can also be written as a direct sum of right ideals $J_1, \ldots, J_{d_\lambda}$. For each J_t and each partial 1-factor a, let $V_{\mathrm{R}}(J_t, a)$ be the linear span of all $a \otimes b \otimes x$, where b is arbitrary and x is in J_t. Define $W_{\mathrm{R}}(J_t, a) \subset V_{\mathrm{R}}(J_t, a)$ to be the linear span of all

$$\sum (u)_{c,i} a \otimes b \otimes A_{j,t},$$

where $u^t T_{m,k}(x)^{\lambda} = 0$ and where $A_{j,t}$ is as before.

The same proofs used in Propositions 4.5 and 4.6 show that

(1) $W_{\mathrm{R}}(J_t, a) \circ (c \otimes d \otimes \sigma) = 0$,

(2) $V_{\mathrm{R}}(J_t, a)/W_{\mathrm{R}}(J_t, a)$ is an irreducible right $\mathfrak{D}_f^{(x)}(2k)$ module.

Define $W_{\mathrm{R}}^{\lambda} = \bigoplus W_{\mathrm{R}}(J_t, a)$, and define W^{λ} to be the nilpotent 2-sided ideal

$$W^{\lambda} = W_{\mathrm{L}}^{\lambda} + W_{\mathrm{R}}^{\lambda}$$

Definition 4.7. Define $\mathfrak{D}^{\lambda}$ to be the 2-sided ideal of $\mathfrak{D}_f^{(x)}(2k)$ given by the linear span of all vectors

$$a \otimes b \otimes x,$$

where a and b are arbitrary and $x \in (\mathrm{Sym}(m))^{\lambda}$.

Note that $\mathfrak{D}^{\lambda}$ is a 2-sided ideal of $\mathfrak{D}_f^{(x)}(2k)$ which contains the left ideals $V_{\mathrm{L}}(I_t, d)$, the right ideals $V_{\mathrm{R}}(J_t, a)$, and the 2-sided ideal W^{λ}. Note also that $\mathfrak{D}_f^{(x)}(2k)$ is the direct sum of the $\mathfrak{D}^{\lambda}$.

Proposition 4.8. *$\mathfrak{D}^{\lambda}/W^{\lambda}$ is canonically isomorphic to the full matrix ring* $\mathrm{End}(R^{\lambda})$. *Recall that R^{λ} is the range of $T_{m,k}(x)^{\lambda}$.*

Proof. Given eigenvectors $v^{(r)}$ and $v^{(s)}$ define

$$Z(v^{(r)}, v^{(s)}) = (\mu^{(r)} \mu^{(s)})^{-1} \sum (v^{(r)})_{a,i} (v^{(s)})_{b,j} a \otimes b \otimes x_i y_j.$$

Taking the product of $Z(v^{(r)}, v^{(s)})$ and $Z(v^{(t)}, v^{(u)})$ we obtain

$$\begin{aligned} &Z(v^{(r)}, v^{(s)})\, Z(v^{(t)}, v^{(u)}) \\ &\quad= (\mu^{(r)}\mu^{(s)}\mu^{(t)}\mu^{(u)})^{-1} \\ &\qquad \sum (v^{(r)})_{a,i}\, (v^{(u)})_{d,l}\, a \otimes d \otimes (v^{(s)})_{b,j}\, (v^{(t)})_{c,k}\, \{x_i y_j \langle b, c\rangle x_k y_l\} \\ &\quad= (\mu^{(r)}\mu^{(s)}\mu^{(t)}\mu^{(u)})^{-1} \sum (v^{(r)})_{a,i}\, (v^{(u)})_{d,l} \\ &\qquad \times a \otimes d \otimes x_i y_j \left(\sum (v^{(s)})_{b,j} \left\{\sum (v^{(t)})_{c,k} \langle b, c\rangle x_k\right\}\right) y_l. \end{aligned}$$

Now

$$\sum (v^{(t)})_{c,k} \langle b, c\rangle x_k = \sum \gamma_r x_r,$$

where γ_r is the b, r coefficient of $(T_{m,k}(x)^{\lambda})\, v^{(t)}$. In this case, $\gamma_r = \mu^{(t)}(v^{(t)})_{b,r}$ as $v^{(t)}$ is an eigenvector with eigenvalue $\mu^{(t)}$. So,

$$\begin{aligned} &x_i y_j \left(\sum (v^{(s)})_{b,j} \left\{\sum (v^{(t)})_{c,k} \langle b, c\rangle x_k\right\}\right) y_l \\ &\qquad = \mu^{(t)} \sum (v^{(s)})_{b,j}\, (v^{(t)})_{b,r}\, \{x_i y_j x_r y_l\}. \end{aligned}$$

But recall that

$$x_i y_j x_r y_l = \begin{cases} x_i y_l & \text{if } j = r \\ 0 & \text{otherwise.} \end{cases}$$

