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Representations of Algebraic Groups of Type F_4 in Characteristic 2

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Making use of results of Curtis [4] on the representations of Lie algebras of classical type, Steinberg [12] has shown that all irreducible representations of a semisimple algebraic group in characteristic p can be obtained in a simple way from a finite number of basic representations. To obtain these basic representations, one can make use of reduction mod. p of representations in characteristic 0, which is possible after choice of a Chevalley basis in the representation space [3]. In this paper we apply this to the simple algebraic group of type F_4 in characteristic 2. From the representations of F_4 we determine the 2-modular (Brauer) characters of the Ree group of type F_4 parametrized by the field of 2 elements.

In Section 1 we recall results of Steinberg and Chevalley, in section 2 we apply these to obtain the representations of a group of type F_4 in characteristic 2. In section 3 we compute the 2-modular characters and the Cartan matrix of the Ree group of type F_4 over the field of 2 elements.

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1. Let G be a connected semisimple linear algebraic group over an algebraically closed field k; we shall identify G with the group of its k-rational points. We refer to [2] for results on algebraic groups and their representations which will be used throughout in this paper. Let T be a maximal torus of G, $X(T)$ the character group of T, $V = X(T) \otimes_{\mathbf{Z}} \mathbf{R}$ the real vector space generated by $X(T)$. We choose an ordering on the set Σ of roots relative to T, and denote the simple roots by $\alpha_1, \ldots, \alpha_l$. Let W be the Weyl group of G and let $(\cdot\,,\cdot)$ denote an inner product on V invariant under W.

An irreducible rational projective representation of G is uniquely determined by its highest weight. The set of possible highest weights consists of all linear combinations with non-negative integral coefficients of the l fundamental highest weights $d_1 ,..., d_l$.

Now let σ be a rational endomorphism of G such that G_σ is finite. We may assume the maximal torus T and the ordering of the roots to be chosen so that σ leaves T invariant and that the transpose of the restriction of σ to T induces a linear transformation σ^* on V which permutes positive integral multiples of the positive roots (see [13] 10.10). Then there exist a permutation ρ of the roots and for each root α a power $q(\alpha)$ of $p = \text{char}(k)$ ($\neq 0$ in this situation) such that ρ permutes the positive roots and $\sigma^* \rho\alpha = q(\alpha)\alpha$ for every root α (see [13], §§11 and 13 for these and the following facts). Assume G, moreover, to be simply connected; then every projective representation is induced by a linear one. Let $\mathscr{R}$ denote the set of irreducible rational linear representations of G for which the highest weight $g = \sum_{i=1}^{l} n_i d_i$ satisfies $0 \leqslant n_i < q(\alpha_i)$ for $1 \leqslant i \leqslant l$. Then the collection $\prod_{i=0}^{\infty} R_i \circ \sigma^i$ (tensor product, $R_i \in \mathscr{R}$, most R_i trivial) is a complete set of irreducible rational linear representations of G, each counted once. The restrictions to G_σ of the representations $\in \mathscr{R}$ form a complete set of irreducible representations of the finite group G_σ (besides [13], see [12], §§1 and 12).

There is an affine group scheme G_0, of finite type and smooth over $\mathbf{Z}$, such that $G = G_0 \times_{\mathbf{Z}} k$. If $\text{char}(k) = 0$, any irreducible representation of G comes from a representation of G_0 over $\mathbf{Z}$ (see [3, 9]). If $\text{char}(k) = p$, any irreducible representation π of G can be obtained as follows. Let g be the highest weight of π, π_0 an irreducible representation of G_0 over $\mathbf{Z}$ with highest weight g. The tensor product over $\mathbf{Z}$ of π_0 with k is a representation ϑ of G such that π occurs as an irreducible constituent of ϑ.

Assume $k = \mathbf{C}$, and let π be an irreducible representation of G. The representation π_0 of G_0 which induces π is obtained in the following way.

In the Lie algebra $L(G)$ of G a system of root vectors X_α can be chosen which satisfy the following equations.

(1.1)

(a) $[X_\alpha , X_{-\alpha}] = H_\alpha$, which is an integral linear combination of the $H_i = H_{\alpha_i}$ ($i = 1,..., l$).

(b) $H_1 ,..., H_l$ form a basis for the Cartan subalgebra $L(T)$ of $L(G)$.

(c) $[H_\alpha , X_\beta] = \beta(H_\alpha)\, X_\beta$, $\beta(H_\alpha) = 2((\alpha, \beta)/(\alpha, \alpha))$.

(d) Whenever α, β and $\alpha + \beta$ are roots,

$[X_\alpha , X_\beta] = \pm(p(\alpha, \beta) + 1)\, X_{\alpha+\beta}$, $p(\alpha, \beta)$ denoting the largest integer $i \geqslant 0$ such that $\beta - i\alpha$ is a root. $[X_\alpha , X_\beta] = 0$ if $\alpha + \beta$ is not a root.

$H_1 ,..., H_l$, $X_\alpha (\alpha \in \Sigma)$ form a basis for $L(G)$, called a *Chevalley basis* for $L(G)$.

