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A geometric formula for Haefliger knots

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Dedicated to Professor Yukio Matsumoto for his sixtieth birthday

Abstract

Haefliger has shown that a smooth embedding of the $(4k-1)$-sphere in the $6k$-sphere can be knotted in the smooth sense. In this paper, we give a formula with which we can detect the isotopy class of such a Haefliger knot. The formula is expressed in terms of the geometric characteristics of an extension, analogous to a Seifert surface, of the given embedding. In particular, the Hopf invariant associated to the extension plays a crucial role. This leads us to a new characterisation of Haefliger knots.

Keywords: Haefliger knot; Geometric formula; Differentiable isotopy; Hopf invariant; Szűcs' linking

MSC: primary 57R40; 57R52; secondary 57R45

1. Introduction

Let C_n^q be the group of C^∞-isotopy classes of smooth embeddings of the oriented n-sphere S^n in the oriented $(n+q)$-sphere S^{n+q} $(q>1)$. Haefliger has shown in [8,9] that C_{4k-1}^{2k+1} is isomorphic to $\mathbb{Z}$ for $k \geqslant 1$. This is quite in contrast with Zeeman's result [26] which claims that any n-sphere is unknotted in the combinatorial sense in the $(n+q)$-sphere if $q>2$ (see Stallings [19] for the topological cases).

Haefliger has actually given an isomorphism $\Omega : C_{4k-1}^{2k+1} \to \mathbb{Z}$ and an explicit embedding representing a generator of C_{4k-1}^{2k+1} (see [8], and [9, 5.16 and Corollary 8.14]). This isomorphism assigns to an embedding $S^{4k-1} \hookrightarrow S^{6k}$ half the square of a certain $2k$-dimensional cohomology class of a normally

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framed $4k$-submanifold $(V^{4k}, \partial V^{4k}) \subset (D^{6k+1}, S^{6k})$ with signature zero, which extends the given embedding.

In this paper, we give a new formula for the Haefliger invariant $\Omega : C_{4k-1}^{2k+1} \to \mathbb{Z}$. We extend an embedding $S^{4k-1} \hookrightarrow S^{6k}$ to an immersion (or an embedding) of an oriented $4k$-manifold in S^{6k}, similar to a Seifert surface in the codimension two case. Then our formula is expressed in terms of the geometric characteristics of such an extension. From this point of view, our formula can be considered as an analogue of the geometric formulae for the Smale invariant given in [13,4] (see also [17,22]).

As in [4], we make use of a formula (see Theorem 2.3) for generic maps, due to Szűcs [21] and to Ekholm–Szűcs [5,4]. In our study, however, *the linking of the map* (see Definition 2.2) is interpreted in terms of the Hopf invariant associated to the extension of the given embedding, and this interpretation will be a crucial step in our argument.

The following is our first formula. Note that for an embedding $F : S^{4k-1} \hookrightarrow S^{6k}$, by taking an orientation-preserving diffeomorphism $S^{6k} \setminus F(S^{4k-1}) \cong \operatorname{Int} D^{4k} \times S^{2k}$, we obtain a homotopy equivalence $S^{6k} \setminus F(S^{4k-1}) \simeq S^{2k}$ and hence $H_{2k}(S^{6k} \setminus F(S^{4k-1})) = H_{2k}(S^{2k}) = \mathbb{Z}$.

Theorem 4.6. *For any embedding $F : S^{4k-1} \hookrightarrow S^{6k}$, there exists a self-transversal immersion $\widetilde{F} : V^{4k} \looparrowright S^{6k}$ of a compact oriented $4k$-manifold V^{4k} with $\partial V^{4k} = S^{4k-1}$ such that $\widetilde{F}|_{\partial V^{4k}} = F$ and $\widetilde{F}(\operatorname{Int} V^{4k}) \cap F(S^{4k-1}) = \emptyset$. Furthermore, the following is an isomorphism*:

$$\begin{array}{rccc} \Omega : & C_{4k-1}^{2k+1} & \longrightarrow & \mathbb{Z} \\ & F & \longmapsto & -\frac{1}{24}(-\bar{p}_k[\widehat{V}^{4k}] + 3t(\widetilde{F}) + 3[\Delta_{\widetilde{F}}] + 3H_{\widetilde{F}}), \end{array}$$

where $\bar{p}_k[\widehat{V}^{4k}]$ is the normal Pontrjagin number of the closed manifold $\widehat{V}^{4k}$ obtained by capping off V^{4k} with a disc, $t(\widetilde{F})$ is the algebraic number of triple points of $\widetilde{F}(V^{4k})$, $[\Delta_{\widetilde{F}}] \in H_{2k}(S^{6k} \setminus F(S^{4k-1})) = \mathbb{Z}$ is the homology class represented by the set $\Delta_{\widetilde{F}} \subset \widetilde{F}(V^{4k})$ of double points of $\widetilde{F}$, and $H_{\widetilde{F}} \in \mathbb{Z}$ is the Hopf invariant of the map from S^{4k-1} into $S^{6k} \setminus F(S^{4k-1}) \simeq S^{2k}$, determined by the outward normal field of $F(S^{4k-1}) \subset \widetilde{F}(V^{4k})$.

Naturally, the Hopf invariants in the theorem above can be considered for oriented $4k$-manifolds with spherical boundaries embedded in S^{6k}. In such a context, we can relate the Hopf invariants to the normal Euler classes of the embedded $4k$-manifolds (Theorem 5.1), and are thus led to the following characterisation of Haefliger knots.

Corollary 6.2. *Let $\widetilde{E}_{a,b} : S^{2k} \times S^{2k} \setminus \operatorname{Int} D^{4k} \hookrightarrow S^{6k}$ be an embedding of the punctured $S^{2k} \times S^{2k}$ with normal Euler class $(2a, 2b) \in H^{2k}(S^{2k} \times S^{2k} \setminus \operatorname{Int} D^{4k}) = H^{2k}(S^{2k} \times S^{2k}) \approx \mathbb{Z} \oplus \mathbb{Z}$. Then*,

$$E_{a,b} := \widetilde{E}_{a,b}|_{\partial(S^{2k} \times S^{2k} \setminus \operatorname{Int} D^{4k})} : S^{4k-1} \hookrightarrow S^{6k}$$

represents $-ab \in C_{4k-1}^{2k+1} = \mathbb{Z}$. In particular, $E_{\pm 1, \mp 1} : S^{4k-1} \hookrightarrow S^{6k}$ represents the generator of C_{4k-1}^{2k+1}.

This theorem implies that the immersion $\widetilde{F} : V^{4k} \looparrowright S^{6k}$ in Theorem 4.6 can actually be chosen to be an embedding. Thus we obtain the following simpler formulae.

Corollary 6.3. (a) *For an arbitrary embedding* $F: S^{4k-1} \hookrightarrow S^{6k}$, *there exists an embedding* $\widetilde{F}: V^{4k} \hookrightarrow S^{6k}$ *of a compact oriented* $4k$*-manifold* V^{4k} *with* $\partial V^{4k} = S^{4k-1}$ *such that* $\widetilde{F}|_{\partial V^{4k}} = F$. *Furthermore*,

$$\begin{aligned}\Omega(F) &= -\tfrac{1}{24}(-\bar{p}_k[\widehat{V}^{4k}] + 3H_{\widetilde{F}}) \\ &= -\tfrac{1}{24}(-\bar{p}_k[\widehat{V}^{4k}] + 3e_{\widetilde{F}} \smile e_{\widetilde{F}})\end{aligned}$$

gives an isomorphism $\Omega: C^{2k+1}_{4k-1} \to \mathbb{Z}$, *where* $e_{\widetilde{F}} \in H^{2k}(V^{4k}) = H^{2k}(\widehat{V}^{4k})$ *is the normal Euler class for* $\widetilde{F}$ *and* $e_{\widetilde{F}} \smile e_{\widetilde{F}} \in H^{4k}(\widehat{V}^{4k}) = \mathbb{Z}$ *is the cup product.*

(b) *Actually, we can extend any embedding* $F: S^{4k-1} \hookrightarrow S^{6k}$ *to an embedding* $\widetilde{F}: S^{2k} \times S^{2k} \setminus \operatorname{Int} D^{4k} \hookrightarrow S^{6k}$ *and then*

$$\begin{array}{cccc}\Omega: & C^{2k+1}_{4k-1} & \longrightarrow & \mathbb{Z} \\ & F & \longmapsto & -\frac{1}{8}H_{\widetilde{F}}\end{array}$$

is an isomorphism.

In the particular case C^3_3 ($k=1$), where $-\bar{p}_1[\widehat{V}^4] = p_1[\widehat{V}^4] = 3\sigma(V^4)$, we can rewrite our formula in terms of the signature of V^4 (Corollary 6.5).

In the last section, we notice the similarity between our formulae and the formula by Hughes and Melvin [13] in the codimension two case, and determine the image of the map $C^2_{4k-1} \to C^{2k+1}_{4k-1}$ induced by the inclusion (Theorem 7.1).

Throughout this paper, we work in the smooth category; all manifolds, embeddings and immersions considered are supposed to be differentiable of class C^∞, unless otherwise explicitly stated. We use the symbol '$\approx$' for a group isomorphism and '$\cong$' for a diffeomorphism between manifolds; the symbol '$\simeq$' means a homotopy equivalence between two topological spaces. The homology and cohomology theories are supposed to be with integer coefficients unless otherwise explicitly noted.

We will suppose the spheres are oriented. If M is an oriented manifold with boundary, then for the *induced orientation of* ∂M we adopt the *outward vector first* convention: we say an ordered basis of $T_p(\partial M)$ ($p \in \partial M$) is positively oriented if an outward vector followed by the basis is a positively oriented basis of T_pM.

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2. Preliminaries

2.1. Basic conventions

We briefly review the cup and cap products, the intersection pairing, etc. Most of the materials here are based on Greenberg's book [6] and are rather elementary, but we need to review them in order to establish the *sign conventions*, which will be particularly important later.

For a topological space X, let $C^*(X)$ be the singular cochain complex and $H^*(X)$ the singular cohomology ring. If $f: Y \to X$ is a continuous map between topological spaces, let $f^\sharp: C^*(X) \to$

$C^*(Y)$ and $f^*: H^*(X) \to H^*(Y)$) be the induced homomorphisms. We similarly define $C_*(X)$, $H_*(X)$, $f_\sharp$ and f_*.

