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SYMMETRY INDUCED BY ECONOMY

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Abstract—Various extremum problems are presented which lead to highly symmetric geometrical configurations.

1. INTRODUCTION

The word "symmetry" evokes the amazing structure of the bees' honeycomb, the wall ornaments of ancient Egyptians and Moors, the Platonic solids, and other regular figures produced by man or nature which all incarnate perfection of symmetry. Due to their intrinsic beauty, and to their close connection with natural science and mathematics, regular figures attracted attention through the ages.

An intelligent schoolboy, trying to construct cardboard models of solids with equal regular faces would hardly fail to rediscover the regular solids. He also would come across the trigonal and pentagonal dipyramids and the few further convex solids with regular triangular faces. But he certainly would realize the higher degree of symmetry of the five solids with equivalent vertices. This may have been the way the Greeks discovered the regular solids.

Generally, a configuration is said to be regular if it consists of equivalent components. The trigonal dipyramid consists of equivalent faces but is not regular according to the traditional definition of regular solids. In the classical theory of regular figures we start with a definition of regularity, and try to give a complete enumeration of the respective figures. This theory, which may be called the *systematology* of regular figures, is contrasted by the *genetics* of regular figures which is based on the perception that certain economy postulates, in a sufficiently wide sense, imply regularity. Here regular figures are not defined but they come into being from irregular figures, and unordered chaotic sets in virtue of the ordering effect of an extremum requirement. In what follows we try to illustrate this theory by some examples.

We shall use the Schläfli symbol $\{p, q\}$ both for a regular polyhedron and a regular tiling with p-gonal faces and q-valent vertices. Polyhedra and tilings with regular faces and equivalent vertices are called uniform. They are denoted by a symbol $(l, m, \ldots)$ giving the number of sides of the faces around one vertex in their cyclic order.

2. POLYHEDRA AND SPHERICAL CONFIGURATIONS

The ancient legend about surrounding the site of Carthage by straps cut out of the skin of a steer suggests the problem of maximizing the area of an n-gon of given perimeter. The solution to this problem is known to be the regular n-gon. We phrase two analogous problems in space: Among the convex polyhedra of given surface-area having (i) a given number f of faces, (ii) a given number v of vertices find that one of maximal volume. It is known[9,11,15,21] that for $f = 4$, 6, and 12 the solutions are given by the respective regular solids ($\{3, 3\}$, $\{4, 3\}$, $\{5, 3\}$), and the same is conjectured to be true for $v = 6$, and 12 ($\{3, 4\}$, $\{3, 5\}$).

J. Steiner (1796–1863) proved a weaker statement for the octahedron: Among all polyhedra of the topologic type of the regular octahedron the regular one is the best. All attempts to prove Steiner's conjecture about the same extremum property of the regular icosahedron failed so far. This shows the difficulties often involved in similar problems.

However several inequalities are known which express extremum properties of all five Platonic solids[11, 15, 20]. We recall the simplest one: Let R be the circumradius and r the inradius of a convex polyhedron with f faces, v vertices, and $e(= f + v - 2)$ edges. Let $p = 2e/f$ be the average number of sides of the faces, and $q = 2e/v$ the average number of edges meeting at the vertices. Then

$$\frac{R}{r} \geq \tan \frac{\pi}{p} \tan \frac{\pi}{q}$$

and equality holds only for the five regular solids. This implies that among the convex polyhedra with either 4, 6 or 12 faces or 4, 6, or 12 vertices the respective regular solid approximates best the shape of the sphere in the sense of minimizing R/r.

There are some extremum requirements posed on all convex polyhedra irrespective of the number of faces and vertices which is fulfilled by one or another Platonic solid. As an example we mention the following theorem: Among all convex polyhedra containing a ball, the circumscribed cube has the least total edge-length[2].

On the pollen-grains of flowers there are small orifices. When the pollen-grain sticks to the stigma from an orifice near the point of adhesion a tube outgrows to the female nucleus enabling the process of fertilization. Some flowers have spherical pollen-grains on which the orifices are rather uniformly distributed. Trying to explain the peculiar arrangements of the orifices, the Dutch biologist Tammes made the hypothesis that on each grain nature tries to produce the maximal number n of orifices under the condition that no two orifices are allowed to get nearer to one another than a certain distance depending only on the species. Now the question arises about the smallest sphere which accommodates n orifices under the above condition. This problem is equivalent with the following: On the unit sphere distribute n points so as to maximize the least distance between pairs of them. This problem is often referred to as the problem of Tammes.