Using this fact in the previous equation we have

$$\begin{aligned} &x_i y_j \left(\sum (v^{(s)})_{b,j} \left\{\sum (v^{(t)})_{c,k} \langle b, c\rangle x_k\right\}\right) y_l \\ &\qquad = \mu^{(t)} x_i y_l \left\{\sum (v^{(s)})_{b,j}\, (v^{(t)})_{b,j}\right\}. \end{aligned}$$

By the orthonormality of the $v^{(i)}$ we have

$$\sum (v^{(t)})_{b,j}\, (v^{(s)})_{b,j} = \delta_{s,t},$$

where $\delta_{s,t}$ is the Kronecker delta. Substituting above we obtain

$$Z(v^{(r)}, v^{(s)})\, Z(v^{(t)}, v^{(u)}) = \delta_{s,t} Z(v^{(r)}, v^{(u)}),$$

which shows that the subspace of $\mathfrak{D}^{\lambda}$ spanned by the $Z(v^{(r)}, v^{(s)})$ is isomorphic to $\mathrm{End}(R^{\lambda})$.

The ideal $\mathfrak{D}^\lambda = V_{m,k} \otimes V_{m,k} \otimes (\mathrm{Sym}(m))^\lambda$ is isomorphic as a vector space to $(V_{m,k} \otimes S^\lambda) \otimes (V_{m,k} \otimes S^\lambda)$ via the linear map f sending $(c \otimes A_i) \otimes (d \otimes A_j)$ to $c \otimes d \otimes x_i y_j$. Writing $V_{m,k} \otimes S^\lambda$ as $N^\lambda \oplus R^\lambda$ we have, from Propositions 4.5, 4.6, and 4.8, that

(A) $f(N^\lambda \otimes (V_{m,k} \otimes S^\lambda) + (V_{m,k} \otimes S^\lambda) \otimes N^\lambda)$ is contained in the radical of $\mathfrak{A}_f^{(x)}(2k)$,

(B) $f(R^\lambda \otimes R^\lambda)$ is a full matrix ring.

The next theorem follows immediately from (A) and (B).

THEOREM 4.9. *With notation as above*:

(A) *Let* $W^\lambda = f(N \otimes (V_{m,k} \otimes S^\lambda) + (V_{m,k} \otimes S^\lambda) \otimes N)$. Then W^λ is the intersection of the radical of $\mathfrak{D}_f^{(x)}(2k)$ with $\mathfrak{D}^\lambda$.

. (B) $\mathfrak{D}^\lambda / W^\lambda$ *is a full matrix ring which is canonically isomorphic to* $\mathrm{End}(R^\lambda)$.

We are able to approach Theorem 4.9 with an alternative treatment which does not produce the explicit algebra isomorphism given above. It also does not require the use of eigenvectors. Note first of all that $V_{\mathrm{L}}(I_t, d)$ is isomorphic to $V_{\mathrm{L}}(I_{k'}, d)$ as I_t and $I_{k'}$ are isomorphic as left S_m modules. The bases $c \otimes d \otimes A_{jt}$ and $c \otimes d \otimes A_{jk'}$ exhibit this isomorphism. Suppose d' is another m, k partial 1-factor. There is a permutation τ in S_f for which $\tau\delta_i = \delta_i'$ and $\tau d = d'$. Here δ_i, δ_i' are the free points in d, d' and if m is joined to n in d, τm is joined to τn in d'. Let δ_τ be the element in $\mathfrak{A}_f^{(x)}(0)$ for which the point $f+m$ in the bottom row is joined to τm in the top row. Then

$$(c \otimes d \otimes \sigma)\, \delta_\tau = c \otimes d' \otimes \sigma.$$

In particular $V_{\mathrm{L}}(I_t, d)$ and $V_{\mathrm{L}}(I_t, d')$ are isomorphic as $\mathfrak{A}_f^{(x)}(2k)$ modules.

We know that a composition series for $V_{\mathrm{L}}(I_t, d)$ contains the irreducible $V_{\mathrm{L}}(I_t, d)/W_{\mathrm{L}}(I_t, d)$ with multiplicity one. The remaining composition factors are all the zero representation. As $V_{\mathrm{L}}(I_t, d)$ and $V_{\mathrm{L}}(I_t, d')$ are isomorphic, $\mathfrak{D}^\lambda$ has a composition series with the one irreducible $V_{\mathrm{L}}(I_t, d)/W_{\mathrm{L}}(I_t, d)$ appearing pd_λ times and the remaining factors all zero. This means that $\mathfrak{D}^\lambda$ modulo its radical is isomorphic to $\mathrm{End}(R^\lambda)$. The codimension of the radical is $(\dim R^\lambda)^2$. But W^λ has this codimension, is in the radical, and so must be the radical. This is Theorem 4.9.