Let $\lambda_p : \Delta_p \to \Delta_{p+q}$ be the inclusion of the singular p-simplex Δ_p on the "front" p-face of the singular $(p+q)$-simplex Δ_{p+q}. Similarly let $\rho_q : \Delta_q \to \Delta_{p+q}$ be the inclusion on the "back" q-face. For $c \in C^p(X)$ and $d \in C^q(X)$ *the cup product* $c \smile d$ is characterised by

$$\langle \sigma, c \smile d \rangle = \langle \sigma\lambda_p, c \rangle \langle \sigma\rho_q, d \rangle$$

for a singular $(p+q)$-simplex σ, where we denote by $\langle z, c \rangle$ the value of the cochain c on the chain z. The cup product gives rise to a product operation on the cohomology.

For $c \in C^p(X)$ and $z \in C_{p+q}(X)$, *the cap product* $z \frown c$ is defined to be the unique q-chain satisfying

$$\langle z \frown c, d \rangle = \langle z, c \smile d \rangle$$

for all q-cochains $d \in C^q(X)$. Then, with respect to the boundary and coboundary operators, we have

$$\partial(z \frown c) = (-1)^p((\partial z) \frown c - z \frown \delta c).$$

We also have the following properties of cup and cap products:

$$f_\sharp(w \frown f^\sharp c) = f_\sharp w \frown c,$$

$$w \frown (c \smile d) = (w \frown c) \frown d$$

for chains and cochains in appropriate dimensions. The cap product induces a bilinear pairing

$$\frown : H_{p+q}(X) \times H^p(X) \to H_q(X).$$

For an oriented manifold M^n, we denote the fundamental homology class $[M^n, \partial M^n]$ by μ_{M^n}. In what follows, although we write down only the case $\partial M^n = \emptyset$, the similar argument also holds when M^n has non-empty boundary. We consider *the Poincaré–Lefschetz duality* to be the isomorphism

$$\mu_{M^n} \frown : H^p(M^n) \to H_{n-p}(M^n),$$

whose inverse we denote by D. Namely $\mu_{M^n} \frown D(a) = a$ for $a \in H^*(M^n)$. The following properties will be used later:

$$\partial([M^n] \frown c^p) = (-1)^{p+1}[M^n] \frown \delta c^p \quad \text{for } c^p \in C^p(M^n). \tag{1}$$

For $a, b \in H_*(M^n)$, *the intersection pairing* $a \cdot b$ is defined by

$$a \cdot b = D^{-1}(D(a) \smile D(b))$$
$$= a \frown D(b).$$

If a and b are represented by submanifolds V and W of M^n, respectively, then $a \cdot b$ is represented by the intersection $V \cap W$.

For $a = [a'] \in H_p(M^n)$ (the homology class represented by the cycle a') and $b = [b'] \in H_q(M^n)$ where $p + q = n - 1$, *the linking number* $\mathrm{lk}(a, b)$ is defined by

$$\mathrm{lk}(a, b) = [A \cap b'] \in H_0(M^n) = \mathbb{Z},$$

where A is a $(p+1)$-chain such that $\partial A = a'$, intersecting b' transversally. Then we can check the following property which will often be used:

$$\mathrm{lk}(a, b) = (-1)^{pq+1}\mathrm{lk}(b, a).$$

2.2. The Hopf invariant and the Steenrod functional cup product

Here we review the Steenrod functional cup product [20], according to the definition and the notation given in [23, p. 368] (see also [25, Chapter XI]).

Let $f : Y \to X$ be a continuous map between topological spaces and $u = [u'] \in H^p(X)$ (the cohomology class represented by the cocycle u') and $v = [v'] \in H^q(X)$ be cohomology classes satisfying

$$u \smile v = 0 \quad \text{and} \quad f^* u = 0.$$

Then there exists a cochain $a \in C^{p+q-1}(X)$ such that $\delta a = u' \smile v'$ and a cochain $b \in C^{p-1}(Y)$ such that $\delta b = f^\sharp u'$. Furthermore the cochain

$$z' = f^\sharp a - b \smile f^\sharp v' \tag{2}$$

is actually a cocycle, whose cohomology class we denote by $z = [z'] \in H^{p+q-1}(Y)$. Thus *the left functional cup product* $L_f(u,v) \in H^{p+q-1}(Y)$ is defined to be the coset of z in

$$H^{p+q-1}(Y)/(f^* H^{p+q-1}(X) + H^{p-1}(Y) \smile f^* v).$$

Similarly for $u \in H^p(X)$ and $v \in H^q(X)$ with $u \smile v = 0$ and $f^* v = 0$, *the right functional cup product* $R_f(u,v) \in H^{p+q-1}(Y)$ is defined in

$$H^{p+q-1}(Y)/(f^* H^{p+q-1}(X) + f^* u \smile H^{p-1}(Y)).$$

The functional cup products $L_f(u,v)$ and $R_f(u,v)$ are invariant under homotopy of $f : Y \to X$.

In a sense, the functional cup product is a generalisation of the Hopf invariant. Let $f : S^{4k-1} \to S^{2k}$ be a continuous map. Since we have supposed that S^{4k-1} and S^{2k} are oriented, we can choose fixed generators $\eta \in H^{4k-1}(S^{4k-1})$ and $\omega \in H^{2k}(S^{2k})$. Then, since $f^*\omega = 0$ and $\omega \smile \omega = 0$, we can consider the functional cup product $R_f(\omega,\omega) = L_f(\omega,\omega) \in H^{4k-1}(S^{4k-1})$ (with no indeterminacy since $H^{2k-1}(S^{4k-1}) = H^{4k-1}(S^{2k}) = 0$). Actually in this case, the functional cup product is nothing but the Hopf invariant. More precisely, there exists an integer h_f such that $R_f(\omega,\omega) = L_f(\omega,\omega) = -h_f \cdot \eta$ for the fixed generator η. The integer h_f is equal to *the Hopf invariant* of f (see [20, Section 17] and also [12, p. 379, Theorem 9.5.6]). Note that the minus sign here is for coincidence with the usual definition of the Hopf invariant as the linking between the inverse images of two regular points in S^{2k}. Note also that the choice of the generator $\omega \in H^{2k}(S^{2k})$ does not affect the Hopf invariant, while that of $\eta \in H^{4k-1}(S^{4k-1})$ changes the sign.

2.3. Triple points and Szűcs' linking

For a (self-transversal) immersion of a manifold in a euclidian space, the relation between its normal characteristic classes and the homology classes of its multiple points has been extensively studied. For example, the following theorem is well known ([24], see also [11, p. xiii, Corollary 6]).

Theorem 2.1. *Let $G : W^{4k} \looparrowright \mathbb{R}^{6k}$ be a self-transversal immersion of a closed oriented $4k$-manifold in $6k$-space. Then three times the algebraic number $t(G)$ of triple points of G is equal to $\bar{p}_k[W^{4k}]$, where $\bar{p}_k[W^{4k}]$ denotes the kth normal Pontrjagin class of W^{4k} evaluated on the fundamental class.*

In [21], Szűcs has introduced the notion of *linking of singular maps* to generalise this formula to the case of generic maps $W^{4k} \to \mathbb{R}^{6k}$. Roughly speaking, Szűcs' linking measures the linking between the image of the map and the singularities (pushed out of the image), and turns out to be very useful (see [4,5,17] for example).

Definition 2.2 (Szűcs' linking). Let $G: W^{4k} \to \mathbb{R}^{6k}$ be a generic map of an oriented $4k$-manifold. Then the closure of the set Δ_G of double points of G is an immersed $2k$-manifold of $\mathbb{R}^{6k}$, whose boundary the image of the set S_G of singular points of G lies in. Since the codimension of G is even, there is an induced orientation on Δ_G and hence on S_G. Let S'_G be a copy of S_G shifted slightly along the outward vector field of S_G in Δ_G. Note that $S'_G \cap G(W^{4k}) = \emptyset$. Define *the linking number* $l(G)$ *of* G to be the linking number of $G(W^{4k})$ and S'_G in $\mathbb{R}^{6k}$.

Then we have (see Ekholm and Szűcs [4, Lemma 5.3] for a precise proof):

Theorem 2.3 (Lemma 5.3 in Ekholm and Szűcs [4]). *Let $G: W^{4k} \to \mathbb{R}^{6k}$ be a generic map of a closed oriented $4k$-manifold. Then*

$$-\bar{p}_k[W^{4k}] + 3t(G) - 3l(G) = 0,$$

where $t(G)$ is the algebraic number of triple points of G.

3. Embeddings of the $(4k-1)$-sphere in the $6k$-sphere

We study several specific situations of embeddings of S^{4k-1} into S^{6k} and give some definitions which will be used later. Also, we look into the explicit generator of C^{2k+1}_{4k-1} given by Haefliger and compute the Hopf invariant associated to its extension (see Definition 3.2) using the Steenrod functional cup product.

3.1. Seifert immersions

The following is an analogue of the notion of Seifert surface in the codimension two case.

Definition 3.1 (Seifert immersion). Let $F: S^{4k-1} \hookrightarrow S^{6k}$ be an embedding. Consider an immersion $\widetilde{F}: V^{4k} \looparrowright S^{6k}$ of a compact oriented $4k$-manifold V^{4k} with $\partial V^{4k} = S^{4k-1}$ such that

(1) $\widetilde{F}|_{\partial V^{4k}} = F: S^{4k-1} \hookrightarrow S^{6k}$ and
(2) $\widetilde{F}(\mathrm{Int}\, V^{4k}) \cap F(S^{4k-1}) = \emptyset$.

Assume furthermore that $\widetilde{F}$ is self-transversal and is an embedding on a collar neighbourhood of $S^{4k-1} = \partial V^{4k} \subset V^{4k}$. We call such an immersion $\widetilde{F}: V^{4k} \looparrowright S^{6k}$ *a Seifert immersion for* F.

The existence of such immersions will be ensured in Section 3.3, where we show that an explicit generator of $C^{2k+1}_{4k-1} \approx \mathbb{Z}$ has a Seifert immersion.

Every embedding $F : S^{4k-1} \hookrightarrow S^{6k}$ has trivial normal bundle (see [14]). Let N be a tubular neighbourhood of $F(S^{4k-1}) \subset S^{6k}$. Then N is diffeomorphic to $S^{4k-1} \times D^{2k+1}$ and the exterior $X := S^{6k} \setminus \operatorname{Int} N$ is diffeomorphic to $D^{4k} \times S^{2k}$ (see [18, Theorem 5.2]). Let $\psi : X \to D^{4k} \times S^{2k}$ be an orientation-preserving diffeomorphism.