The problem was investigated by several authors[5,8,26,29], and completely solved for $n \leq 12$, and $n = 24$. We emphasize the cases of $n = 3, 4, 5$, and 12 points when the best arrangements are given by the vertices of $\{3, 2\}$, $\{3, 3\}$, $\{3, 4\}$ and $\{3, 5\}$, and the cases of $n = 8$, and 24 points which lead as solutions to the vertices of the uniform polyhedra (3, 3, 3, 4) and (3, 3, 3, 3, 4) (see Fig. 1).

Let p_n be the convex hull of the extremal set of n points. The sequence $P_4, P_5, \ldots$ can be considered as a natural extension of the set of the trigonal Platonic solids. Polarity with respect to the sphere yields a similar extension of the set of the trihedral Platonic solids. However, there is a great variety of other extremum problems which provide essentially different "generalisations" of the notion of regular solids.

The problem of Tammes can be reformulated as follows: On the sphere find the densest packing of n equal circles (spherical caps). The *density* of a set of domains lying on the sphere is defined by the total area of the domains divided by the surface-area of the sphere. If no two domains overlap then they are said to form a *packing*. Thus the problem is to find the biggest circle whose n congruent replicas can be placed on the sphere without overlapping each other.

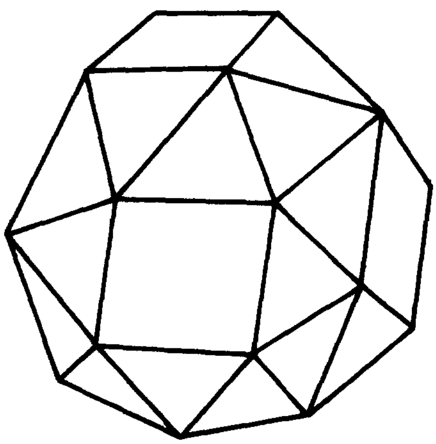

Fig. 1. The uniform solid (3, 3, 3, 3, 4).

The following formulation of the problem suggests its connection with stereochemistry: What is the maximal number of non-overlapping equal balls of prescribed radius which can be brought in contact with a unit ball?

Analogous problems arise in higher dimensions in information theory. For the efficient transmission of informations through a noisy channel we have to design a code. Suppose that we need n code words each consisting of a sequence of d pulses of discrete voltage levels. The code words can be represented in the d-dimensional space as points whose coordinates are the d voltage levels. The energy needed to transmit a pulse is proportional to the square of the voltage level. So the total power required to transmit one code word is proportional to the square of the distance between the origin and the point representing the code word. It is convenient to choose code words whose transmission requires the same energy. This means that we have to choose the n points on the boundary of a d-dimensional ball. Another requirement is to choose the code words so that they could be well distinguished from one another. Under the supposition of a background noise of constant intensity this requirement turns out to be equivalent with distributing the points so that no two should get closer to each other than a prescribed distance. Adding the last condition of minimizing the total energy which is needed to transmit an information, we have the problem of finding the smallest d-dimensional ball whose boundary can hold n points under the above condition.

It is interesting to note that the problem of Tammes, as well as its d-dimensional analogue were brought up independently of their applications by pure geometric considerations, and were solved in some highly interesting cases by geometrical methods. Among others it turned out[3] that the extremal distribution of 120 points on a four-dimensional ball is given by the vertices of the regular 600-cell, one of the four-dimensional analogues of the Platonic solids discovered by L. Schläfli in the middle of the last century. But recently even more efficient methods were developed based on sophisticated considerations in analysis. We will return to some results obtained by this method later.

We still recall a jocular interpretation of the problem of Tammes[23]: Over a planet n inimical dictators bear rule. How should the residences of these gentlemen be distributed so as to get as far from one another as possible? Our next problem is the problem of the allied dictators, who want to set up their residences so as to control the planet as well as possible. More exactly, on a sphere mark n points so as to minimize the greatest distance between a point of the sphere and the mark nearest to it.