This completes Part A of Section 4 in which we analyzed the structure of the ring $\mathfrak{D}_f^{(x)}(2k)$. In Part B of this section we will relate the results in Part A to the structure of the ideals $\mathfrak{A}_f^{(x)}(2k)$.

B. *The Representations of* $\mathfrak{A}_f^{(x)}(2k)$ *and* $\mathfrak{B}_f^{(x)}(2k)$

We continue describing results for $\mathfrak{A}_f^{(x)}(2k)$ only. The results for $\mathfrak{B}_f^{(x)}(2k)$ follow from Theorem 2.10. Let $\mathfrak{A} = \mathfrak{A}_f^{(x)}(0)$. The information we have obtained is enough to determine the irreducible representations of $\mathfrak{A}$. In particular let V be an irreducible $\mathfrak{A}$ module. Recall that an irreducible $\mathfrak{A}$ module is one for which $\mathfrak{A}V$ is not zero and V has no proper invariant subspaces. Let K be the kernel of the action. In other words, K is the set of elements, α, of $\mathfrak{A}$ such that $\alpha v = 0$ for all v in V. Let k be the integer such that $\mathfrak{A}_f^{(x)}(2k)$ is not in K but $\mathfrak{A}_f^{(x)}(2k+2)$ is in K. By the definition of irreducibility, $\mathfrak{A}$ is not K. If $r = [f/2]$, $\mathfrak{A}_f^{(x)}(2r+2)$ is 0 and so such a k exists. Now V can be considered as an $\mathfrak{A}/\mathfrak{A}_f^{(x)}(2k+2)$ module. To simplify notation further, let $\mathfrak{A}_f^{(x)}(2k)$ be denoted by $\mathfrak{A}(k)$, $\mathfrak{A}_f^{(x)}(2k+2)$ by $\mathfrak{A}(k+1)$, and $\mathfrak{O}_f^{(x)}(2k)$ by $\mathfrak{O}(k)$. Now $\mathfrak{O}(k)$ is an ideal in $\mathfrak{A}/\mathfrak{A}(k+1)$.

Some of the results we use follow from general theorems about representations of finite-dimensional algebras or Artinian rings. To the extent possible, we show these directly. In particular, the restriction of the representation of $\mathfrak{A}/\mathfrak{A}(k+1)$ to $\mathfrak{O}(k)$ remains irreducible. To see this, note the restriction is a nonzero representation by our choice of k. The set of v in V for which $\mathfrak{O}(k)v = 0$ is an $\mathfrak{A}$ invariant subspace as for any such v, $\omega\alpha v = \omega_1 v = 0$, where α is in $\mathfrak{A}$, ω is in $\mathfrak{O}(k)$, and $\omega_1 = \omega\alpha$ is in $\mathfrak{O}(k)$. As V is an irreducible $\mathfrak{A}$ module, this set must be 0. Now $\mathfrak{O}(k)v$ is an $\mathfrak{A}$ invariant submodule which is nonzero if v is not zero. This means that $\mathfrak{O}(k)v = V$ for all nonzero v and so V is an irreducible $\mathfrak{O}(k)$ module. As such we may index this module by a partition λ of m, where $m = f - 2k$.

It follows from general results about representations of algebras that any such representation may be extended to a unique representation of $\mathfrak{A}$. We present a specific module. If λ is a partition of m, let V be the $\mathfrak{O}(k)$ module $V_{\mathrm{L}}(I_t, d)/W_{\mathrm{L}}(I_t, d)$ as in Proposition 4.6. We will show that $V_{\mathrm{L}}(I_t, d)$ and $W_{\mathrm{L}}(I_t, d)$ are $\mathfrak{A}$ invariant and so V is an irreducible $\mathfrak{A}$ module. Let $c \otimes d \otimes y$ correspond as in 4.2 to an element in $V_{\mathrm{L}}(I_t, d)$. This means that y is in I_t and is a linear combination of permutations in S_m. By the terms in $c \otimes d \otimes y$ we mean the individual partial 1-factors on $2f$ points which correspond as in 4.2 to terms of the form $c \otimes d \otimes \sigma$. They all have the same top, c, and the same bottom, d. The free points of d are joined to the free points of c via a permutation σ in S_m. They differ in their permutations. Let δ be a 1-factor on $2f$ points with s vertical lines. Let $a \otimes b \otimes \pi$ correspond to δ as in 4.2. This means that δ can be pictured with a top a, a bottom b, s vertical lines, and $f-s$ horizontal lines. Here a and b are in $P_{s,u}$, where $u = (f-s)/2$. If the free points of a are $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ and those of b are $\beta_1 < \beta_2 < \cdots < \beta_s$, the vertical lines join β_i to $\alpha_{\pi(i)}$. As $\mathfrak{A}(k)$ and $\mathfrak{A}(k+1)$ are ideals, δ acts on $\mathfrak{A}(k)/\mathfrak{A}(k+1) = \mathfrak{O}(k)$. Suppose $\delta(c \otimes d \otimes y)$ is not zero in $\mathfrak{O}(k)$. Then $\delta(c \otimes d \otimes \sigma)$ is not zero in $\mathfrak{O}(k)$ for some σ in S_m. Let δ' be the 1-factor on $2f$ points corresponding to $c \otimes d \otimes \sigma$. This means δ' has c as its top, d as its bottom, and the free point δ_i of d is joined to the free point $\psi_{\sigma(i)}$ of c. As $\delta\delta'$ is not zero in $\mathfrak{O}(k)$, $\delta\delta'$ must have k horizontal lines and the bottom of $\delta\delta'$ must be d. The free points of c, $\psi_1, \ldots, \psi_m$, must all be joined to free points of b in $b \cup c$.