Let M be the representation space for π. $d\pi$ is a representation of $L(G)$ in M. A basis $(m_1 ,..., m_n)$ of M is called a *Chevalley basis* for π if every m_i

is a weight vector, and if for any root α and any integer $i > 0$, $(i!)^{-1}\, d\pi(X_\alpha)^i$ maps

$$M_0 = \mathbf{Z}m_1 \oplus \cdots \oplus \mathbf{Z}m_n$$

into itself. M_0 is a representation module for π_0 (see [3, 9]).

Now assume k has characteristic p and π is an irreducible representation of $G = G_k$ with highest weight $g = \sum_{i=1}^{l} n_i d_i$ such that $0 \leqslant n_i < p$ for $i = 1,..., l$. Then π is an irreducible constituent of the representation θ obtained by reduction mod p of a representation π_0 of G_0 with highest weight g. Let the module M_0 for π_0 and the Chevalley basis $m_1 ,..., m_n$ be as above. $M = M_0 \otimes k$ is the representation module of θ. Let $M_1 \supset M_2$ be G-submodules of M such that θ induces π in $M_1/M_2 = N$. Let d be an extreme weight of π (hence also of θ and of π_0). The corresponding weight spaces N_d in N and M_d in M (and M_1) are 1-dimensional, and $N_d = (M_d + M_2)/M_2$. Thus N_d is spanned by some $v_d = (m_i \bmod p) + M_2$. We shall say that such a vector v_d is *obtained from the Chevalley basis* $m_1 ,..., m_n$.

2. Now assume char$(k) = 2$. Let G be the simple algebraic group of type F_4 over k and let σ be the endomorphism of G such that, in the above notation, ρ interchanges long and short roots and with $q(\alpha_1) = q(\alpha_2) = 1$, $q(\alpha_3) = q(\alpha_4) = 2$, where α_1 and α_2 are the long simple roots, α_3 and α_4 the short ones. Then G_σ is the Ree group of type F_4 parametrized by $\mathbf{F}_2$, the field of two elements (see [8] and [13], 11.6).

Let $n_1n_2n_3n_4$ denote the weight $\sum_{i=1}^{4} n_i d_i$ and also the representation having this as highest weight. $\mathscr{R}$ consists of the representations 0000, 0001, 0010 and 0011.

We first determine a table of representations in characteristic 0, that is, for G_0 or, which amounts to the same, for the complex group $G_{\mathbf{C}} = G_0 \times_{\mathbf{Z}} \mathbf{C}$. The multiplicities of the weights in an irreducible representation can be

TABLE I

Multiplicities in Characteristic 0 for the Group of Type F_4

g \ a	0000	0001	1000	0010	0002	1001	0100	0011	$(g, g+2\rho)$	dim
0000	1								0	1
0001	2	1							12	26
1000	4	1	1						18	52
0010	9	5	2	1					24	273
0002	12	5	3	1	1				26	324
1001	21	14	6	4	1	1			32	1053
0100	26	13	10	4	3	1	1		36	1274
0011	64	40	24	14	8	4	2	1	39	4096

computed by means of Freudenthal's formula (see [6], [7], Ch. VIII, or [10], 3.10). In table 1 we have listed the irreducible representations with highest weight g satisfying $(g, g + 2\rho) \leqslant 39 = (g_0, g_0 + 2\rho)$, where $g_0 = 0011$ and where $\rho = \frac{1}{2} \sum_{\alpha \mathrm{root} > 0} \alpha = \sum_{i=1}^{4} d_i$. The author's computations for table 1 have been checked by computations on the $X8$ of the Electronic Computing Department of the University of Utrecht by Mr. M.I. Krusemeyer, to whom the author wishes to express his gratitude.

Before we proceed to determine the representations of G in characteristic 2, we give some lemmas. We use the following notation. If V is the space of a representation of a semisimple algebraic group G (in any characteristic) and if d is a weight of this representation, then V_d is the space of weight vectors of weight d. In $L(G_C)$ we assume some Chevalley basis is chosen. Then H_α, X_α have the same meaning as in (1.1), but they also denote the corresponding vectors $H_\alpha \bmod p$, $X_\alpha \bmod p$, resp., in $L(G)$ if the characteristic of the ground-field k is $p > 0$.

(2.1) Lemma. *If d is a weight of a representation of G in V, α a root of G and $d + \alpha$ not a weight, then $X_\alpha V_d = 0$.*

(2.2) Lemma. *If d is an extreme weight of an irreducible representation of G in V and α a root of G such that $2((d, \alpha)/(\alpha, \alpha)) \geqslant 0$, then $X_\alpha V_d = 0$.*

Proof. d is conjugate under the Weyl group W to the highest weight g. $2((g, \alpha)/(\alpha, \alpha)) \geqslant 0$ implies that $\alpha \geqslant 0$, hence $X_\alpha V_g = 0$ as is well known.