The solution is known for $n \leq 7$ and $n = 10$, 12 and 14. For $n = 3$, 4, 6 and 12 we have the vertices of {3, 2}, {3, 3}, {3, 4} and {3, 6}, similarly as in the problem of the inimical dictators. The solution of the remaining cases can be summarized along with the cases when $n = 6$ and 12 as follows. For $n = 5$, 6 and 7 we have the vertices of a dipyramid, and for $n = 10$, 12 and 14 the vertices of an "antiprismatic dipyramid" (Fig. 2) with the respective number of vertices[6, 28].

It is not difficult to show that there are only finitely many numbers n such that the problems of n inimical and n allied dictators have identical solutions, and it is conjectured that the only such numbers are 2, 3, 4, 5, 6 and 12[7].

Some dome structures and geometrical sculptures consist of spherical circle packings[30]. Here the problem arises of minimizing the total material of the circles simultaneously requiring

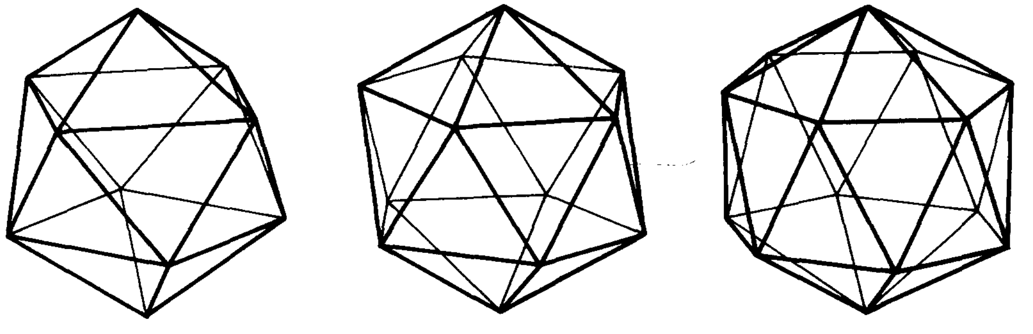

Fig. 2. Antiprismatic dipyramids with 10, 12, and 14 vertices.

a prescribed stability of the framework. The circles need not to be congruent but for practical reasons their size must be bounded from below. In order to formulate an exact problem we introduce a notion.

On the unit sphere let $P = \{c_1, \ldots, c_n\}$ be a *locally stable packing* of circles defined by the property that each circle is fixed by the others so that no circle of P can be moved alone without overlapping another circle of P. On the boundary of c_i let a be a greatest arc which is not touched by a circle c_j, $j \neq i$. Let $2\lambda_i$ be the angle subtended by a at the center of c_i. We call λ_i the *lability* of c_i, and define the lability λ of P by $\lambda = \max_{1 \le i \le n} \lambda_i$. Since the smallest circle is touched by at most five, and at least two circles, we have $\pi/5 \le \lambda \le \pi/2$.

The problem we are interested in is to find among all locally stable packings of circles with lability not exceeding a prescribed bound λ, and radii not less than a given value r that one of minimal density. For various particular values of λ and r this problem leads to many regular configurations. We consider the locally stable packing consisting of equal circles centered at the vertices of any of the following tilings: $\{4, 3\}$, $\{5, 3\}$, (3, 4, 4), (3, 6, 6), (3, 8, 8), (3, 10, 10), (3, 4, 4, 4), (4, 6, 6), (5, 6, 6). Let λ_0 and r_0 be the lability and radius of the circles. Then for $\lambda = \lambda_0$ and $r = r_0$ the solution to the problem is the respective packing under consideration[18]. The packing generated by the "football tiling" (5, 6, 6) is exhibited in Fig. 3.

Before discussing our last problem about spherical arrangements we make a digression. In an Euclidean or non-Euclidean space let B be a ball. I called the maximal number of congruent non-overlapping copies of B which can touch B the *Newton number* of B[16]. This name refers to the controversy between Newton and D. Gregory about the Newton number of an ordinary ball which, 180 years later, was proved to be twelve as claimed by Newton. In a non-Euclidean space the Newton number depends besides the dimension on the size of the ball.