Otherwise $\delta\delta'$ would be zero in $\mathfrak{D}(k)$ as if ψ_i were joined to $\psi_{i'}$, $\delta\delta'$ would be in $\mathfrak{A}(k+1)$. This means that s is at least m. Suppose ψ_i is joined to $\beta_{j(i)}$ in $b \cup c$. From the definition of π, $\beta_{j(i)}$ is joined to $\alpha_{\pi j(i)}$. The free points of the top of $\delta\delta'$ will be the set $\alpha_{\pi j(i)}$ for i from 1 to m. Relabel them as $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. Define σ' in S_m by $\sigma'(i) = i'$, where $\alpha_{i'}$ is joined to $\beta_{j(i)}$. Then $\delta\delta'$ corresponds to $x^{\gamma}(a' \otimes d \otimes \sigma'\sigma)$, where a' is a top whose free points are $\alpha_1, ..., \alpha_m$ and whose horizontal lines depend only on the horizontal lines in a and paths in $b \cup c$ disjoint from ψ_i and $\beta_{j(i)}$. The power γ is the number of internal cycles in $b \cup c$. Suppose now δ'_1 corresponding to $c \otimes d \otimes \sigma_1$ is another term in $c \otimes d \otimes y$. Then $\delta\delta'_1$ corresponds to $x^{\gamma}(a' \otimes d \otimes \sigma'\sigma_1)$. It follows now that $\delta(c \otimes d \otimes y) = a' \otimes d \otimes \sigma' y$ and $\delta(c \otimes d \otimes y)$ is in $V_L(I_t, d)$.

Now let w be in $W_L(I_t, d)$. Recall that $W_L(I_t, d)$ is the set of elements of $V_L(I_t, d)$ which are mapped to zero by $\mathfrak{D}(k)$. If δ' is in $\mathfrak{D}(k)$, $\delta'(\delta w) = (\delta'\delta) w = 0$ as $\delta'\delta$ is in $\mathfrak{D}(k)$. In particular δw is in $W_L(I_t, d)$. This means $\mathfrak{A}$ acts on V. The action is an extension of the irreducible action of $\mathfrak{D}(k)$ to $\mathfrak{A}$.

We have shown that the irreducible representations of $\mathfrak{A}$ can be indexed by partitions of m where $m = f$, $f-2$, $f-4, ...,$ $f - 2[f/2]$. Each such representation restricts to an irreducible representation of $\mathfrak{A}(k)/\mathfrak{A}(k+1)$ for $m = f - 2k$. Each representation extends to one of $\mathfrak{A}$. Corresponding results follow for the ideals $\mathfrak{A}(k)$.

We use some facts about the radical of an Artinian ring. In particular suppose that I is an ideal in an Artinian ring R. If Rad R denotes the radical of R, the following relations hold;

$$\text{Rad } R \cap I = \text{Rad } I$$

$$(\text{Rad } R + I)/I = \text{Rad } R/I.$$

To see the second note that Rad R is a nilpotent ideal and so $(\text{Rad } R + I)/I$ is a nilpotent ideal and so is in Rad R/I. Also $R/(\text{Rad } R + I)$ is a homomorphic image of $R/\text{Rad } R$ by $(\text{Rad } R + I)/\text{Rad } R$. But $R/\text{Rad } R$ is semisimple and so any homomorphic image is also semisimple. This means the quotient of R/I by $(\text{Rad } R + I)/I$ is semisimple and so equality holds.

We now examine $\text{Rad}(\mathfrak{A}_f^{(x)}(2k))$. We keep the notation above. Of course $\text{Rad}(\mathfrak{A}) \subseteq \mathfrak{A}(1)$ as $\mathfrak{A}/\mathfrak{A}(1)$ is isomorphic to S_f and so is semisimple. We know $\text{Rad}(\mathfrak{D}(k)) = \sum W^{\lambda}$. The arguments above show

$$(\text{Rad}(\mathfrak{A}(k)) + \mathfrak{A}(k+1)/\mathfrak{A}(k+1)) = \sum W^{\lambda}$$

$$\text{Rad } \mathfrak{A}(k)/\text{Rad } \mathfrak{A}(k+1) = \sum W^{\lambda}.$$

This shows that the dimensions of the radicals of the various $\mathfrak{A}(k)$ are determined by the dimensions of the nullspaces of the matrices $T_{m,k'}(x)^{\lambda}$ for k' greater than or equal to k.