(2.3) Lemma. *Let d be an extreme weight of an irreducible representation of G in V with highest weight $g = \sum_{i=1}^{l} n_i d_i$, and let α be a root of G such that $2((d, \alpha)/(\alpha, \alpha)) = -1$. Assume in case the ground-field K has characteristic p that $0 \leqslant n_i < p$ for $i = 1, \ldots, l$. Then $d + \alpha$ is an extreme weight and $X_\alpha v_d = \pm v_{d+\alpha}$, where $v_d \in V_d$ and $v_{d+\alpha} \in V_{d+\alpha}$ belong to a Chevalley basis if k has characteristic 0, or are obtained from a Chevalley basis if k has characteristic p.*

Proof. Ler $r_\alpha \in W$ be the reflection in the hyperplane orthogonal to α. Then $r_\alpha d = d + \alpha$, hence $d + \alpha$ is extreme and $\dim V_d = \dim V_{d+\alpha} = 1$. $X_\alpha v_d = m v_{d+\alpha}$ for some $m \in \mathbf{Z}$. $X_{-\alpha} v_{d+\alpha} = n v_d$ for some $n \in \mathbf{Z}$. Hence

$$\begin{aligned} mnv_d &= X_{-\alpha} X_\alpha v_d \\ &= -H_\alpha v_d, \qquad \text{since} \quad X_{-\alpha} v_d = 0 \quad \text{by Lemma (2.2)}, \\ &= -2\frac{(d, \alpha)}{(\alpha, \alpha)} v_d \\ &= v_d . \end{aligned}$$

Hence $m = \pm 1$.

(2.4) LEMMA. *Let L be the complex Lie algebra of type F_4. Assume rootvectors X_α have been chosen in L satisfying conditions* (1.1). *Let $\Gamma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4\}$, where α_1, α_2 are the long simple roots of L and α_3, α_4 the short ones. Let Θ be a set of positive roots containing Γ and having the property: For $\alpha, \beta \in \Theta$, $[X_\alpha, X_\beta] = \pm X_\gamma$ for some $\gamma \in \Theta$. Then Θ consists of all positive roots.*

Proof. From (1,1)(d) it follows: if α and β are positive roots such that $\alpha + \beta$ is a root but $\alpha - \beta$ is not, then $[X_\alpha, X_\beta] = \pm X_{\alpha+\beta}$. Apply this to the roots in Θ, starting from the roots in Γ. Then Θ is seen to contain the following roots

$$abcd = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 .$$

1000, 0100, 0010, 0001, 0120, 0122.

1100, 0110, 0011, 0121, 1120, 1122.

1110, 0111, 1220, 1222, 1121, 1221.

1111, 1231.

These are all positive roots of L (see e.g. [14]).

(2.5) LEMMA. *Let d be a weight of an irreducible representation with highest weight g of F_4 in a space V. Assume $d \neq g$. Then V_d is generated by the vectors $X_{-\alpha}v_{d-\alpha}$, where α runs over $\Gamma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4\}$ and where for each α such that $d + \alpha$ is a weight, $v_{d+\alpha}$ runs over a basis of $V_{d+\alpha}$. If $v \in V_d$ is such that $X_\alpha v = 0$ for all $\alpha \in \Gamma$, then $v = 0$. In case the groundfield has characteristic p we assume, again, that $g = \sum_{i=1}^4 n_i d_i$ with $0 \leqslant n_i < p$ for $1 \leqslant i \leqslant 4$.*

Proof. It is known (see e.g. [12]) that V_d is generated by the vectors $X_{-\gamma_1}X_{-\gamma_2} \cdots X_{-\gamma_t}v_g$ with $d + \gamma_1 + \cdots + \gamma_t = g$, γ_i positive roots, v_g a nonzero weightvector for g. It follows from the previous lemma that $X_{-\gamma_1}$ is a linear combination of vectors of the form $X_{-\beta_1}X_{-\beta_2} \cdots X_{-\beta_s}$ with $\beta_1, \ldots, \beta_s \in \Gamma$. This proves the first statement. If $v \in V_d$ has the property $X_\alpha v = 0$ for all $\alpha \in \Gamma$, then, again by the previous lemma, $X_\gamma v = 0$ for all positive roots γ. Since the scalar multiples of v_g are the only vectors in an irreducible representation having this property (see [4], or [12], 2.7 and theorem 5.1), it follows that $v = 0$.

Now we shall determine a table of multiplicities for the representations in characteristic 2 of a simple algebraic group G of type F_4. It suffices to consider the representations 0010, 0001 and 0011, since the other ones can be derived from these. The simple roots and fundamental highest weights can be given in the following form ([2], pp. 19-10 and 11).

$\alpha_1 = \omega_1 - \omega_2\,,\quad \alpha_2 = \omega_2 - \omega_3\,,\quad \alpha_3 = \omega_3\,,\quad \alpha_4 = \frac{1}{2}(-\omega_1 - \omega_2 - \omega_3 + \omega_4).$

$d_1 = \omega_1 + \omega_4 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\,.$

$d_2 = \omega_1 + \omega_2 + 2\omega_4 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4\,.$

$d_3 = \frac{1}{2}(\omega_1 + \omega_2 + \omega_3 + 3\omega_4) = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4\,.$

$d_4 = \omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4\,.$

Invariant metric: $(\omega_i\,, \omega_j) = \delta_{ij}\,.$

(i) The representation with highest weight $g = 0011$. In table 1 one sees that for any weight $a \neq g$ in this representation,

$$(a, a + 2\rho) \not\equiv (g, g + 2\rho) \bmod 2,$$

hence the multiplicities in characteristic 2 are the same as in characteristic 0 by Theorem 4.2 of [10]. This result also follows from [11] since 0011 induces on G_σ the representation whose dimension equals the order of a Sylow 2-subgroup of G_σ.