Definition 3.2 (the Hopf invariant for a Seifert immersion). Let $\widetilde{F} : V^{4k} \looparrowright S^{6k}$ be a Seifert immersion for an embedding $F : S^{4k-1} \hookrightarrow S^{6k}$. Then, the outward normal field of $F(S^{4k-1}) \subset \widetilde{F}(V^{4k})$ determines the map $v_{\widetilde{F}} : S^{4k-1} \to X$, whose homotopy class $[v_{\widetilde{F}}]$ we consider to be lying in $\pi_{4k-1}(X) = \pi_{4k-1}(S^{2k})$, via the homotopy equivalence $p \circ \psi : X \xrightarrow{\cong} D^{4k} \times S^{2k} \xrightarrow{\simeq} S^{2k}$, where p is the projection. We define *the Hopf invariant* $H_{\widetilde{F}}$ *for* $\widetilde{F}$ to be the Hopf invariant of $[v_{\widetilde{F}}]$.

Remark 3.3. From now on, we will always consider that the exterior space X is homotopy equivalent to S^{2k} as in Definition 3.2. Furthermore, we consider $H_{2k}(X)$ isomorphic to $\mathbb{Z}$ by assigning to $[z^{2k}] \in H_{2k}(X)$ the linking number $\operatorname{lk}(z^{2k}, F(S^{4k-1}))$ in S^{6k}.

We further consider the following integer, associated to a Seifert immersion.

Definition 3.4. Let $\widetilde{F} : V^{4k} \looparrowright S^{6k}$ be a Seifert immersion for an embedding $F : S^{4k-1} \hookrightarrow S^{6k}$. Then, the set $\Delta_{\widetilde{F}} \subset \widetilde{F}(V^{4k})$ of double (including triple) points of $\widetilde{F}$ is a $2k$-dimensional immersed manifold lying in X. We denote its homology class by $[\Delta_{\widetilde{F}}] \in H_{2k}(X) = \mathbb{Z}$.

3.2. Haefliger's construction of the generator E

In [8] and [9, Section 5.16], Haefliger has given an explicit generator of C^{2k+1}_{4k-1}. First we recall Haefliger's construction of the generator $E : S^{4k-1} \hookrightarrow S^{6k}$ of C^{2k+1}_{4k-1} described in [8, Section 4]. Here we consider S^{6k} as $\mathbb{R}^{6k} \cup \{\infty\}$.

Consider the following three embedded $(4k-1)$-spheres in

$$\mathbb{R}^{6k} = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x_1, \ldots, x_{2k}, y_1, \ldots, y_{2k}, z_1, \ldots, z_{2k})\} :$$

$$S_1 : \; \mathbf{x} = 0, \frac{\mathbf{y}^2}{\alpha^2} + \frac{\mathbf{z}^2}{\beta^2} = 1 \subset \mathbb{R}^{4k}_1 := \{\mathbf{x} = 0\},$$

$$S_2 : \; \mathbf{y} = 0, \frac{\mathbf{z}^2}{\alpha^2} + \frac{\mathbf{x}^2}{\beta^2} = 1 \subset \mathbb{R}^{4k}_2 := \{\mathbf{y} = 0\},$$

$$S_3 : \; \mathbf{z} = 0, \frac{\mathbf{x}^2}{\alpha^2} + \frac{\mathbf{y}^2}{\beta^2} = 1 \subset \mathbb{R}^{4k}_3 := \{\mathbf{z} = 0\},$$

where α and β are real numbers with $\alpha > \beta > 0$ and $\mathbf{x}^2$ means $x_1^2 + \cdots + x_{2k}^2$. We obtain an embedded $(4k-1)$-sphere $\mathtt{S}$ after joining S_1 to S_2 and S_2 to S_3 orientation-preservingly by thin tubes T_1 and T_2 (we take these tubes so that they do not cross the inside of $S_i \subset \mathbb{R}^{4k}_i$) (see Fig. 1). Then (the isotopy class of) the embedding $E : S^{4k-1} \hookrightarrow S^{6k}$ whose image is $\mathtt{S}$ generates the group $C^{2k+1}_{4k-1} \approx \mathbb{Z}$ (see [8, Section 4] and [9, Section 5.16]).

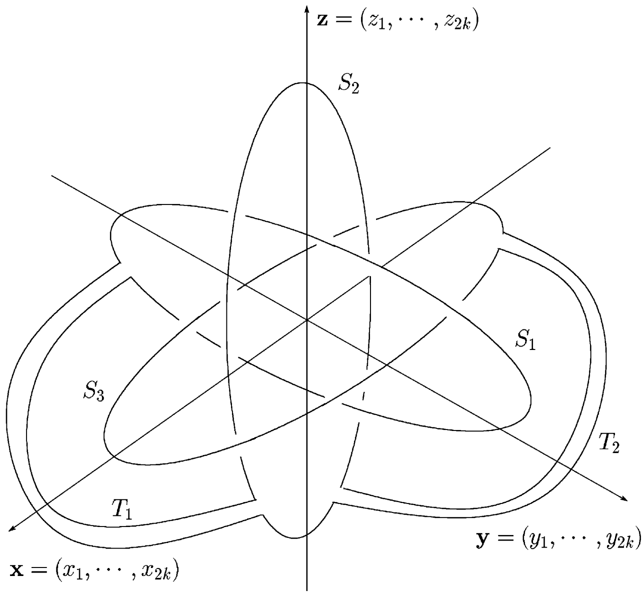

Fig. 1.

3.3. The Seifert immersion $\widetilde{E}$ for the generator

For the embedding $E: S^{4k-1} \hookrightarrow S^{6k}$ with $E(S^{4k-1}) = \mathtt{S}$, representing the generator of $C^{2k+1}_{4k-1} = \mathbb{Z}$, we can construct a Seifert immersion $\widetilde{E}: V_E^{4k} \looparrowright S^{6k}$ as follows.

First by using the three $4k$-discs:

$$D_1\colon\ \mathbf{x} = 0, \frac{\mathbf{y}^2}{\alpha^2} + \frac{\mathbf{z}^2}{\beta^2} \leqslant 1,$$

$$D_2\colon\ \mathbf{y} = 0, \frac{\mathbf{z}^2}{\alpha^2} + \frac{\mathbf{x}^2}{\beta^2} \leqslant 1,$$

$$D_3\colon\ \mathbf{z} = 0, \frac{\mathbf{x}^2}{\alpha^2} + \frac{\mathbf{y}^2}{\beta^2} \leqslant 1$$

and two $4k$-discs filling the tubes T_1 and T_2, we can easily construct an immersed $4k$-disc D_E bounded by $\mathtt{S} = E(S^{4k-1})$. But $\mathrm{Int}\, D_E$ is not immersed in the exterior $\mathbb{R}^{6k} \backslash \mathtt{S}$, i.e. it has double points which are intersections of the boundary and the interior of the $4k$-disc. Therefore D_E is *not* the image of a Seifert immersion for E as it is. We perform a "surgery" on D_E so that it becomes the image of some Seifert immersion for E in the following way.

The set of double points of D_E is

$$\{\mathbf{x} = \mathbf{y} = 0, \mathbf{z}^2 \leqslant \beta^2\} \cup \{\mathbf{y} = \mathbf{z} = 0, \mathbf{x}^2 \leqslant \beta^2\} \cup \{\mathbf{z} = \mathbf{x} = 0, \mathbf{y}^2 \leqslant \beta^2\},$$

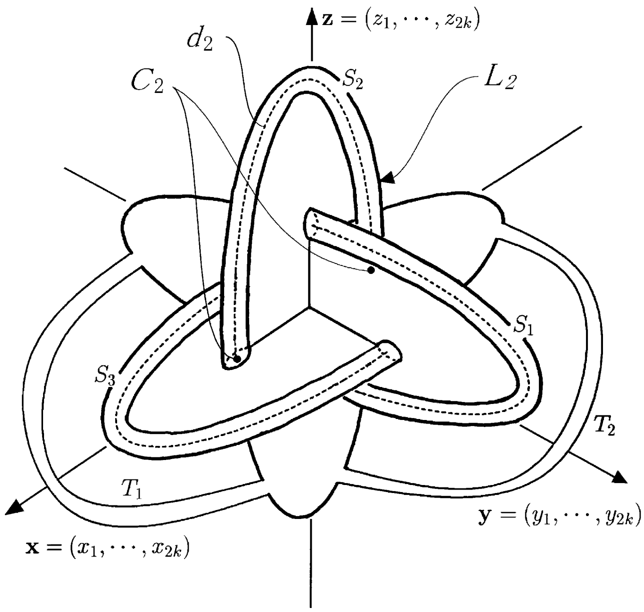

Fig. 2.

in which the "unfavourable" double points—the intersection points of the boundary and the interior of the $4k$-disc—consist of the three $(2k-1)$-spheres:

$$C_1 := \partial D_1 \cap D_2 = \{\mathbf{x} = \mathbf{y} = 0, \mathbf{z}^2 = \beta^2\},$$

$$C_2 := \partial D_2 \cap D_3 = \{\mathbf{y} = \mathbf{z} = 0, \mathbf{x}^2 = \beta^2\},$$

$$C_3 := \partial D_3 \cap D_1 = \{\mathbf{z} = \mathbf{x} = 0, \mathbf{y}^2 = \beta^2\}.$$

Consider the three $2k$-discs bounded by C_i $(i = 1, 2, 3)$:

$$d_1\colon \{\mathbf{x} = 0, y_1 = \cdots = y_{2k-1} = 0, y_{2k} \geqslant 0\} \cap S_1,$$

$$d_2\colon \{\mathbf{y} = 0, z_1 = \cdots = z_{2k-1} = 0, z_{2k} \geqslant 0\} \cap S_2,$$

$$d_3\colon \{\mathbf{z} = 0, x_1 = \cdots = x_{2k-1} = 0, x_{2k} \geqslant 0\} \cap S_3.$$

Note that each d_i is bounded in $\partial D_i = S_i$ by C_i.