5. Conclusion

The degrees of the representations and the dimension of the radical of $\mathfrak{A}_f^x(2k)$ depend on the dimensions of the nullspaces of the matrices $T_{m,k}(x)^\lambda$. Here $f = m + 2k$. In this section we summarize information for $m = 0$ and 1 and give some conjectures about the eigenvalues when m is larger than 1.

Theorem 3.1 gives information from which the eigenvalues of $T_{0,2k}(x)$ may be determined. Here $f = 2k$, $m = 0$, and the superscript λ is omitted. Note that $T_{0,2k}(x)$ is the matrix $T_k(x)$ in Theorem 3.1. For each even partition μ of $2f$, there are dim S^μ eigenvalues $d_\mu(x) = \Pi(x + a_{ij})$, where a_{ij} is the entry in the (ij) position of Δ and (ij) is in the diagram of shape μ. The number of 1-factors of $2f$ points is $(2f-1)(2f-3)\cdots 1 = (2f-1)!!$. The dimension of the associated irreducible is $(2f-1)!! - \sum \dim S^\mu$, where the sum is over all even partitions μ of $2f$ in which some a_{ij} is $-x$ for an (ij) in the diagram of shape μ. In particular $\mathfrak{A}_{2k}^{(x)}(2k)$ is semisimple if x is not an integer. If x is an integer in the interval $[-2k+2, k-1]$, $\mathfrak{A}_{2k}^{(x)}(2k)$ has a nontrivial radical. If x is an integer not in this interval, $\mathfrak{A}_{2k}^{(x)}(2k)$ is semisimple.

Because of the isomorphism between $\mathfrak{A}_f^{(x)}$ and $\mathfrak{B}_f^{(-x)}$, $\mathfrak{B}_{2k}^{(x)}(2k)$ also has only one nontrivial irreducible representation. Its degree is $(2k-1)!! - \sum \dim S^\mu$, where the sum is over all even partitions μ of $2k$ in which some a_{ij} is x for an (ij) in the diagram of shape μ.

Theorem 3.1 may also be used in the case $m = 1$, $f = 1 + 2k$. Here the number of 1, k partial 1-factors is $f!!$. This is the same as the number of 1-factors of a set with $f + 1 = 2k + 2$ elements. The 1, k partial 1-factors can be naturally mapped to the partial 1-factors of $2k + 2$ points by joining the free point of a 1, k partial 1-factor to an extra point. Let δ_1, δ_2 be 1, k partial 1-factors and δ_1', δ_2' the corresponding 1-factors on $2k + 2$ points obtained by adding a line from the free point to this new point. The cycles of $\delta_1' \cup \delta_2'$ not including the new point are all internal cycles of $\delta_1 \cup \delta_2$. The path in $\delta_1 \cup \delta_2$ joining the free points together with the new point is a cycle in $\delta_1' \cup \delta_2'$. This means

$$xT_{1,k}(x) = T_{k+1}(x).$$

In particular, the eigenvalues and the nullity of $T_{1,k}(x)$ are determined by Theorem 3.1. Again the nullspace is zero unless x is integral between certain values related to k. If x is integral, the eigenvalues are integers.

If m is at least two, the eigenvalues of $T_{m,k}(x)^\lambda$ cannot be determined except for certain λ. We conjectured that the determinant of $T_{m,k}(x)^\lambda$ has only integral roots. This implies $U_f^{(x)}$ is semisimple unless x is an integer. Recently this conjecture was proved by Hans Wenzl [15] who used methods completely different than those used in this paper.

APPENDIX: A CONJECTURE OF R. P. STANLEY GENERALIZING THEOREM 3.1

In the course of work on multidimensional generalizations of the chi-squared distribution, A. T. James defined a set of symmetric functions $\{Z_\lambda(a_1, ..., a_n): \lambda \text{ is an even partition}\}$ which he called *zonal polynomials* (see James [6–8]). Amongst the properties enjoyed by zonal polynomials is a factorization of their value at (1, 1, ..., 1) (see James [7]):

$$\text{(A1)} \qquad Z_\lambda(1, 1, ..., 1) = \prod \prod (n + (2j - i - 1))(i, j) \text{ within } \lambda.$$