In a packing of circles $c_1, \ldots, c_n$ of radius r let c_i be touched by k_i circles. Our problem is to find the maximum of the average number of points of contact:

$$M(r) = \max \frac{1}{n} (k_1 + \ldots + k_n)$$

extended over all packings of circles of radius r.

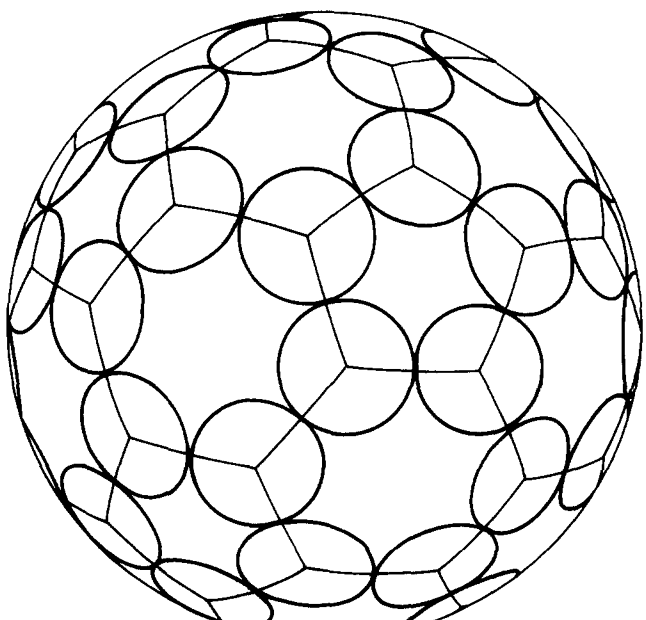

Fig. 3. A locally stable packing of circles centered at the vertices of the tiling (5, 6, 6).

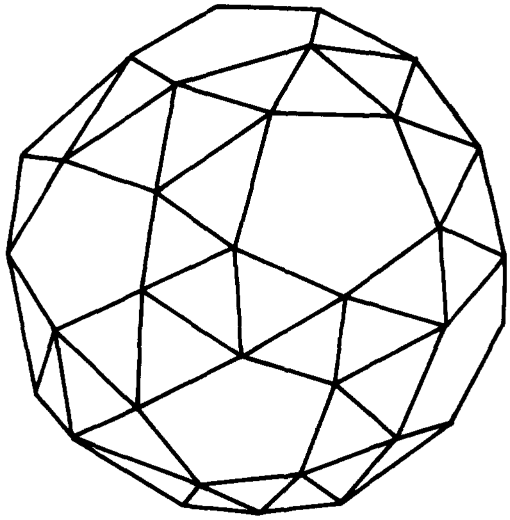

Fig. 4. The uniform solid (3, 3, 3, 3, 5).

Among the varied extremal packings special attention is due to those in which each circle is touched by as many circles as its Newton number. We call such a packing maximal neighbour packing, in short *maximal packing*.

A maximal packing of n equal circles exists only if $n = 2, 3, 4, 6, 8, 9, 12, 24, 48, 60$, or 120[16,27]. For $4 \leq n \leq 24$ the circles constitute a densest packing, and the same is conjectured to be true for $n = 48$ and 120. For $n = 60$ the circles are centered at the vertices of (3, 3, 3, 3, 5) (Fig. 4). The maximal packings of 48 and 120 circles have the same symmetry groups as the maximal packings of 24 and 60 circles, notably the rotation groups of {3, 4} and {3, 5}, respectively. Accordingly, they exist in two enantiomorphous varieties.

We still emphasize a further particular case: we have $M(\pi/6) = 4$. The extremal packing is not unique. The centers of six circles are equally spaced on the equator, leaving room for three further circles on both hemispheres in two different ways: the packing is either symmetric with respect to the equator or with respect to the center of the sphere. The two configurations of unit balls which touch the central unit ball at the centers of these circles occur in nature in the crystal structure of some metals.