(ii) The representation with highest weight $g = 0001$. Let 0 have multiplicity m in this representation. From the table of multiplicities in characteristic 0 it follows that $m \leqslant 2$. Choose a highest weight-vector v_g. Write

$$g = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \alpha + \beta$$

where

$$\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 \quad \text{and} \quad \beta = \alpha_4 \quad \text{are roots.}$$

Set

$$\begin{aligned} v_1 &= X_{-\alpha}X_{-\beta}v_g\,, \\ v_2 &= X_{-\beta}X_{-\alpha}v_g\,. \end{aligned}$$

v_1 and v_2 are vectors of weight 0. Now

$$\begin{aligned} X_\alpha X_\beta v_1 &= X_\alpha X_{-\alpha} X_\beta X_{-\beta} v_g \\ &= H_\alpha H_\beta v_g \quad \text{since} \quad X_\alpha v_g = X_\beta v_g = 0 \qquad \text{and} \\ &\qquad\qquad [X_\gamma\,, X_{-\gamma}] = H_\gamma \quad \text{for any root } \gamma, \\ &= \left(2\frac{(g, \alpha)}{(\alpha, \alpha)}\right)\left(2\frac{(g, \beta)}{(\beta, \beta)}\right) v_g \\ &= v_g\,. \end{aligned}$$

Similar computations yield:

$$\begin{aligned} X_\alpha X_\beta v_2 &= 0 \\ X_\beta X_\alpha v_1 &= 0 \\ X_\beta X_\alpha v_2 &= v_g\,. \end{aligned}$$

Hence v_1 and v_2 are linearly independent, which implies that $m = 2$.

(iii) The representation with highest weight $g = 0010$. From the multiplicities in characteristic 0 it follows that the dominant weights are 1000, 0001 and 0000 with multiplicities $m_1 \leqslant 2$, $m_2 \leqslant 5$ and $m_3 \leqslant 9$, respectively. The conjugates of g under the Weyl group are easily seen to be

$$\tfrac{1}{2}(\pm\omega_i \pm \omega_j \pm \omega_k \pm 3\omega_l), \qquad \pm\omega_i \pm \omega_j \pm \omega_k ,$$

with i, j, k and l distinct, all combinations of signs being permitted (see [2], p. 19-10). The conjugates of d_1 are all possible

$$\pm\omega_i \pm \omega_j , \qquad i \neq j.$$

(a) Computation of m_1. $d_1 + \alpha_3 + \alpha_4 = \frac{1}{2}(\omega_1 - \omega_2 + \omega_3 + 3\omega_4)$ is conjugate to g under W, i.e. extreme. Consider the following vectors of weight d_1.

$$v_1 = X_{-\alpha_3}X_{-\alpha_4}v_{d_1+\alpha_3+\alpha_4}$$

$$v_2 = X_{-\alpha_4}X_{-\alpha_3}v_{d_1+\alpha_3+\alpha_4} ,$$

where for any extreme weight d, v_d denotes the element in V_d obtained from a Chevalley basis which we assume chosen once and for all in part (iii).

By similar computations as in (ii), using Lemma (2.2), we find

$$X_{\alpha_3}X_{\alpha_4}v_1 = v_{d_1+\alpha_3+\alpha_4} , \qquad X_{\alpha_4}X_{\alpha_3}v_1 = 0,$$

$$X_{\alpha_3}X_{\alpha_4}v_2 = 0, \qquad X_{\alpha_4}X_{\alpha_3}v_2 = v_{d_1+\alpha_3+\alpha_4} .$$

So v_1 and v_2 are independent, hence $m_1 = 2$.

(b) Computation of m_2. $d_4 + \alpha_2 + 2\alpha_3 + 2\alpha_4 = -\omega_1 + 2\omega_4$ is not a weight, since it is neither conjugate to $g = d_3$ nor to d_1 under the Weyl group. $d_4 + \alpha_1$, $d_4 + \alpha_2$, $d_4 + \alpha_4$ and $d_4 + \alpha_2 + 2\alpha_3$ are extreme weights, and $d_4 + \alpha_3 = \omega_3 + \omega_4 = r_2 r_1 d_1$, where $r_i \in W$ denotes the reflection in the hyperplane orthogonal to α_i. Let w_i be the (unique) realization of r_i in G_{σ^2}, the Chevalley group over $\mathbf{F}_2$. From Lemma (2.5) it follows that V_d is generated by the vectors

$$x_1 = X_{-\alpha_1}v_{d_4+\alpha_1} , \qquad x_2 = X_{-\alpha_2}v_{d_4+\alpha_2} , \qquad x_3 = X_{-\alpha_4}v_{d_4+\alpha_4} ,$$

$$x_4 = X_{-\alpha_3}w_2w_1v_1 , \qquad x_5 = X_{-\alpha_3}w_2w_1v_2 , \qquad x_6 = X_{-\alpha_2-2\alpha_3}v_{d_4+\alpha_2+2\alpha_3} ,$$

with v_1 and v_2 as under (a).