Let K_i be the total space of the normal disc (with small radius ε) bundle of $C_i \subset D_j$, where $(i, j) = (1, 2), (2, 3), (3, 1)$; each K_i is diffeomorphic to $S^{2k-1} \times D^{2k+1}$. Let N be the normal disc (with radius ε) bundle of $\mathrm{S} \subset \mathbb{R}^{6k}$; the total space of N is diffeomorphic to $S^{4k-1} \times D^{2k+1}$.

Then, near the "unfavourable" double points of D_E, we operate a surgery (see Fig. 2). Namely we put $L_i := \partial(\text{the total space of } N|_{d_i})$ and consider

$$V := \left(D_E \setminus \bigcup_{i=1,2,3} K_i \right) \cup \bigcup_{i=1,2,3} L_i.$$

Note that each L_i is diffeomorphic to $D^{2k} \times S^{2k}$. Then, after a suitable smoothing process, V becomes the image of some immersion $\widetilde{E} : V_E^{4k} \looparrowright S^{6k}$ of a compact oriented $4k$-manifold V_E^{4k}.

Thus the immersion $\widetilde{E}$ is a Seifert immersion for the generator E, and consequently we see that an arbitrary embedding $S^{4k-1} \hookrightarrow S^{6k}$ has a Seifert immersion.

3.4. The Hopf invariant for $\widetilde{E}$

Let $\mathtt{S}$ be the embedded $(4k-1)$-sphere constructed in Section 3.2 by connecting the three embedded $(4k-1)$-spheres S_1, S_2 and S_3. Since each S_i is standardly embedded and bounds a $4k$-disc D_i in $\mathbb{R}_i^{4k} \subset \mathbb{R}^{6k}$, we can consider the outward normal field of $S_i \subset D_i$. Denote by $\mathtt{S}'$ the embedded $(4k-1)$-sphere obtained from $\mathtt{S}$ slightly shifted along this normal field (extended on the connecting tubes T_i outward of the boundary of the $4k$-discs filling them).

Proposition 3.5. *The Hopf invariant of the element represented by $\mathtt{S}' \subset S^{6k} \backslash \mathtt{S}$ in $\pi_{4k-1}(S^{6k} \backslash \mathtt{S}) = \pi_{4k-1}(S^{2k})$ is equal to -6. In other words, $H_{\widetilde{E}} = -6$ for the Seifert immersion $\widetilde{E}$.*

Proof. We imitate the calculation of the Massey triple product given in [15, Sections 4 and 6]. Namely we first consider the Hopf invariant to be (minus) a functional cup product for the inclusion $\mathtt{S}' \to S^{6k} \backslash \mathtt{S}$ and then we calculate this functional cup product by translating the cup product operation on cohomology into the intersection theory on homology using the duality theorems.

Let $\xi : \mathtt{S}' \hookrightarrow S^{6k} \backslash \mathtt{S}$ be the inclusion. Let

$$\omega \in H^{2k}(S^{6k} \backslash \mathtt{S}) = H^{2k}(D^{4k} \times S^{2k}) = H^{2k}(S^{2k})$$

denote the generator. Then we have to calculate the functional cup product

$$R_\xi(\omega, \omega) = L_\xi(\omega, \omega) \in H^{4k-1}(\mathtt{S}') = \mathbb{Z}.$$

Let N, N' be sufficiently small (i.e. $N \cap N' = \emptyset$) tubular neighbourhoods of $\mathtt{S}$, $\mathtt{S}'$, respectively, and put $X := S^{6k} \backslash \mathrm{Int}\, N$. Then, consider the following diagram:

$$\begin{array}{ccc} H^*(S^{6k} \backslash \mathtt{S}) & \xrightarrow{\xi^*} & H^*(\mathtt{S}') \\ \approx \downarrow & & \uparrow \approx \\ H^*(X) & \xrightarrow{\zeta^*} & H^*(N'), \end{array}$$

where all homomorphisms are induced by the inclusion maps and the vertical arrows are isomorphisms. By this diagram, it follows that we have only to calculate

$$R_\zeta(\omega, \omega) = L_\zeta(\omega, \omega) \in H^{4k-1}(N') = \mathbb{Z}$$

with respect to the inclusion $\zeta : N' \hookrightarrow X$, instead of $R_\xi(\omega, \omega) = L_\xi(\omega, \omega) \in H^{4k-1}(\mathtt{S}') = \mathbb{Z}$, where we identify ω with its image (also denoted by ω) under the isomorphism $H^{2k}(S^{6k} \backslash \mathtt{S}) \to H^{2k}(X)$.

Furthermore, by the duality and the excision theorems, we have the following:

$$
\begin{array}{ccc}
H^i(X) & \xrightarrow{\zeta^*} & H^i(N') \\
\approx\big\downarrow & & \big\downarrow\approx \\
H_{6k-i}(X, \partial X) & \xrightarrow{\zeta_!} & H_{6k-i}(N', \partial N') \\
\approx\big\uparrow & & \\
H_{6k-i}(S^{6k}, \mathtt{S}), & &
\end{array}
$$

where all vertical arrows are isomorphisms.

Now we calculate the desired functional cup product in terms of intersections. Let w' be the relative $4k$-chain realised by the immersed 4-disc D_E (see Section 3.3). Perturb D_E slightly in the direction $(-1, \ldots, -1) \in \mathbb{R}^{6k}$; denote this perturbed copy of D_E by $\tilde{D}_E$. If the perturbation is sufficiently small, we can take the relative $4k$-chain $w'' \in H_{4k}(X, \partial X) \approx H_{4k}(S^{6k}, \mathtt{S})$ realised by $\tilde{D}_E \cap X$. Denote by ω' and ω'' the dual $2k$-cochains of w' and w'', respectively. Then, the homology class $[w'] = [w''] \in H_{4k}(X, \partial X) \approx H_{4k}(S^{6k}, \mathtt{S})$ is dual to $[\omega'] = [\omega''] = \omega \in H^{2k}(X)$.

First we compute the second term of (2) in Section 2.2—the equation defining the functional cup product. The relative $4k$-chain dual to $\zeta^\sharp \omega'$ is represented by the intersection I of N' and D_E, which consists of the three components $I_i = D_i \cap N'$ $(i = 1, 2, 3)$. We see that each I_i is a relative boundary of $D^{2k} \times D^{2k+1}$ lying in

$$(\text{the northern hemisphere of } S_i) \times D^{2k+1} \subset S^{4k-1} \times D^{2k+1} \cong N'$$

and that this $D^{2k} \times D^{2k+1}$ does not intersect $\tilde{I} := \tilde{D}_E \cap N'$. Hence the term corresponding to the second term of (2) in Section 2.2 vanishes.

Next we compute the first term of (2) in Section 2.2. The relative $2k$-chain dual to $\omega' \smile \omega''$ is represented by the intersection of D_E and $\tilde{D}_E$, which essentially consists of two copies of the self-intersection of the relative $4k$-chain D_E, homologous to the union L of the three embedded $2k$-discs described as the bold lines in Fig. 3 (if the connecting tubes T_i are appropriately placed). Thus, by property (1) in Section 2.1, we see that the dual chain of the cochain a cobounded by $\omega' \smile \omega''$ is homologous to ($(-1)^{4k-1+1}$ time) the relative $(2k+1)$-chain represented by two copies of the union of the three embedded $(2k+1)$-discs $M = M_1 \amalg M_2 \amalg M_3$ shadowed in Fig. 3, where each M_i lies in the northern half of D_i $(i = 1, 2, 3)$.

Finally, the image $\zeta^\sharp a$ itself should be a cocycle and its cohomology class $[\zeta^\sharp a] \in H^{4k-1}(N')$ is equal to the functional cup product $L_\zeta(\omega, \omega)$. If we denote by $j : \mathtt{S}' \hookrightarrow N'$ and put $\zeta' := j \circ \zeta : \mathtt{S}' \hookrightarrow X$, then, to determine $[\zeta^\sharp a] \in H^{4k-1}(N')$, we have only to compute its image $\zeta'_*(\mu_{\mathtt{S}'} \frown j^*[\zeta^\sharp a]) \in H_0(X)$ under the composition of the isomorphisms:

$$\zeta'_* \circ (\mu_{\mathtt{S}'} \frown) \circ j^* : H^{4k-1}(N') \to H^{4k-1}(\mathtt{S}') \to H_0(\mathtt{S}') \to H_0(X).$$

Since we see that

$$
\begin{aligned}
\zeta'_*(\mu_{\mathtt{S}'} \frown j^*[\zeta^\sharp a]) &= \zeta'_*(\mu_{\mathtt{S}'} \frown [\zeta'^\sharp a]) \\
&= [\zeta'_\sharp(\mathtt{S}' \frown \zeta'^\sharp a)]
\end{aligned}
$$

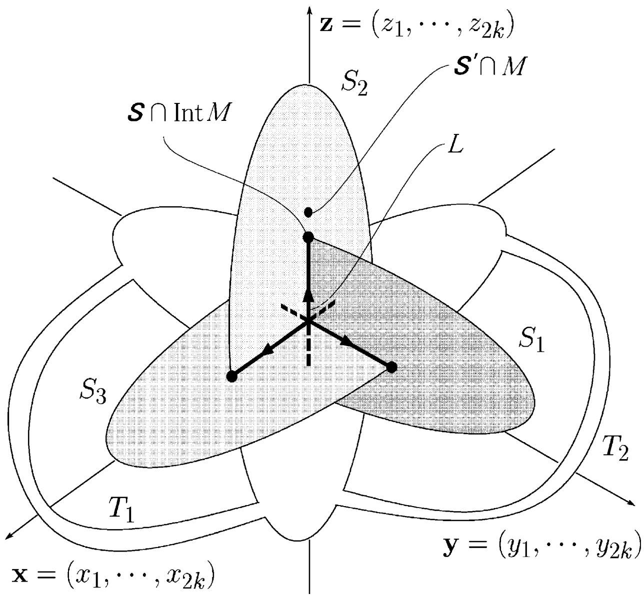

Fig. 3.

$$= [\zeta'_{\#}\mathtt{S}' \frown a]$$
$$= [\zeta'_{\#}\mathtt{S}' \cap 2M]$$
$$= \zeta'_{*}[\mathtt{S}' \cap 2M],$$

we need only to compute twice the intersection of $\mathtt{S}'$ and M as a 0-cycle of $\mathtt{S}'$. Furthermore, if the shift in obtaining $\mathtt{S}'$ is sufficiently small, this is the same as twice the intersection of $\mathtt{S}$ and $\mathrm{Int}\,M$, which consists of the three points seen in Fig. 3. To find the signs of these intersection points, we have only to check the intersection of S_1 and

$$M_2\colon \left\{\mathbf{y}=0, z_1=\cdots=z_{2k-1}=0, z_{2k}\geqslant 0, \frac{z_{2k}^2}{\alpha^2}+\frac{\mathbf{x}^2}{\beta^2}\leqslant 1\right\}$$

for example. We can easily check that at $p \in S_1 \cap M_2$, the $6k$-frame:

$$\left\langle \left(-\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_{2k}},\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_{2k-1}}\right),\left(-\frac{\partial}{\partial z_{2k}},\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_{2k}}\right)\right\rangle_p,$$

where the first $4k-1$ vectors belong to $T_p(S_1)$ and the rest to $T_p(M_2)$, has the same orientation as the standard frame. Namely the intersection points of $\mathtt{S}$ and M have positive signs: $[\mathtt{S}\cap 2M]=6\in H_0(\mathtt{S})$. Therefore, $L_\xi(\omega,\omega)=L_\zeta(\omega,\omega)=6$; hence $H_{\widetilde{E}}=-L_\xi(\omega,\omega)=-6$. □

Remark 3.6. In [6, p. 420], Haefliger has stated that for $k=1$ the above $H_{\widetilde{E}}$ is equal to 6 up to sign.