We will define a multivariable generalization of the matrix $T_f(x)$. Given a set $U = \{u_1, ..., u_t\}$ of positive integers, let $p_U(a_1, ..., a_n)$ be the power sum symmetric function indexed by U, i.e.,

$$p_U(a_1, ..., a_n) = \prod (a_1^{u_i} + a_2^{u_i} + \cdots + a_n^{u_i}).$$

We will consider a matrix $T_f(a_1, ..., a_n)$ whose rows and columns are indexed by the 1-factors on $2f$ points. Given two 1-factors δ_1 and δ_2 let $2u_1, ..., 2u_t$ be the sizes of the connected components of $\delta_1 \cup \delta_2$ (here $t = \gamma(\delta_1, \delta_2)$). Define the (δ_1, δ_2) entry of $T_f(a_1, ..., a_n)$ to be

$$(T_f(a_1, ..., a_n))_{\delta_1, \delta_2} = p_{\{u_1, ..., u_t\}}(a_1, ..., a_n).$$

Note that $p_{\{u_1, ..., u_t\}}(1, ..., 1) = n^{\gamma(\delta_1, \delta_2)}$ so $T_f(1, ..., 1) = T_f(n)$, where $T_f(n)$ means the matrix $T_f(x)$ defined in Section 2 evaluated at $x = n$.

Let λ be a partition of $2f$. Observe that the right-hand side of (A1) is the eigenvalue of $T_f(n)$ indexed by λ. This observation along with some computational evidence led Richard Stanley to make the following conjecture.

Conjecture A1 (R. P. Stanley). The eigenvalues of $T_f(a_1, ..., a_n)$ are the zonal polynomials $Z_\lambda(a_1, ..., a_n)$ for λ a partition of $2f$. Moreover, the multiplicity of the eigenvalue $Z_\lambda(a_1, ..., a_n)$ is the dimension of the irreducible S_{2f} module S^λ indexed by λ.

Theorem 3.1 shows that this conjecture is true when $a_1 = a_2 = \cdots a_n = 1$. In this appendix we verify the conjecture in general. Before going on with the proof of this conjecture we need to define the zonal polynomials. Although these polynomials have several equivalent definitions, the most convenient for our purposes is the one given by James [8].

Let $u_{i,j}$ $(1 \leqslant i, j \leqslant n)$ be the entries of a real symmetric matrix U having eigenvalues $a_1, ..., a_n$. We think of these $u_{i,j}$ as being real-valued indeterminates satisfying $u_{i,j} = u_{j,i}$. The a_i are functions of the $u_{i,j}$. Let $P^{(f)}$ denote the vector space of homogeneous polynomials of degree f in the $u_{i,j}$. Define an action of $Gl(n, \mathbb{R})$ on $P^{(f)}$ as follows: given $A \in Gl(n, \mathbb{R})$ and $q(u) \in P^{(f)}$ let $A \cdot q(u) = q(v)$, where $v_{i,j}$ is the linear function of the $u_{k,l}$ found in the i, j entry of A^tUA. Thrall [13, p. 378] proved that as a $Gl(n, \mathbb{R})$ module, $P^{(f)}$ decomposes as

$$P^{(f)} = \sum W^{\lambda},$$

where the sum is over all even partitions λ of $2f$ with no more than n rows and where W^{λ} is the irreducible $Gl(n, \mathbb{R})$ module with high weight λ (see also Macdonald [12]). Define $C_{\lambda}(u_{i,j})$ to be the W^{λ} component of $(\mathrm{tr}\, U)^f$. As $\mathrm{tr}\, U = a_1 + \cdots + a_n$, we can write $C_{\lambda}(u_{i,j})$ as a symmetric function in the a_i. If $C_{\lambda}(a_1, ..., a_n)$ evaluated at $a_1 = \cdots = a_n = 1$ is 0, define $Z_{\lambda}(a_1, ..., a_n)$ to be 0. Otherwise, define $Z_{\lambda}(a_1, ..., a_n)$ to be the unique multiple of $C_{\lambda}(a_1, ..., a_n)$ satisfying (A1).

In order to identify the subspaces W^{λ} and the polynomials $C_{\lambda}(u_{i,j})$ we will use an alternative description of the $Gl(n, \mathbb{R})$ module $P^{(f)}$. Let H_f denote the hyperoctahedral group of all f by f signed permutation matrices, and let $e_1, ..., e_f$ denote the unit coordinate vectors in $\mathbb{R}^f$. For $i = 1, 2, ..., f$ let $s_{2i-1} = e_i$ and let $s_{2i} = -e_i$. Then H_f faithfully permutes the set $\{s_1, ..., s_{2f}\}$ which gives an embedding of H_f in S_{2f}. We henceforth think of H_f as a subgroup of S_{2f}.

The group S_{2f} acts on $T^{2f}\mathbb{R}^n$ by permutation of tensor positions. This restricts to an action of H_f on $T^{2f}\mathbb{R}^n$. The group $Gl(n, \mathbb{R})$ also acts on $T^{2f}(\mathbb{R}^n)$ via the diagonal action

$$A \cdot (v_1 \otimes \cdots \otimes v_{2f}) = (Av_1) \otimes \cdots \otimes (Av_{2f}).$$

It is easy to check that the actions of S_{2f} and $Gl(n, \mathbb{R})$ on $T^{2f}(\mathbb{R}^n)$ commute. In particular, $Gl(n, \mathbb{R})$ acts on the space of H_f-invariants in $T^{2f}(\mathbb{R}^n)$.