Since $d_4 + \alpha_2 + 2\alpha_3 + 2\alpha_4$ is not a weight,

$$X_{\alpha_2+2\alpha_3+2\alpha_4}x_i = 0, \qquad i = 1,\dots, 6.$$

The action of the root vectors X_α, $\alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_4$ or $\alpha_2 + 2\alpha_3$, on x_i is given in the following table.

	X_{α_1}	X_{α_2}	X_{α_3}	X_{α_4}	$X_{\alpha_2+2\alpha_3}$
$x_1 = X_{-\alpha_1}v_{d_4+\alpha_1}$	0	$v_{d_4-\alpha_2}$	0	0	$v_{d_4+\alpha_2+2\alpha_3}$
$x_2 = X_{-\alpha_2}v_{d_4+\alpha_2}$	$v_{d_4+\alpha_1}$	0	$w_2w_1v_1$	0	0
$x_3 = X_{-\alpha_4}v_{d_4+\alpha_4}$	0	0	$w_2w_1v_2$	$v_{d_4+\alpha_4}$	$v_{d_4+\alpha_2+2\alpha_3}$
$x_4 = X_{-\alpha_3}w_2w_1v_1$	0	0	0	$v_{d_4+\alpha_4}$	0
$x_5 = X_{-\alpha_3}w_2w_1v_2$	0	$v_{d_4+\alpha_2}$	0	0	$v_{d_4+\alpha_2+2\alpha_3}$
$x_6 = X_{-\alpha_2-2\alpha_3}v_{d_4+\alpha_2+2\alpha_3}$	$v_{d_4+\alpha_1}$	0	$w_2w_1v_1$	$v_{d_4+\alpha_4}$	0

The results of this table will be proved at the end of the present section (iii)(b). We first proceed to determine the value of m_2. Let

$$\sum_{i=1}^{6} c_i x_i = 0$$

be any linear relation. From the action of X_α, $\alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_2 + 2\alpha_3$, on this equation we deduce

$$c_2 + c_6 = 0, \quad c_1 + c_5 = 0, \quad c_2 + c_6 = 0, \quad c_3 = 0,$$
$$c_3 + c_4 + c_6 = 0, \quad c_1 + c_3 + c_5 = 0,$$

respectively. Hence

$$(*) \qquad c_3 = 0, \quad c_1 = c_5, \quad c_2 = c_4 = c_6.$$

Consider, conversely, any $v = \sum_{i=1}^{6} c_i x_i$ with $c_1, \ldots, c_6$ satisfying (*). Then $X_\alpha v = 0$ for $\alpha \in \Gamma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4\}$, hence $v = 0$ by Lemma (2.5). It follows that the subspace V_{d_4}, which is generated by $x_1, \ldots, x_6$, has dimension 4, i.e., $m_2 = 4$.

Now we shall give proofs for the above table of $X_\alpha x_i$, $\alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2 + 2\alpha_3$.

$$X_{\alpha_1}x_1 = X_{\alpha_1}X_{-\alpha_1}v_{d_4+\alpha_1} = H_{\alpha_1}v_{d_4+\alpha_1}, \qquad \text{by Lemma (2.2)},$$
$$= 2\frac{(d_4 + \alpha_1, \alpha_1)}{(\alpha_1, \alpha_1)} v_{d_4+\alpha_1} = 0.$$

Similarly for $X_{\alpha_2}x_2$, $X_{\alpha_4}x_3$, $X_{\alpha_2+2\alpha_3}x_6$.

$$X_{\alpha_1}x_2 = X_{\alpha_1}X_{-\alpha_2}v_{d_4+\alpha_2}$$
$$= X_{-\alpha_2}X_{\alpha_1}v_{d_4+\alpha_2}, \qquad \text{since } \alpha_1 - \alpha_2 \text{ is not a root},$$
$$= X_{-\alpha_2}v_{d_4+\alpha_1+\alpha_2}, \qquad \text{by Lemma (2.3)},$$
$$= v_{d_4+\alpha_1}, \qquad \text{by Lemma (2.3)}.$$

In a similar way one obtains by application of Lemma (2.3), $X_{\alpha_1}x_6$, $X_{\alpha_2}x_1$, $X_{\alpha_4}x_6$, $X_{\alpha_2+2\alpha_3}x_1$, $X_{\alpha_2+2\alpha_3}x_3$.

$$X_{\alpha_1}x_3 = X_{\alpha_1}X_{-\alpha_4}v_{d_4+\alpha_4} = X_{-\alpha_4}X_{\alpha_1}v_{d_4+\alpha_4} = 0$$

by Lemma (2.2). In the same way we get

$$X_{\alpha_2}x_3\,, \qquad X_{\alpha_2}x_6\,, \qquad X_{\alpha_3}x_1\,, \qquad X_{\alpha_4}x_1\,, \qquad X_{\alpha_4}x_2\,, \qquad X_{\alpha_2+2\alpha_3}x_2\,.$$