4. A geometric formula

In this section, we give a formula for Haefliger knots, with which we can read off the isotopy class of an embedding $S^{4k-1} \hookrightarrow S^{6k}$, via the geometric characteristics of its Seifert immersion.

First we want to define a homomorphism $\Theta : C^{2k+1}_{4k-1} \to \mathbb{Z}$.

Definition 4.1. Let $F : S^{4k-1} \hookrightarrow S^{6k}$ $(k \geqslant 1)$ be an embedding and $\widetilde{F} : V^{4k} \looparrowright S^{6k}$ be a Seifert immersion for F. Then, define

$$\Theta(F) := -\bar{p}_k[\widehat{V}^{4k}] + 3t(\widetilde{F}) + 3[\Delta_{\widetilde{F}}] + 3H_{\widetilde{F}} \in \mathbb{Z},$$

where $\bar{p}_k[\widehat{V}^{4k}]$ is the normal Pontrjagin number of the closed manifold $\widehat{V}^{4k}$ obtained by capping off V^{4k} with a disc, $t(\widetilde{F})$ is the algebraic number of triple points of $\widetilde{F}(V^{4k})$, $[\Delta_{\widetilde{F}}] \in H_{2k}(X) = \mathbb{Z}$ is the homology class represented by the set $\Delta_{\widetilde{F}} \subset X$ of double points of $\widetilde{F}$ (see Definition 3.4), and $H_{\widetilde{F}} \in \mathbb{Z}$ is the Hopf invariant for $\widetilde{F}$ (see Definition 3.2).

Actually Θ gives a well-defined homomorphism $C^{2k+1}_{4k-1} \to \mathbb{Z}$. To show this, we need the following lemma.

Lemma 4.2. *Let $\tilde{\alpha} : S^{4k-1} \to S^{4k-1} \times S^{2k}$ be a continuous map such that $p_1 \circ \tilde{\alpha} : S^{4k-1} \to S^{4k-1} \times S^{2k} \to S^{4k-1}$ is a degree one map, where p_i is the projection to the ith factor for $i = 1, 2$. Let $B : S^{4k-1} \times S^{2k} \to S^{2k}$ be a continuous map such that $B|_{\{*\} \times S^{2k}} : S^{2k} \to S^{2k}$ is a degree one map. Then,*

$$[B \circ \tilde{\alpha}] = [\beta] + [\alpha] \in \pi_{4k-1}(S^{2k}),$$

where $\alpha = p_2 \circ \tilde{\alpha} : S^{4k-1} \to S^{2k}$ and $\beta = B|_{S^{4k-1} \times \{*\}} : S^{4k-1} \to S^{2k}$.

Proof. Consider the induced homomorphisms

$$\begin{array}{ccccc} \pi_{4k-1}(S^{4k-1}) & \xrightarrow[\tilde{\alpha}_*]{} & \pi_{4k-1}(S^{4k-1} \times S^{2k}) & \xrightarrow[B_*]{} & \pi_{4k-1}(S^{2k}) \\ & & \approx \Big\downarrow p_{1*} \oplus p_{2*} & & \\ & & \pi_{4k-1}(S^{4k-1}) \oplus \pi_{4k-1}(S^{2k}). & & \end{array}$$

Take the generator $[\mathrm{id}] \in \pi_{4k-1}(S^{4k-1})$, then under the isomorphism $p_{1*} \oplus p_{2*}$, we have

$$\begin{aligned} [B \circ \tilde{\alpha}] &= B_* \circ \tilde{\alpha}_*([\mathrm{id}]) \\ &= B_*([\mathrm{id}] \oplus [\alpha]) \\ &= [\beta] + [\alpha]. \end{aligned}$$

This completes the proof. □

Proposition 4.3. *$\Theta(F)$ is well-defined on the isotopy class of F.*

Proof. Take two Seifert immersions $\widetilde{F}_i : V_i^{4k} \looparrowright S^{6k}$ $(i = 0, 1)$ for $F : S^{4k-1} \hookrightarrow S^{6k}$. Denote by $W^{4k} := V_1^{4k} \cup (-V_0^{4k})$ the closed oriented $4k$-manifold obtained by pasting V_0^{4k} and V_1^{4k} together along their boundaries. Then, after a suitable smoothing process, we can obtain a smooth generic map

$$G := \widetilde{F}_1 \cup (-\widetilde{F}_0) : W^{4k} \to S^{6k}.$$

We want to apply Theorem 2.3 to the map G, and show that $\Theta(F)$ does not depend on the choice of the Seifert immersion.

The map G may have new triple points other than those of each $\widetilde{F}_i$, and furthermore, new singularities may appear in pasting V_i^{4k} along $S^{4k-1} \subset V_1^{4k} \cup (-V_0^{4k})$.

First, the new triple points consist of

$$\widetilde{F}_1(\mathrm{Int}\, V_1^{4k}) \cap \Delta_{\widetilde{F}_0} - \widetilde{F}_0(\mathrm{Int}\, V_0^{4k}) \cap \Delta_{\widetilde{F}_1},$$

since $F(S^{4k-1})$ does not intersect $\widetilde{F}_i(\mathrm{Int}\, V_i^{4k})$ $(i=0,1)$. Furthermore, the algebraic number of points in $\widetilde{F}_i(\mathrm{Int}\, V_i^{4k}) \cap \Delta_{\widetilde{F}_j}$ is nothing but the linking number

$$\mathrm{lk}(F(S^{4k-1}), \Delta_{\widetilde{F}_j}) = -\mathrm{lk}(\Delta_{\widetilde{F}_j}, F(S^{4k-1}))$$

$((i,j) = (0,1), (1,0))$ and hence is equal to $-[\Delta_{\widetilde{F}_j}] \in H_{2k}(X) = \mathbb{Z}$ (see Remark 3.3). Thus we see that the algebraic number of "new" triple points is equal to

$$[\Delta_{\widetilde{F}_1}] - [\Delta_{\widetilde{F}_0}] \in H_{2k}(X) = \mathbb{Z}.$$

Next, we look into the newly appeared singularities. We can assume that, in the (oriented) tubular neighbourhood $N \subset S^{6k}$, each $\widetilde{F}_i(V_i^{4k}) \cap N$ is an embedded image of collar neighbourhoods $S^{4k-1} \times [-1, 0] \subset V_0^{4k}$ and $S^{4k-1} \times [0, 1] \subset V_i^{4k}$. If we take a trivialisation $N \cong S^{4k-1} \times D^{2k+1}$, then the inward normal fields of $F(S^{4k-1}) \subset \widetilde{F}_i(V_i^{4k})$ determine the maps $v_i : S^{4k-1} \to \partial N \cong S^{4k-1} \times S^{2k} \to S^{2k}$ $(i = 0, 1)$.

Let $\mathtt{NH}$ and $\mathtt{SH}$ be the northern and the southern hemispheres (including the equator) of S^{4k-1}, respectively. Denote the antipodal points of q_0 and $q_1 \in S^{2k}$ by $-q_0$ and $-q_1 \in S^{2k}$, respectively. We may assume that

(1) $v_0(x) = q_0$ $(x \in \mathtt{NH})$,
(2) $v_1(x) = q_1$ $(x \in \mathtt{SH})$,
(3) q_1 and $-q_1$ are regular values of v_0 and
(4) q_0 and $-q_0$ are regular values of v_1.

Note that, each homotopy class $[v_i] \in \pi_{4k-1}(S^{2k})$ may possibly be different from $[v_{\widetilde{F}_i}] \in \pi_{4k-1}(X) = \pi_{4k-1}(S^{2k})$, but the difference $[v_1] - [v_0]$ is equal to $[v_{\widetilde{F}_1}] - [v_{\widetilde{F}_0}]$ by Lemma 4.2 (see Definition 3.2 for the term $v_{\widetilde{F}_i}$, which was independent of the trivialisation of N).

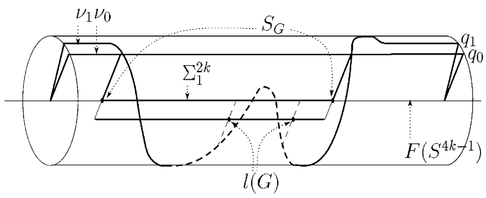

Fig. 4.

Then, on the collar neighbourhoods restricted to the northern part $\mathtt{NH}\times[-1,0]\subset S^{4k-1}\times[-1,0]\subset V_0^{4k}$ and to $\mathtt{NH}\times[0,1]\subset S^{4k-1}\times[0,1]\subset V_1^{4k}$, $\widetilde{F}_0$ and $\widetilde{F}_1$ can be written as follows:

$$\begin{aligned}\widetilde{F}_0: \quad \mathtt{NH}\times[-1,0) &\to S^{4k-1}\times D^{2k+1}\subset S^{6k}\\ (x,t) &\mapsto (F(x),(q_0,-t)),\\ \widetilde{F}_1: \quad \mathtt{NH}\times(0,1] &\to S^{4k-1}\times D^{2k+1}\subset S^{6k}\\ (x,t) &\mapsto (F(x),(v_1(x),t)),\end{aligned}$$

$$\widetilde{F}_0(x,0)=\widetilde{F}_1(x,0)=F(x) \text{ for } (x,0)\in\mathtt{NH}\times[0,1],$$

where we consider D^{2k+1} as $S^{2k}\times(0,1]\cup\{\mathbf{0}\}$ on the right-hand side. On the southern part the situation is similar by alternating the indices (Fig. 4 describes the situation on the northern part).