3. ARRANGEMENTS IN THE PLANE

We start with some fundamental concepts. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be vectors which span the d-dimensional Euclidean space. Applying the translations $k_1\mathbf{e}_1 + \ldots + k_d\mathbf{e}_d$ with all possible d-tuples of integers $k_1, \ldots, k_d$ to a body B, we obtain a lattice of translates of B. Lattice-translations are fundamental symmetry operations of all regular arrangements which extend through the whole space.

The *density* of an infinite set of bodies scattered through the whole space is defined by a limiting value. Instead of the exact definition we confine ourselves to the vivid interpretation of the density as the total volume of the bodies divided by the volume of the whole space.

After these general remarks we consider arrangements in the ordinary plane.

Let D be an arbitrary centro-symmetric convex disc. Among all possible regular or completely irregular packings of congruent copies of D we want to find a packing of maximal density. The answer is given by the following theorem[10,11,15]: The density of a packing of equal centro-symmetric convex discs never exceeds the density of the densest lattice-packing. Vividly expressed this means that at the command to fill the greatest possible part of the plane the disorderly lying discs will get into parallel position, and align in a lattice.

It must be noted that this interpretation is rough because the requirement of maximal density does not determine the packing uniquely. The regularity of the packing can be disturbed by "breaking lines" and other kinds of irregularities without changing the density in the whole plane. In addition there are special discs which allow a regular non-lattice-packing having the same density as the densest lattice-packing.

For general convex discs a similar theorem holds only for packings of translates of the disc[25].

In the 1930s German scientists studied the problem of drawing up the plan of economic human settlements laying the foundation of the so called location theory. Among others they raised the following problem. In a uniformly populated big country we want to plant a certain number n of factories which produce the same kinds of goods. Each point of the country is provided by the factory nearest to it. How should the factories be distributed so as to minimize the total haulage?

We try to formulate the problem exactly. Let D be a domain, P a point, and $f(x)$ a strictly increasing function defined for $x \geq 0$. We consider the moment of D with respect to P defined by $M(D, P) = \int_D f(PA)da$, where da is the area element at the point A. Let $P_1, \ldots, P_n$ be n points. Let D_i be the Dirichlet cell of P_i consisting of those points of D which are nearer to P_i than to any other point P_j. The problem is to distribute the points $P_1, \ldots, P_n$ so as to minimize the sum $\Sigma_{i=1}^{n} M(D_i, p_i)$.

It was conjectured that for great values of n we obtain the best distribution by putting the points in the vertices of a tiling {3, 6}. Prompted by purely geometrical considerations, the problem was raised and studied again confirming the correctness of the above conjecture[9,11].

J. Nigli gave a complete survey over the infinite connected regular circle-packings enumerating 31 types of such packings. Four of these packings are solutions to the problem of minimizing the density of an artibrary packing of circles under the condition that the lability of the packing (defined as on the sphere) should not exceed a prescribed value[15]. The respective packings consist of equal circles centered at the vertices of the tilings (3, 12, 12), (4, 8, 8), {6, 3} and {4, 4} (Fig. 5).

We call a circle-packing in which each circle is touched by at least k circles *k-neighbour packing*. Two consecutive rows in the densest lattice-packing of circles form a 4-neighbour packing with zero density. But any 5-neighbour packing of equal circles has positive density. What is the thinnest 5-neighbour packing of equal circles? The solution is another of the packings enumerated by Nigli[17] in which the circles are centered at the vertices of the tiling (3, 3, 3, 3, 6) (Fig. 6).

The regular shape of the honeycomb fascinated man ever and again. According to a widely spread (but questionable[14]) hypothesis, the bees aim at using the minimum amount of wax per cell. Since the bee-cells are deep as compared with the diameter of their openings, the problem of constructing not necessarily congruent "bee-cells" of given volume so as to minimize the total area of the cell-walls can be approximated by the problem of decomposing a plane region into a great but given number of convex polygons of equal area having minimal total perimeter. This problem leads to the tiling {6, 3}, in accordance with the shape of the honeycomb[13].

Similar problems are suggested by succulent vegetable tissues in which the cells are crammed tightly together in a part of space without filling it completely. Suppose that: (1) under the condition of equal constant surface-area the cells try to expand so as to maximize their total volume and (2) under the condition of equal constant volume the cell-walls try to contract so as to minimize their total surface-area. What shape and arrangement will the cells assume under these conditions?