PROPOSITION A.2. *The space of H_f-invariants in $T^{2f}(\mathbb{R}^n)$ is isomorphic to $P^{(f)}$ as a $Gl(n, \mathbb{R})$ module.*

Proof. There is an obvious vector space isomorphism Φ between $P^{(f)}$ and $(T^{2f}\mathbb{R}^n)^{H_f}$ given by

$$\Phi(u_{b_1,b_2} u_{b_3,b_4} \cdots u_{b_{2f-1},b_{2f}}) = (1/H_f) \sum \sigma(e_{b_1} \otimes e_{b_2} \otimes \cdots \otimes e_{b_{2f}}).$$

We will show that the actions of $Gl(n, \mathbb{R})$ commute with Φ. Clearly it is enough to consider the case $f = 1$, where Φ is the map

$$\Phi(u_{r,s}) = \tfrac{1}{2}(e_r \otimes e_s + e_s \otimes e_r).$$

For any $M=(m_{i,j})$ in $Gl(n, \mathbb{R})$ we have

$$\begin{aligned}
\Phi(M\cdot u_{r,s}) &= \Phi(M^tUM)\\
&= \Phi\left(\sum m^t_{r,c}u_{c,d}m_{d,s}\right)\\
&= \Phi\left(\sum m_{c,r}m_{d,s}u_{c,d}\right)\\
&= \sum m_{c,r}m_{d,s}(\tfrac{1}{2}(e_c\otimes e_d+e_d\otimes e_c))\\
&= \tfrac{1}{2}(Me_r\otimes Me_s+Me_s\otimes Me_r)\\
&= M\cdot(\tfrac{1}{2}(e_r\otimes e_s+e_s\otimes e_r))\\
&= M\cdot(\Phi(u_{r,s})),
\end{aligned}$$

which completes the proof of Proposition A.2.

We will use several facts which are either well known or follow directly from well-known results. The first is half of the famous Double Centralizer Theorem.

PROPOSITION A.3. *For* $z=(z_1, ..., z_n)$ *a weight of* $Gl(n, \mathbb{R})$ *and* V *a* $Gl(n, \mathbb{R})$ *module, let* V_z *denote the z-weight space of* V. *Then as an* S_{2f} *module,* $(T^{2f}\mathbb{R}^n)_z$ *decomposes as*

$$(T^{2f}\mathbb{R}^n)_z=\bigoplus\,(\dim(W^\alpha_z))\,S^\alpha,$$

where $(\dim(W^\alpha_z))\,S^\alpha$ denotes the direct sum of $\dim(W^\alpha_z)$ copies of S^α. In particular,

$$(T^{2f}\mathbb{R}^n)=\bigoplus\,\dim(W^\alpha)\,S^\alpha.$$

The next result concerns the action of H_f on the irreducible S_{2f} modules. Recall that we consider H_f to be a subgroup of S_{2f}. Thus if α is a partition of $2f$ then H_f acts by restriction on S^α.

PROPOSITION A.4. *Let* α *be a partition of* $2f$ *and let* $(S^\alpha)^{H_f}$ *denote the space of* H_f*-invariants in* S^α. *Then*

$$\dim((S^\alpha)^{H_f})=\begin{cases}1 & \text{if } \alpha \text{ is even}\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Frobenius reciprocity (see [5]) we have

$$\dim((S^\alpha)^{H_f}) = \langle \varepsilon_{H_f}, \chi^\alpha \downarrow_{H_f} \rangle = \langle \varepsilon_{H_f} \uparrow^{S_{2f}}, \chi^\alpha \rangle = \begin{cases} 1 & \text{if } \alpha \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION A.5. Define V^α to be the space of H_f-invariants in the χ^α isotypic component of $T^{2f}\mathbb{R}^n$.

Note that $V^\alpha = 0$ unless α is even and $l(\alpha) \leqslant n$. If α is even and $l(\alpha) \leqslant n$ then it follows from Propositions A.4 and A.5 that

$$\dim(V_z^\alpha) = \dim(W_z^\alpha)$$

for all weights z. So by the weight theory for $Gl(n, \mathbb{R})$, we conclude that $V^\alpha = W^\alpha$.

The matrix $T_f(a_1, ..., a_n)$ commutes with the action of S_{2f} on the set of 1-factors F_f. Recall that this permutation character decomposes as the direct sum of S^α over even partitions α of $2f$. Since the character is multiplicity free, $T_f(a_1, ..., a_n)$ restricted to the subspace of F_f isomorphic to S^α is a scalar, which we denote $q_\alpha(a_1, ..., a_n)$.