Hence, in $S^{4k-1}\times D^{2k+1}\subset S^{6k}$, we see the end-part of the set Δ_G of double points of G

$$F(v_1^{-1}(q_0))\times(\{q_0\}\times(0,1])\cup -F(v_0^{-1}(q_1))\times(\{q_1\}\times(0,1]).$$

Note that the orientation on $v_1^{-1}(q_0)$ (resp. on $v_0^{-1}(q_1)$), induced by v_1 (resp. by v_0) from the orientation on $S^{2k}=\partial D^{2k+1}\subset\{*\}\times D^{2k+1}\subset S^{4k-1}\times D^{2k+1}$, together with the positive direction of the interval $[0,1]$ (resp. $[-1,0]$), is coherent with the orientation on Δ_G (see Definition 2.2). Therefore, we see that the $(2k-1)$-dimensional set S_G of singular (Whitney umbrella) points of G is

$$S_G=F(v_1^{-1}(q_0))\times\{\mathbf{0}\}\cup -F(v_0^{-1}(q_1))\times\{\mathbf{0}\}\subset S^{4k-1}\times D^{2k+1}.$$

Now we compute Szűcs' linking $l(G)$ of G. The linking $l(G)$ is equal to the linking number of $G(W^{4k})$ and the union of each $F(v_j^{-1}(q_i))\times\{\mathbf{0}\}$ shifted slightly in the direction antipodal to $q_i\in S^{2k}$ for $(i,j)=(0,1),(1,0)$. Namely $l(G)$ is equal to

$$\begin{aligned}&\mathrm{lk}(G(W^{4k}),F(v_1^{-1}(q_0))\times\{(-q_0,\varepsilon)\}\cup -F(v_0^{-1}(q_1))\times\{(-q_1,\varepsilon)\})\\ &\quad=\mathrm{lk}(G(W^{4k}),F(v_1^{-1}(q_0))\times\{(-q_0,\varepsilon)\})-\mathrm{lk}(G(W^{4k}),F(v_0^{-1}(q_1))\times\{(-q_1,\varepsilon)\}),\end{aligned}$$

where $\varepsilon \in (0,1)$ is a small number. If we take a $2k$-chain Σ_1^{2k} in S^{4k-1} bounded by $v_1^{-1}(q_0)$, then we see that

$$\begin{aligned}
&\mathrm{lk}(G(W^{4k}), F(v_1^{-1}(q_0)) \times \{(-q_0,\varepsilon)\}) \\
&\quad = -\mathrm{lk}(F(v_1^{-1}(q_0)) \times \{(-q_0,\varepsilon)\}, G(W^{4k})) \\
&\quad = -[F(\Sigma_1^{2k}) \times \{(-q_0,\varepsilon)\} \cap \widetilde{F}_1(S^{4k-1} \times (0,1])] && (\in H_0(F(S^{4k-1}) \times D^{2k+1})) \\
&\quad = -[F(\Sigma_1^{2k}) \times \{(-q_0,\varepsilon)\} \cap F(v_1^{-1}(-q_0)) \times \{(-q_0,\varepsilon)\}] && (\in H_0(F(S^{4k-1}) \\
&&& \quad \times \{(-q_0,\varepsilon)\})) \\
&\quad = -[F(\Sigma_1^{2k}) \cap F(v_1^{-1}(-q_0))] && (\in H_0(F(S^{4k-1}))) \\
&\quad = -[\Sigma_1^{2k} \cap v_1^{-1}(-q_0)] && (\in H_0(S^{4k-1})) \\
&\quad = -\text{ the linking number of } v_1^{-1}(q_0) \text{ and } v_1^{-1}(-q_0) \text{ in } S^{4k-1},
\end{aligned}$$

which is nothing but the Hopf invariant of v_1. Since we have a similar argument on the southern part, we have

$$\begin{aligned}
-l(G) &= \text{the Hopf invariant of } v_1 - \text{the Hopf invariant of } v_0 \\
&= H_{\widetilde{F}_1} - H_{\widetilde{F}_0}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\bar{p}_k[W^{4k}] = \bar{p}_k[\widehat{V}_1^{4k}] - \bar{p}_k[\widehat{V}_0^{4k}], \\
&t(G) = t(\widetilde{F}_1) - t(\widetilde{F}_0) + [\Delta_{\widetilde{F}_1}] - [\Delta_{\widetilde{F}_0}], \\
&-l(G) = H_{\widetilde{F}_1} - H_{\widetilde{F}_0}.
\end{aligned}$$

By applying Theorem 2.3 to G, we have

$$-\bar{p}_k[W^{4k}] + 3t(G) - 3l(G) = 0.$$

Therefore, we see that

$$-\bar{p}_k[\widehat{V}_1^{4k}] + 3t(\widetilde{F}_1) + 3[\Delta_{\widetilde{F}_1}] + 3H_{\widetilde{F}_1} = -\bar{p}_k[\widehat{V}_0^{4k}] + 3t(\widetilde{F}_0) + 3[\Delta_{\widetilde{F}_0}] + 3H_{\widetilde{F}_0}.$$

This implies that the value $\Theta(F)$ does not depend on the choice of a Seifert immersion for F. Thus the map Θ is well-defined for the map F and clearly for the isotopy class of F. □

Remark 4.4. Each term of Θ is additive with respect to the connected sum of embeddings $S^{4k-1} \hookrightarrow S^{6k}$ and the boundary connected sum of their Seifert immersions. Therefore, we have actually the homomorphism $\Theta : C_{4k-1}^{2k+1} \to \mathbb{Z}$.

The following proposition implies that $\Theta : C_{4k-1}^{2k+1} \to \mathbb{Z}$ is a *non-trivial* homomorphism.

Proposition 4.5. *$\Theta(E) = -24$ for the generator E of C_{4k-1}^{2k+1} in Section* 3.2.

Proof. We compute each term of $\Theta(E)$ with respect to the Seifert immersion $\widetilde{E}: V_E^{4k} \looparrowright S^{6k}$ constructed in Section 3.3. We use the notation in Section 3.3.

(a) The $4k$-manifold V_E^{4k} is obtained by attaching to D^{4k} three $2k$-handles with trivial framings. Therefore we see that $\bar{p}_k[\widehat{V}_E^{4k}] = 0$.
(b) Clearly $\widetilde{E}: V_E^{4k} \looparrowright S^{6k}$ has one triple point with positive sign at the origin of $\mathbb{R}^{6k}$; hence $t(\widetilde{E}) = 1$.
(c) The set of double points of $\widetilde{E}$ is

$$\bigcup_{(i,j)=(1,2),(2,3),(3,1)} [(L_i \cap D_i) \cup ((D_j \setminus K_i) \cap D_i)],$$

(see Section 3.3 and Fig. 2). For $(i,j) = (1,2), (2,3), (3,1)$, put

$$J_i := (L_i \cap D_i) \cup ((D_j \setminus K_i) \cap D_i).$$

Then each J_1, J_2 and J_3 bounds $(2k+1)$-discs in the exterior $\mathbb{R}^{6k} \setminus \mathtt{S}$

$$M_1\colon \left\{ \mathbf{x} = 0, y_1 = \cdots = y_{2k-1} = 0, y_{2k} \geqslant 0, \frac{y_{2k}^2}{\alpha^2} + \frac{\mathbf{z}^2}{\beta^2} \leqslant (1-\varepsilon)^2 \right\},$$

$$M_2\colon \left\{ \mathbf{y} = 0, z_1 = \cdots = z_{2k-1} = 0, z_{2k} \geqslant 0, \frac{z_{2k}^2}{\alpha^2} + \frac{\mathbf{x}^2}{\beta^2} \leqslant (1-\varepsilon)^2 \right\},$$

$$M_3\colon \left\{ \mathbf{z} = 0, x_1 = \cdots = x_{2k-1} = 0, x_{2k} \geqslant 0, \frac{x_{2k}^2}{\alpha^2} + \frac{\mathbf{y}^2}{\beta^2} \leqslant (1-\varepsilon)^2 \right\},$$

where ε is the radius of a fibre of the normal disc bundle N of $\mathtt{S} \subset \mathbb{R}^{6k}$.

Therefore, to compute $[\Delta_{\widetilde{E}}] \in H^2(S^{6k} \setminus \mathtt{S})$, we have to consider the intersection of $\bigcup_i M_i$ and $\mathtt{S}$. The situation is very similar to the last step of the computation of the Hopf invariant for $\widetilde{E}$ (see the end of the proof of Proposition 3.5 and Fig. 3), but the orders of the considered intersections are opposite and this changes the signs of the intersection points. We can check that each intersection point in $M_i \cap \mathtt{S}$ has negative sign and hence that $[\Delta_{\widetilde{E}}] = -3$.
(d) By Proposition 3.5, $H_{\widetilde{E}} = -6$.
(e) Finally we have

$$\begin{aligned} \Theta(E) &= -\bar{p}_k[\widehat{V}^{4k}] + 3t(\widetilde{E}) + 3[\Delta_{\widetilde{E}}] + 3H_{\widetilde{E}} \\ &= 0 + 3 \times 1 - 3 \times 3 - 3 \times 6 \\ &= -24. \end{aligned}$$

This completes the proof. □

Since we already know that C_{4k-1}^{2k+1} is isomorphic to $\mathbb{Z}$ by the result of Haefliger, as an easy consequence of Propositions 4.3 and 4.5, we have:

Theorem 4.6. *The following is an isomorphism*:

$$\begin{aligned} \Omega : C_{4k-1}^{2k+1} &\to \mathbb{Z}, \\ F &\mapsto -\tfrac{1}{24}(-\bar{p}_k[\widehat{V}^{4k}] + 3t(\widetilde{F}) + 3[\Delta_{\widetilde{F}}] + 3H_{\widetilde{F}}), \end{aligned}$$

where $\widetilde{F}: V^{4k} \looparrowright S^{6k}$ *is a Seifert immersion for* F.

As a corollary, we obtain the following observation on Haefliger's generator.