In the stem of plants the cells are largely elongated in the axial direction. This propounds the two-dimensional analogues of the above problems in which the part of volume and surface-area are played by area and perimeter. In both problems we have an array of extremal packings depending on the prescribed perimeter p and area a of the cells which we consider as plastic convex discs packed into a domain. For a great number of cells the asymptotic behaviour of the two arrays are similar[12,15,19]. For small values of p and a the cells are equal small circles packed anyhow. Along with p and a the circles will increase, and at certain values of p and a will get into the densest packing forming the incircles of the faces of a tiling {6, 3}. Increasing forth p and a, the discs will turn into smooth hexagons which arise from the faces of {6, 3} by rounding off their corners with equal circular arcs. Finally these arcs will shrink to points so that the cells will fill the whole available room.

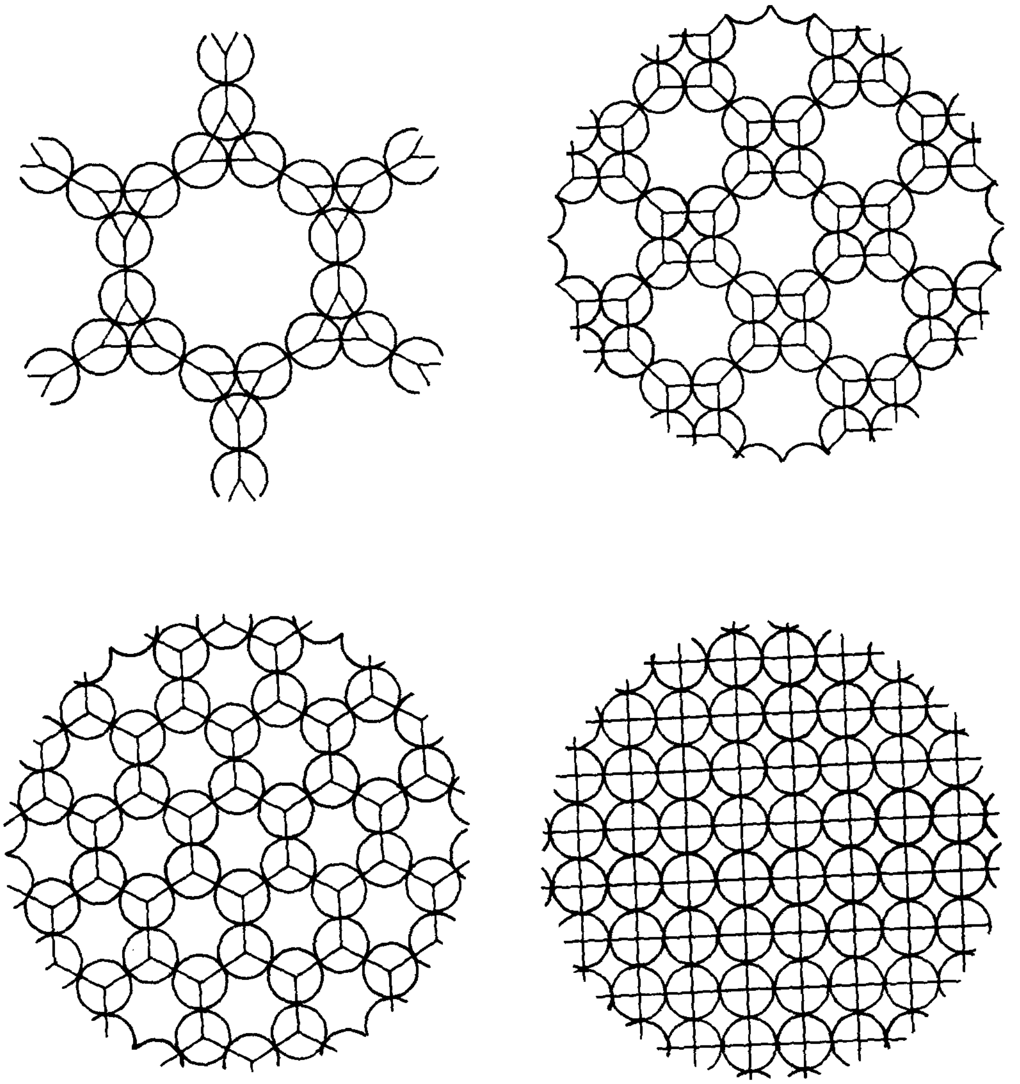

Fig. 5. Locally stable packings of circles centered at the vertices of (3, 12, 12), (4, 8, 8), {6, 3}, and {4, 4}.