In the proof of Theorem 3.1 we gave a method for computing the scalar $q_\alpha(x)$ that $T_f(x)$ restricts to on the same subspace. This method works just as well to compute $q_\alpha(a_1, ..., a_n)$. Unfortunately, the simple recursion we found holding amongst the $q_\lambda(x)$'s fails to hold amongst the $q_\lambda(a_1, ..., a_n)$, making it impossible to carry out the same induction we used in the proof of Theorem 3.1. However, using our method to compute $q_\alpha(a_1, ..., a_n)$ we are able to extract one very important piece of information.

Let $\alpha = 2(r_1, ..., r_s)$ and let t be a standard Young tableau with the numbers $1, 2, ..., 2r_1$ in row 1, $2r_1 + 1, ..., 2r_1 + 2r_2$ in row 2, and so on. Let R_t and C_t be the row and column stabilizers of t and let δ_0 be the 1-factor on $2f$ points having an edge from $2l - 1$ to $2l$ for $l = 1, 2, ..., f$. According to our method of computation, $q_\alpha(a_1, ..., a_n)$ is a multiple of

$$\left\{\sum \sum \varepsilon(\sigma)(T_f(a_1, ..., a_n)_{\delta_0, \sigma\tau\delta_0})\right\}.$$

For our purposes, the exact multiple is unimportant.

As in Section 3 we let e_t denote $\sum \varepsilon(\sigma)\, \sigma\tau$. Also we let Γ be the projection operator onto the H_f-invariants,

$$\Gamma = (1/H_f) \sum \sigma,$$

and let v be the following vector in $T^{2f}\mathbb{R}^n$:

$$v=\left(\sum e_{i_1}\otimes e_{i_1}\right)\otimes\left(\sum e_{i_2}\otimes e_{i_2}\right)\otimes\cdots\otimes\left(\sum e_{i_f}\otimes e_{i_f}\right).$$

We will compute $\Gamma e_t v$, which is an element of $(T^{2f}\mathbb{R}^n)^{H_f}$.

Fix $\sigma\in C_t$ and $\tau\in R_t$ and suppose that $\delta_0\cup\sigma\tau\delta_0$ has cycles of lengths $2\mu_1, ..., 2\mu_t$. Let $\mathfrak{C}=(c_1, c_2, ..., c_{2\mu})$ be one of these cycles written so that the edges of δ_0 join to c_2, c_3 to c_4, ..., and $c_{2\mu-1}$ to $c_{2\mu}$. Thus

$$\begin{aligned} c_2&=c_1+1\\ c_4&=c_3+1\\ &\vdots\\ c_{2\mu}&=c_{2\mu-1}+1. \end{aligned}$$

Note that when v is written out as a sum of pure tensors, all of the terms have the property that their $2a-1$ and $2a$ tensor positions are identical. So in tensor positions $(c_1, c_2, ..., c_{2\mu})$, $\sigma\tau v$ looks like

$$\sum\cdots e_{j_1}\otimes e_{j_2}\cdots e_{j_2}\otimes e_{j_3}\cdots e_{j_3}\otimes e_{j_4}\cdots e_{j_4}\otimes e_{j_1}\cdots,$$

where the sum is over all sequences with $1\leqslant j_s\leqslant n$.

Applying Γ, we see that $\Gamma(\sigma\tau v)$ is the product, over cycles $\mathfrak{C}$ in $\delta_0\cup(\sigma\tau\delta_0)$ of length 2μ, of the quantity Q_μ:

$$Q_\mu=\sum u_{j_1,j_2}u_{j_2,j_3}\cdots u_{j_\mu,j_1}.$$

On the other hand, for fixed j_1 the j_1, j_1, entry in U^μ is

$$\sum u_{j_1,j_2}u_{j_2,j_3}\cdots u_{j_\mu j_1}.$$

Thus $Q_\mu=\operatorname{tr}(U^\mu)=a_1^\mu+\cdots+a_n^\mu$.

In summary, if $\delta_0\cup(\sigma\tau\delta_0)$ has cycles of lengths $2\mu_1, 2\mu_2, ..., 2\mu_t$ then $\Gamma(\sigma\tau v)=p_\mu(a_1, ..., a_n)$. Hence $\Gamma(e_t v)$ is a multiple of $q_\alpha(a_1, ..., a_n)$.

Also recall that Γ and e_t commute and note that $\Gamma v=(\operatorname{tr}(U))^f$. So, $\Gamma(e_t v)=e_t(\Gamma v)$ is a multiple of $C_\alpha(u_{i,j})$, which is a multiple of $Z_\alpha(a_1, ..., a_n)$. So $q_\alpha(a_1, ..., a_n)=y_\alpha Z_\alpha(a_1, ..., a_n)$ for some $y_\alpha\in\mathbb{R}$. But by Theorem 3.1 and (A1) we have

$$q_\alpha(1, 1, ..., 1)=Z_\alpha(1, 1, ..., 1)$$

so $y_\alpha=1$, which proves Stanley's conjecture.

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