Corollary 4.7. *In the construction of the generator E of C^{2k+1}_{4k-1} given in Section* 3.2, *if we change one of the two tubes joining S_1, S_2, S_3 into the orientation-reversing one, then the resulting embedding E' represents $-1 \in C^{2k+1}_{4k-1} = \mathbb{Z}$.*

Proof. We can construct a Seifert immersion $\widetilde{E}'$ for E' in the same way as in Section 3.3. In the computations in Propositions 3.5 and 4.5, we can check that, with respect to this Seifert immersion $\widetilde{E}'$, all the terms of Ω, except the normal Pontrjagin number, have the opposite signs of those for $\widetilde{E}$. Thus we have $\Omega(E') = -1$. □

5. The Hopf invariants for punctured $4k$-manifolds embedded in $6k$-space

It is clear that the Hopf invariant in our formula can be considered for oriented $4k$-manifolds with spherical boundaries embedded in S^{6k}. In this section, for such an embedded $4k$-manifold, we develop a method to compute its Hopf invariant by means of the normal Euler class. The purpose of this section is to show the following theorem.

Theorem 5.1. *Let $\widehat{V}^{4k}$ be a closed oriented $4k$-manifold and put $V^{4k} := \widehat{V}^{4k} \setminus \mathrm{Int}\, D^{4k}$. Let $\widetilde{F} : V^{4k} \hookrightarrow S^{6k}$ be an embedding with normal Euler class $e_{\widetilde{F}} \in H^{2k}(V^{4k}) = H^{2k}(\widehat{V}^{4k})$. Then, the Hopf invariant $H_{\widetilde{F}}$ (along the boundary) is equal to $e_{\widetilde{F}} \smile e_{\widetilde{F}} \in H^{4k}(\widehat{V}^{4k}) = \mathbb{Z}$.*

Theorem 5.1 is obtained from the following two Lemmas 5.2 and 5.3. Put $F := \widetilde{F}|_{\partial V^{4k}} : S^{4k-1} \hookrightarrow S^{6k}$, whose exterior space (as in the proof of Proposition 3.5) we denote by X. Let Σ^{2k} be a $2k$-cycle lying in $\mathrm{Int}\, V^{4k}$ such that $[\Sigma^{2k}] \in H_{2k}(V^{4k}) = H_{2k}(\widehat{V}^{4k})$ is the integral dual to $e_{\widetilde{F}} \in H^{2k}(\widehat{V}^{4k})$; $e_{\widetilde{F}} = D([\Sigma^{2k}])$.

Lemma 5.2. *$H_{\widetilde{F}}$ is equal to $-[\widetilde{F}(\Sigma^{2k})] \in H_{2k}(X) = \mathbb{Z}$.*

Proof. The argument here is similar to the one in the proof of Proposition 3.5; we use the same symbols and diagrams.

The corresponding dual in $H^{4k}(X, \partial X) \approx H^{4k}(S^{6k}, F(S^{4k-1}))$ to the generator $\omega \in H^{2k}(X)$ is represented by $\widetilde{F}(V^{4k})$. Take the inward normal field of $F(S^{4k-1}) \subset \widetilde{F}(V^{4k})$ and extend to a (not necessarily non-zero) vector field on $\widetilde{F}(V^{4k})$. Perturb $\mathtt{V} := \widetilde{F}(V^{4k})$ to $\mathtt{V}'$ along this field. Then $\widetilde{F}^{-1}(\mathtt{V} \cap \mathtt{V}') \subset V^{4k}$, which we can assume to be lying in $\mathrm{Int}\, V^{4k}$, represents the integral dual to $e_{\widetilde{F}}$. Hence, we can put $\Sigma^{2k} = \widetilde{F}^{-1}(\mathtt{V} \cap \mathtt{V}')$.

First, since a sufficiently small tubular neighbourhood of $\partial \mathtt{V}' \subset S^{6k}$ does not intersect $\mathtt{V}$, we see that the term corresponding to $\zeta^{\sharp}\omega'$ and hence the second term of (2) in Section 2.2 vanish.

Next we check the first term of (2) in Section 2.2. Again by the same argument as in the proof of Proposition 3.5, the desired term is represented by the intersection of $F(S^{4k-1})$ and a relative

$(2k+1)$-chain bounded by $\mathtt{V} \cap \mathtt{V}' = \widetilde{F}(\Sigma^{2k})$, which is further equal to

$$-\mathrm{lk}(\widetilde{F}(\Sigma^{2k}), F(S^{4k-1})) = -[\widetilde{F}(\Sigma^{2k})] \in H_{2k}(X) = \mathbb{Z}.$$

Thus, we have $H_{\widetilde{F}} = -[\widetilde{F}(\Sigma^{2k})]$. □

Lemma 5.3. *$-[\widetilde{F}(\Sigma^{2k})] \in H_{2k}(X) = \mathbb{Z}$ is equal to $e_{\widetilde{F}} \smile e_{\widetilde{F}} \in H^{4k}(\widehat{V}^{4k}) = \mathbb{Z}$.*

Proof. Take two copies $\mathtt{V}'$ and $\mathtt{V}''$ of $\mathtt{V} := \widetilde{F}(V^{4k})$ perturbed in appropriate manners, and put $\Sigma := \mathtt{V} \cap \mathtt{V}'$ and $\Sigma' := \mathtt{V} \cap \mathtt{V}''$. Then, we see that

$$\begin{aligned}
-[\widetilde{F}(\Sigma^{2k})] = -[\Sigma] &= -[\Sigma'] && (\in H_{2k}(X) = \mathbb{Z})\\
&= -\mathrm{lk}(\Sigma', F(S^{4k-1})) && (\in H_0(S^{6k}) = \mathbb{Z})\\
&= \mathrm{lk}(F(S^{4k-1}), \Sigma') &&\\
&= [\mathtt{V} \cap \Sigma'] &&\\
&= [\mathtt{V} \cap \mathtt{V}' \cap \mathtt{V}''] &&\\
&= [\Sigma \cap \Sigma'] && (\in H_0(\widetilde{F}(V^{4k})) = \mathbb{Z})\\
&= [\Sigma^{2k}] \cdot [\Sigma^{2k}] && (\in H_0(V^{4k}) = \mathbb{Z})\\
&= D^{-1}(e_{\widetilde{F}} \smile e_{\widetilde{F}}) && .
\end{aligned}$$

This completes the proof. □

6. Punctured $S^{2k} \times S^{2k}$s embedded in S^{6k}

In this section, we apply the result of the previous section to the punctured $S^{2k} \times S^{2k}$. As a consequence, we show that any (isotopy class of) embedding of S^{4k-1} in S^{6k} can be seen as the boundary of the punctured $S^{2k} \times S^{2k}$ embedded in S^{6k} with suitable normal bundle, and hence that every embedding $F : S^{4k-1} \hookrightarrow S^{6k}$ extends to an embedding of a compact oriented $4k$-manifold in S^{6k}.

Throughout this section, we denote the punctured $S^{2k} \times S^{2k}$ by $M^{4k} := S^{2k} \times S^{2k} \setminus \mathrm{Int}\, D^{4k}$. Note that M^{4k} is stably parallelisable and hence parallelisable.

Proposition 6.1. *The normal Euler class of an immersion $M^{4k} \looparrowright \mathbb{R}^{6k}$ is of the form $2c^{2k}$ for some $c^{2k} \in H^{2k}(M^{4k})$. Furthermore, for an arbitrary $c^{2k} \in H^{2k}(M^{4k})$, there exists an embedding $M^{4k} \hookrightarrow \mathbb{R}^{6k}$ with normal Euler class $2c^{2k}$.*

Proof. Since M^{4k} is parallelisable, by taking a trivialisation $TM^{4k} \cong M^{4k} \times \mathbb{R}^{4k}$, we can associate to each immersion $f : M^{4k} \looparrowright \mathbb{R}^{6k}$ the map $\overline{df} : M^{4k} \to V_{6k,4k}$ (the Stiefel manifold of all $4k$-frames in $6k$-space), naturally determined by the differential df. This, by the Smale–Hirsch classification theorem, gives a bijective correspondence between the set of regular homotopy classes of immersions $M^{4k} \looparrowright \mathbb{R}^{6k}$ and the set of homotopy classes of maps $M^{4k} \to V_{6k,4k}$.

Consider the canonical S^{2k-1}-bundle $\pi: V_{6k,4k+1} \to V_{6k,4k}$. Since we can compute $H^1(V_{6k,4k}) = 0$, $H^{2k}(V_{6k,4k}) \approx \mathbb{Z}$, $H^{2k-1}(V_{6k,4k+1}) = 0$ and $H^{2k}(V_{6k,4k+1}) \approx \mathbb{Z}_2$, by the Gysin exact sequence

$$0 \to H^0(V_{6k,4k}) \overset{\smile e_\pi}{\to} H^{2k}(V_{6k,4k}) \to \mathbb{Z}_2 \to 0,$$

we see that the Euler class e_π of π is equal to 2σ for a generator $\sigma \in H^{2k}(V_{6k,4k}) \approx \mathbb{Z}$. This proves the first part of the theorem, since the normal bundle of any immersion $f: M^{4k} \looparrowright \mathbb{R}^{6k}$ is induced from π by the associated map $\overline{df}$.

Denote by $\xi: S^{2k} \to V_{6k,4k}$ a map representing a generator of $\pi_{2k}(V_{6k,4k}) \approx \mathbb{Z}$. Then for any $c^{2k} \in H^{2k}(M^{4k})$ we can construct a map $g: M^{4k} \to V_{6k,4k}$ such that $g^*\sigma = c^{2k}$, by using the map ξ. Again the Smale–Hirsch h-principle implies that there exists an immersion $f: M^{4k} \looparrowright \mathbb{R}^{6k}$ such that the associated map $\overline{df}: M^{4k} \to V_{6k,4k}$ is homotopic to the map g; the normal Euler class of f is equal to $g^*e_\pi = 2g^*\sigma = 2c^{2k}$.

Finally, by Haefliger and Hirsch [10, Theorem 3.1(a)], the immersion f is regularly homotopic to an embedding because $S^{2k} \times S^{2k}$ is $(2k-1)$-connected. Since a regular homotopy does not change the normal bundle, the second part of the theorem is proved. □

The following shows that every embedding $S^{4k-1} \hookrightarrow S^{6k}$ extends to an embedding of M^{4k} with suitable normal bundle, whose Euler class retains complete information on the isotopy class of the given embedding.