$X_{\alpha_1}x_4 = X_{\alpha_1}x_5 = 0$ since $d_4 + \alpha_3 + \alpha_1 = \omega_1 - \omega_2 + \omega_3 + \omega_4$ is not a weight.

$$\begin{aligned}
X_{\alpha_2}x_4 &= X_{\alpha_2}X_{-\alpha_3}w_2w_1v_1 \\
&= w_2w_1X_{r_1r_2\alpha_2}X_{-r_1r_2\alpha_3}v_1 \\
&= w_2w_1X_{-\omega_1+\omega_3}X_{-\omega_1}X_{-\alpha_3}X_{-\alpha_1}v_{d_1+\alpha_3+\alpha_1} \\
&= w_2w_1X_{-\omega_1}X_{-\omega_1+\omega_3}X_{-\alpha_3}v_{d_1+\alpha_3}\,, \qquad \text{by Lemma (2.3),} \\
&= w_2w_1X_{-\omega_1}[X_{-\omega_1+\omega_3}\,, X_{-\alpha_3}]\, v_{d_1+\alpha_3} + w_2w_1X_{-\omega_1}X_{-\alpha_3}X_{-\omega_1+\omega_3}v_{d_1+\alpha_3} \\
&= w_2w_1X^2_{-\omega_1}v_{d_1+\alpha_3} + 0, \qquad \text{by (1.1)(d) and Lemma (2.2),} \\
&= 2nw_2w_1v_{-\omega_1+\omega_3+\omega_4}\,, \qquad \text{since } \tfrac{1}{2}X^2_{-\omega_1} \text{ acts with integral} \\
&\qquad\qquad\qquad\qquad\qquad\quad \text{coefficients on a Chevalley basis,} \\
&= 0.
\end{aligned}$$

Similar arguments take care of $X_{\alpha_2}x_5$, $X_{\alpha_4}x_4$, $X_{\alpha_4}x_5$, $X_{\alpha_2+2\alpha_3}x_4$ and $X_{\alpha_2+2\alpha_3}x_5$.

$X_{\alpha_3}x_2 = X_{\alpha_3}X_{-\alpha_2}v_{d_4+\alpha_2}$ has weight $\omega_3 + \omega_4$, which is the weight of $w_2w_1v_i$, $i = 1, 2$, so consider

$$\begin{aligned}
w_1w_2X_{\alpha_3}X_{-\alpha_2}v_{d_4+\alpha_2} &= X_{r_1r_2\alpha_3}X_{-r_1r_2\alpha_2}v_{r_1r_2(d_4+\alpha_2)} \\
&= X_{\omega_1}X_{\omega_1-\omega_3}v_{-\omega_1+\omega_3+\omega_4} \\
&= X_{\omega_1-\omega_3}X_{\omega_1}v_{-\omega_1+\omega_3+\omega_4} \\
&= X_{\omega_1-\omega_3}X_{\omega_1}(\tfrac{1}{2}X^2_{-\omega_1})\, v_{\omega_1+\omega_3+\omega_4} \qquad \text{(see below)} \\
&= \tfrac{1}{2}X_{\omega_1-\omega_3}X_{-\omega_1}X_{\omega_1}X_{-\omega_1}v_{\omega_1+\omega_3+\omega_4} \\
&\quad + \tfrac{1}{2}X_{\omega_1-\omega_3}H_{\omega_1}X_{-\omega_1}v_{\omega_1+\omega_3+\omega_4} \\
&= \tfrac{1}{2}X_{\omega_1-\omega_3}X_{-\omega_1}X_{\omega_1}X_{-\omega_1}v_{\omega_1+\omega_3+\omega_4}\,, \qquad \text{since}
\end{aligned}$$

$$2\,\frac{(\omega_3 + \omega_4\,, \omega_1)}{(\omega_1\,, \omega_1)} = 0,$$

$$= \tfrac{1}{2} X_{-\omega_3} X_{\omega_1} X_{-\omega_1} v_{\omega_1+\omega_3+\omega_4}, \qquad \text{by (1.1)(d) and Lemma (2.2),}$$

$$= \tfrac{1}{2} X_{-\omega_3} H_{\omega_1} v_{\omega_1+\omega_3+\omega_4} + \tfrac{1}{2} X_{-\omega_3} X_{-\omega_1} X_{\omega_1} v_{\omega_1+\omega_3+\omega_4}$$

$$= X_{-\omega_3} v_{\omega_1+\omega_3+\omega_4} + 0, \qquad \text{by Lemma (2.2),}$$

$$= X_{-\alpha_3} X_{-\alpha_4} v_{d_1+\alpha_3+\alpha_4}, \qquad \text{by Lemma (2.3),}$$

$$= v_1 .$$

We still have to show that

$$(\tfrac{1}{2} X^2_{-\omega_1})\, v_{\omega_1+\omega_3+\omega_4} = v_{-\omega_1+\omega_3+\omega_4} .$$