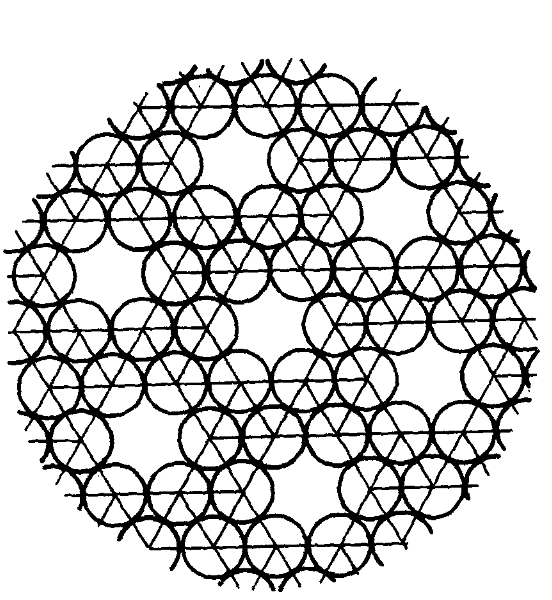

Fig. 6. Thinnest 5-neighbour packing of equal circles.

Observe the general conditions in the above problems: The congruences of the discs, their regular shape and regular arrangement are all induced by one simple and natural extremum requirement.

The configuration of smooth hexagons can be observed in microscopic sections of the stem of some plants.

4. ARRANGEMENTS IN THE SPACE

Little is known about the three-dimensional analogues of the problems considered in the plane. The difficulties inherent in these problems are demonstrated by the fact that even the problem of the densest packing of unit balls is a long-standing unsolved problem.

According to a well-founded conjecture among the solutions there are two regular packings of different types. Both are built up of hexagonal layers consisting of balls centered at the vertices of a tiling {3, 6} of edge-length two. The regular packings in question arise by putting the layers together so as to form a packing in which each ball is touched by twelve others in one of the two configurations described at the end of Sec. 2.

The layers can be put together also by letting the two configurations alternate from layer to layer in any order. Since the Newton number of a ball is twelve all these packings are maximal packings. It is conjectured that in three-dimensional Euclidean space all maximal packings consist of hexagonal layers[16].

Still the proof of this conjecture seems to be difficult. On the other hand, the method we referred to in connection with higher dimensional analogues of the problem of Tammes brought some surprising results.

Let N_d denote the Newton number of a d-dimensional ball. Obviously, we have $N_1 = 2$ and $N_2 = 6$, and we mentioned that $N_3 = 12$. These were the only values of N_d known until quite recently. A remarkable achievement of late years was the determination of the Newton number of the 8- and 24-dimensional ball[22,24]: $N_8 = 240$ and $N_{24} = 196560$. For no other values of $d > 3$ is the value of N_d known.

It turned out that the configurations of 240 and 196560 balls touching a central ball are unique[1]. Each one occurs in a particular lattice-packing which in 8-dimensional space is proved, and in 24-dimensional space is conjectured to be the densest lattice-packing. So these lattices are the unique maximal packings in the respective dimensions. In other dimensions higher than three we do not know whether a maximal packing exists at all or not.

In spite of the difficulties there is a hope of elaborating the genetics of three- and more-dimensional regular distributions. To conclude we present an encouraging result.

In our ordinary space let P be a packing of unit balls. Let r be the least upper bound of the radii of all balls disjoint to the balls of P. We call $1/r$ closeness of P. We want to find among all possible packings of unit balls the closest one, i.e. that one for which r assumes the least possible value. The solution to this problem is a lattice-packing. The centers of the balls form a so called body-centered cubic lattice which consists of the centers and the vertices of cubes constituting a face to face tiling of the space[4].

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