Corollary 6.2. *Let $\widetilde{E}_{a,b}: M^{4k} \hookrightarrow S^{6k}$ be an embedding with normal Euler class $(2a, 2b) \in H^{2k}(M^{4k}) = H^{2k}(S^{2k} \times S^{2k}) \approx \mathbb{Z} \oplus \mathbb{Z}$. Then $E_{a,b} := \widetilde{E}_{a,b}|_{\partial M^{4k}}: S^{4k-1} \hookrightarrow S^{6k}$ represents $-ab \in C_{4k-1}^{2k+1} = \mathbb{Z}$. In particular, $E_{\pm 1,\mp 1}: S^{4k-1} \hookrightarrow S^{6k}$ represents the generator of C_{4k-1}^{2k+1}.*

Proof. By Theorem 5.1, we have

$$\begin{aligned}\Omega(E_{a,b}) &= -\tfrac{1}{24}(0 + 0 + 0 + 3H_{\widetilde{E}_{a,b}}) \\ &= -\tfrac{1}{24} \cdot 3[(2a, 2b) \smile (2a, 2b)] \in H^{4k}(S^{2k} \times S^{2k}) = \mathbb{Z} \\ &= -\tfrac{1}{24} \cdot 3 \cdot 8ab = -ab \in \mathbb{Z}.\end{aligned}$$

Thus $E_{a,b}: S^{4k-1} \hookrightarrow S^{6k}$ is a representative of $-ab \in C_{4k-1}^{2k+1} = \mathbb{Z}$. □

This corollary implies that the Seifert immersion $\widetilde{F}: V^{4k} \looparrowright S^{6k}$ in Theorem 4.6 can actually be chosen to be an embedding. The following is an immediate corollary of Theorems 4.6, 5.1 and Corollary 6.2.

Corollary 6.3. (a) *For an arbitrary embedding $F: S^{4k-1} \hookrightarrow S^{6k}$, there exists an embedding $\widetilde{F}: V^{4k} \hookrightarrow S^{6k}$ of a compact oriented $4k$-manifold V^{4k} with $\partial V^{4k} = S^{4k-1}$ such that $\widetilde{F}|_{\partial V^{4k}} = F$. Furthermore,*

$$\begin{aligned}\Omega(F) &= -\tfrac{1}{24}(-\bar{p}_k[\widehat{V}^{4k}] + 3H_{\widetilde{F}}) \\ &= -\tfrac{1}{24}(-\bar{p}_k[\widehat{V}^{4k}] + 3e_{\widetilde{F}} \smile e_{\widetilde{F}})\end{aligned}$$

gives an isomorphism $\Omega: C_{4k-1}^{2k+1} \to \mathbb{Z}$, where $e_{\widetilde{F}} \in H^{2k}(V^{4k}) = H^{2k}(\widehat{V}^{4k})$ is the normal Euler class for $\widetilde{F}$ and $e_{\widetilde{F}} \smile e_{\widetilde{F}} \in H^{4k}(\widehat{V}^{4k}) = \mathbb{Z}$ is the cup product.

(b) *Actually, we can extend any embedding* $F : S^{4k-1} \hookrightarrow S^{6k}$ *to an embedding* $\widetilde{F} : S^{2k} \times S^{2k} \setminus \operatorname{Int} D^{4k} \hookrightarrow S^{6k}$ *and then*

$$\begin{array}{llll} \Omega : & C^{2k+1}_{4k-1} & \longrightarrow & \mathbb{Z} \\ & F & \longmapsto & -\frac{1}{8} H_{\widetilde{F}} \end{array}$$

is an isomorphism.

Remark 6.4 (see also Section 7). In proving the existence of an "embedded" Seifert immersion for an *arbitrary* embedding $S^{4k-1} \hookrightarrow S^{6k}$, we cannot count on the same method as the construction of a usual Seifert surface in the codimension two case. The usual construction gives rise to a *normally framed* submanifold, but if $F(S^{4k-1})$ indeed bounds a normally framed $V^{4k} \subset S^{6k}$ we have the constraints that $\bar{p}_k[\widehat{V}^{4k}]$ should be divisible by $j_k a_k (2k-1)!$ (Milnor and Kervaire [16], for j_k and a_k see Theorem 7.1) and that $H_{\widetilde{F}} = 0$ (Theorem 5.1) and hence the invariant $\Omega(F)$ is divisible by $j_k a_k (2k-1)!/24$.

In the case C^3_3 ($k = 1$), by the Hirzebruch index theorem, we have $-\bar{p}_1[\widehat{V}^4] = p_1[\widehat{V}^4] = 3\sigma(V^4)$ where $\sigma(V^4)$ is the signature of V^4. Hence, we can rewrite Corollary 6.3 as follows.

Corollary 6.5. *Every embedding* $S^3 \hookrightarrow S^6$ *extends to an embedding* $\widetilde{F} : V^4 \hookrightarrow S^6$ *of a compact oriented* 4*-manifold* V^4*, and*

$$\begin{aligned} \Omega(F) &= -\tfrac{1}{8}(\sigma(V^4) + H_{\widetilde{F}}) \\ &= -\tfrac{1}{8}(\sigma(V^4) + e_{\widetilde{F}} \smile e_{\widetilde{F}}) \end{aligned}$$

gives an isomorphism $\Omega : C^3_3 \longrightarrow \mathbb{Z}$.

7. The suspension of the usual knot theory

Corollary 6.3 reminds us of the similar formula for embeddings in codimension two (up to regular homotopy) given in [13]. In fact, we obtain the following observation in connection with the usual knot theory.

Theorem 7.1. *The image of the map* $i_* : C^2_{4k-1} \to C^{2k+1}_{4k-1}$ *induced by the inclusion* $i : S^{4k+1} \hookrightarrow S^{6k}$ *corresponds to*

$$\frac{j_k a_k (2k-1)!}{24} \mathbb{Z} \subset \mathbb{Z} \approx C^{2k+1}_{4k-1},$$

where j_k *denotes the order of the image of the Hopf–Whitehead* J*-homomorphism* $J : \pi_{4k-1}(SO(4k+1)) \to \pi_{8k}(S^{4k+1})$*, and* $a_k = 2$ *if* k *is odd,* $a_k = 1$ *if* k *is even.*

Proof. Let $f : S^{4k-1} \hookrightarrow S^{4k+1}$ be an embedding and $\widetilde{f} : V^{4k} \hookrightarrow S^{4k+1}$ a (usual) Seifert surface for f. Then, by Hughes and Melvin [13], we see that $\bar{p}_k[\widehat{V}^{4k}]$ is invariant up to regular homotopy and hence up to isotopy. Furthermore, $\widehat{V}^{4k}$ is always almost parallelisable and we have the epimorphism [13]:

$$\Omega' : \quad C^2_{4k-1} \longrightarrow \mathbb{Z}$$
$$f \longmapsto \frac{1}{j_k a_k (2k-1)!}\, \bar{p}_k[\widehat{V}^{4k}].$$

On the other hand, $i \circ \widetilde{f} : V^{4k} \hookrightarrow S^{6k}$ is a Seifert immersion for the embedding $i \circ f : S^{4k-1} \hookrightarrow S^{6k}$, where $i : S^{4k+1} \hookrightarrow S^{6k}$ is the inclusion. Then, by a certain naturality property of the functional cup product (see e.g. [25, p. 498]), we see that the Hopf invariant $H_{i\circ\widetilde{f}}$ for $i \circ \widetilde{f}$ vanishes, since the map $v_{i\circ\widetilde{f}} : S^{4k-1} \to S^{6k} \backslash f(S^{4k-1})$ is factorised by the map to the space $S^{4k+1} \backslash f(S^{4k-1})$, which has the homology type of S^1. Hence we have $\Omega(i \circ f) = \frac{1}{24}\, \bar{p}_k[\widehat{V}^{4k}]$.

Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccc} C^2_{4k-1} & \xrightarrow[\text{epi.}]{\Omega'} & \mathbb{Z} \\ {\scriptstyle i_*}\downarrow & & \downarrow {\scriptstyle \times \frac{j_k a_k (2k-1)!}{24}} \\ C^{2k+1}_{4k-1} & \xrightarrow[\text{iso.}]{\Omega} & \mathbb{Z}, \end{array}$$

which implies that the image of the map $i_* : C^2_{4k-1} \to C^{2k+1}_{4k-1}$ corresponds to $(j_k a_k (2k-1)!/24)\mathbb{Z} \subset \mathbb{Z} = C^{2k+1}_{4k-1}$. □

Remark 7.2. It is well known [1] that j_k is equal to the denominator of $|B_k|/4k$, where B_k is (the reduced expression of) the kth Bernoulli number. Furthermore, as a consequence of the theorem by Clausen and von Staudt, the denominator of B_k is divisible by 6. Therefore, we see that $j_k a_k (2k-1)!$ is always divisible by 24.

Remark 7.3. According to Haefliger [9, Corollary 6.7], C^q_n is finite except that C^q_{4k-1} has a $\mathbb{Z}$-component for $2 < q \leqslant 2k+1$, and $i_* : C^3_{4k-1} \otimes \mathbb{Q} \to C^{2k+1}_{4k-1} \otimes \mathbb{Q}$ is an isomorphism.

Remark 7.4. In the case $k = 1$, the theorem above implies that the image of the map $C^2_3 \to C^3_3$ induced by the inclusion $S^5 \hookrightarrow S^6$ corresponds to $2\mathbb{Z} \subset \mathbb{Z} \approx C^3_3$. This is stated in Haefliger [9, Theorem 5.17].

Remark 7.5. The proof of the theorem above implies that two embeddings f and $g : S^{4k-1} \hookrightarrow S^{4k+1}$ are regularly homotopic if and only if their compositions with the inclusion $i \circ f$ and $i \circ g : S^{4k-1} \hookrightarrow S^{6k}$ are isotopic.

Addendum. Boéchat [2] and Boéchat–Haefliger [3] have related the group C^{2k+1}_{4k-1} to the obstruction to smoothing a “semi-differentiable” embedding of a closed connected oriented $4k$-dimensional manifold in $\mathbb{R}s^{6k+1}$. It seems that Corollaries 6.2, 6.3 and 6.5 can be deduced from their results, combined with Proposition 6.1 (see also [7, p. 95] for the case $k = 1$). Also, the computation $H_{\tilde{E}} = -6$ in Proposition 3.5 has been carried out in [2].

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