We know that, in the $\mathbf{Z}$-module for G_0,

$$X^2_{-\omega_1} v_{\omega_1+\omega_3+\omega_4} = n v_{-\omega_1+\omega_3+\omega_4} .$$

By action of the Weyl group on this equation we get

$$X^2_{\omega_1} v_{-\omega_1+\omega_3+\omega_4} = \pm n v_{\omega_1+\omega_3+\omega_4} .$$

Hence

$$\begin{aligned} 4 v_{\omega_1+\omega_3+\omega_4} &= X^2_{\omega_1} X^2_{-\omega_1} v_{\omega_1+\omega_3+\omega_4} \\ &= \pm n^2 v_{\omega_1+\omega_3+\omega_4}, \end{aligned}$$

(see [12], p. 41, line 3) from which the result follows. This completes the proof for the case $X_{\alpha_3} x_2$. A similar line of reasoning works for $X_{\alpha_3} x_3$ and $X_{\alpha_3} x_6$.

Finally, consider $X_{\alpha_3} x_4$ and $X_{\alpha_3} x_5$. For these cases, we argue as follows.

$$\begin{aligned} X_{\alpha_3} X_{-\alpha_3} w_2 w_1 v_i &= H_{\alpha_3} w_2 w_1 v_i + X_{-\alpha_3} X_{\alpha_3} w_2 w_1 v_i \\ &= 0, \end{aligned}$$

since $w_2 w_1 v_i$ has weight $\omega_3 + \omega_4$ and $2((\omega_3 + \omega_4, \alpha_3)/(\alpha_3, \alpha_3)) = 2$.

(c) Computation of m_3. From the representation 0001, which we have determined in (ii), we can compute the symmetric part of (0001) $\otimes$ (0001) in characteristic 2; for this we find the following table of multiplicities m_d of the dominant weights d.

d	0000	0001	1000	0010	0002
m_d	15	6	3	1	1

In the Grothendieck group of rational representations of G we have

$$[(0001) \otimes (0001)] = [0002] + [0010] + a[1000] + b[0001] + c[0000].$$

On the representation space of 0001 there exists one nondegenerate bilinear form (unique up to scalar multiples) which is invariant under G; hence $c = 1$. This is so because -1 belongs to the Weyl group of F_4, hence any representation is equivalent to its contragredient representation. Comparing multiplicities we find

$$15 = 3 + m_3 + 2a + 2b$$
$$6 = m_2 + b$$
$$3 = m_1 + a$$

It follows that

$$m_3 = 2(m_1 + m_2) - 6 = 6.$$

Thus the representation 0010 in characteristic 2 is completely determined.

In this way we have obtained the set $\mathscr{R}$ of basic, irreducible representations in characteristic 2 from which one can obtain all other irreducible representations. In table 2 we have listed those with highest weight g such that $(g, g + 2\rho) \leqslant (g_0, g_0 + 2\rho)$ for $g_0 = 0011$.

TABLE II

Multiplicities in Characteristic 2 for the Group of Type F_4

g \ a	0000	0001	1000	0010	0002	1001	0100	0011	dim
0000	1								1
0001	2	1							26
1000	2	0	1						26
0010	6	4	2	1					246
0002	2	0	0	0	1				26
1001	4	8	2	3	0	1			676
0100	6	0	4	0	2	0	1		246
0011	64	40	24	14	8	4	2	1	4096

3. From table 2 one can compute the 2-modular characters of G_σ, the Ree group of type F_4 parametrized by $\mathbf{F}_2$. For general information about modular characters we refer to [5]. First we have to determine a representative for each of the conjugacy classes of 2-regular elements of G_σ. We recall that an element x is called 2-regular if its order is not divisible by 2; x is 2-regular if and only if it is semisimple. The semisimple conjugacy classes can be determined with the aid of lemma 3.9 of [12]. This lemma holds for any endomorphism σ such that G_σ is finite; the restriction on σ in [12], 3.9, was

used in step (1) of the proof only, but that step holds in general as follows from [13], Theorem 10.1; this is also implicit in the proof of [13], Theorem 14.8.

Denote an arbitrary element of the maximal torus T of G by (t_1, t_2, t_3, t_4, s), $t_i \in k^*$, $s^2 = t_1 t_2 t_3 t_4$. Each semisimple conjugacy class of G contains a $t \in T$ satisfying $\sigma t = wt$ for some $w \in W$, the Weyl group of G. Let

$$t^{\omega_i} = t_i, \qquad t^{\frac{1}{2}(\omega_1+\omega_2+\omega_3+\omega_4)} = s.$$

$\omega_1, \ldots, \omega_4$ form a basis for $V = X(T) \otimes \mathbf{R}$. The simple roots are acted on by σ^* as follows.

$$\sigma^*\alpha_1 = 2\alpha_4, \quad \sigma^*\alpha_2 = 2\alpha_3, \quad \sigma^*\alpha_3 = \alpha_2, \quad \sigma^*\alpha_4 = \alpha_1.$$

From this it follows that

$$\sigma^*\omega_1 = -\omega_1 + \omega_4, \quad \sigma^*\omega_2 = \omega_2 + \omega_3,$$
$$\sigma^*\omega_3 = \omega_2 - \omega_3, \quad \sigma^*\omega_4 = \omega_1 + \omega_4.$$

Since $(\sigma t)^{\omega_i} = t^{\sigma^*\omega_i}$, we obtain from the above

$$\sigma(t_1, t_2, t_3, t_4, s) = (t_1^{-1}t_4, t_2t_3, t_2t_3^{-1}, t_1t_4, t_2t_4).$$

The Weyl group W, acting on T, contains the transformations

$$(t_1, t_2, t_3, t_4, s) \to \left(t_{\pi(1)}^{\epsilon_1}, t_{\pi(2)}^{\epsilon_2}, t_{\pi(3)}^{\epsilon_3}, t_{\pi(4)}^{\epsilon_4}, s \prod_{1 \leqslant i \leqslant 4} t_{\pi(i)}^{1/2(\epsilon_i - 1)}\right),$$

π any permutation, $\epsilon_i = \pm 1$ (in all possible combinations), and also

$$(t_1, t_2, t_3, t_4, s) \mapsto (t_2^{-1}t_3^{-1}s, t_1^{-1}t_3^{-1}s, t_1^{-1}t_2^{-1}s, s, t_4)$$

(see [2], p. 19-10).

G has order $2^{12}3^3 5^2 13$ and contains 4 semisimple (=2-regular) conjugacy classes by [13], hence these must be C_1, C_3, C_5, C_{13}, where the elements of C_i have order i. It is easily verified that the following elements in T satisfy the equation $\sigma t = wt$ for some $w \in W$.

$$(1, 1, 1, 1, 1), \quad (1, \epsilon_3, \epsilon_3, \epsilon_3^2, \epsilon_3^2), \quad (1, 1, \epsilon_5, \epsilon_5^2, \epsilon_5^4), \quad (\epsilon_{13}, \epsilon_{13}^2, \epsilon_{13}^3, \epsilon_{13}^5, \epsilon_{13}^{12}),$$

where ϵ_j denotes a primitive j-th root of unity in k.

Let φ be the isomorphism of the group of 3.5.13-th roots of unity in k^* into $\mathbf{C}^*$ defined by

$$\varphi(\epsilon_n) = e^{2\pi i n^{-1}}, \qquad n = 3, 5, 13.$$

The 2-modular (Brauer) characters can now be computed from table 2. We omit the tedious but straightforward calculations. In this way one obtains table 3, where $\chi_{n_1 n_2 n_3 n_4}$ denotes the 2-modular character of the representation $n_1 n_2 n_3 n_4$. Notice that these characters turn out to be integral, hence they do not depend on the isomorphism φ from the group of 3.5.13-th roots of unity in k^* onto the analoguous group in $\mathbf{C}^*$.

TABLE III

2-Modular Characters of the Ree Group of Type F_4 over $\mathbf{F}_2$

character \ class	C_1	C_3	C_5	C_{13}
$\phi_1 = \chi_{0000}$	1	1	1	1
$\phi_2 = \chi_{0001}$	26	-1	1	0
$\phi_3 = \chi_{0010}$	246	3	-4	-1
$\phi_4 = \chi_{0011}$	4096	-8	-4	1

We shall, finally, determine the Cartan matrix $C = (c_{ij})$ which expresses the principal indecomposable characters of G_σ in the irreducible ones [1, 5]. ϕ_4 is the only character in its block, as follows from [11], Theorem 4, and [1], Theorem 1, or [5], Theorem (86.3), hence

$$c_{41} = c_{42} = c_{43} = 0, \quad c_{44} = 1.$$

Moreover, C is known to be symmetric. Let $C^{-1} = (\tilde{c}_{ij})$, then

$$\tilde{c}_{41} = \tilde{c}_{42} = \tilde{c}_{43} = 0, \quad \tilde{c}_{44} = 1.$$

Let n denote the order of G_σ, n_l that of the class C_l ($l = 1, 3, 5, 13$). Let g_l be a representative of C_l. The following character relations are known

$$n\tilde{c}_{ij} = \sum_l n_l \phi_i(g_l)\, \phi_j(g_l^{-1}), \qquad 1 \leqslant i, j \leqslant 4. \tag{3.1}$$

Since for $i = 4$ the $\tilde{c}_{ij}$ are known, we can compute the numbers n_l, keeping in mind that $g_l^{-1} \in C_l$ in this case.

$$\begin{aligned} n_1 &= 1 \\ n_3 &= 166\,400 = 2^9 5^2 13 \\ n_5 &= 359\,424 = 2^{10} 3^3 13 \\ n_{13} &= 2\,764\,800 = 2^{12} 3^3 5^2 \end{aligned}$$

The character relations (3.1) then yield C^{-1}, from which C is easily computed.

$$C = \begin{pmatrix} 160 & 528 & 160 & 0 \\ 528 & 1972 & 572 & 0 \\ 160 & 572 & 172 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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