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Generalized Binomial Coefficients and the Subset–Subspace Problem

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Generalized binomial coefficients of the first and second kind are defined in terms of object selection with and without repetition from weighted boxes. The combinatorial definition unifies the binomial coefficients, the Gaussian coefficients, and the Stirling numbers and their recurrence relations under a common interpretation. Combinatorial proofs for some Gaussian coefficient identities are derived and shown to reduce to the ordinary binomial coefficients when $q = 1$. This approach provides a different perspective on the subset–subspace analogy problem. Generating function relations for the generalized binomial coefficients are derived by formal methods. © 1998 Academic Press

1. INTRODUCTION

The *binomial coefficients*, the *Gaussian coefficients*, and the *Stirling numbers* are three fundamental classes of numbers arising frequently in enumerative combinatorics (see, for example, [14]). The binomial coefficients have a well-known interpretation in terms of subset selection with or without repetition. The Gaussian coefficients have a classical interpretation related to counting subspaces of a finite vector space, as well as an interpretation in terms of row-reduced echelon form matrices. The Stirling numbers of the first kind count cycles in the cycle decomposition of permutations, while the Stirling numbers of the second kind count set partitions. Despite the various combinatorial interpretations of these classical numbers, the question still remains: given the *algebraic* similarities of these classes is there a unifying *combinatorial* generalization that captures the intrinsic properties of these numbers? The algebraic unification was started by Rota [12, 13], who gave the general definition of the characteristic polynomial of a geometric lattice. The binomial coefficients, the Gauss-

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ian coefficients, and the Stirling numbers appear as the coefficients in the characteristic polynomials of the following geometric lattices: subsets, subspaces, and set partitions, respectively. In this algebraic interpretation the generalized coefficients vary with the lattice and are called the *Whitney numbers of the first and second kind* (numbers of the first kind are associated with the characteristic polynomial, and numbers of the second kind with the rank polynomial, see [1] and [14]).

In this paper we will define two kinds of generalized binomial coefficients (the combinatorial analogues of the Whitney numbers) and unify *combinatorially* the basic properties of the binomial coefficients, the Gaussian coefficients, and the Stirling numbers. Specifically, the *generalized binomial coefficients of the first kind* will be defined in terms of selections from weighted boxes with the following constraints: no repeated boxes are selected and only one object is chosen from each selected box (i.e., we are counting *choice functions* on weighted boxes). The *generalized binomial coefficients of the second kind* are similarly defined except box repetition is allowed. By varying the weights of the boxes we obtain as special cases the classical combinatorial numbers. The generalized coefficients of the first kind include the ordinary binomial coefficients $\binom{n}{k}$ (box weights are constant), the Gaussian coefficients of the first kind, denoted

$$q^{\binom{k}{2}}\binom{n}{k}_q$$

(box weights are exponential), and the Stirling numbers of the first kind, denoted $\left[{n \atop k}\right]$ (box weights are linear). These numbers can be interpreted combinatorially in terms of general selection without repetition. The generalized coefficients of the second kind can be interpreted in terms of selection with repetition allowed. These coefficients include the binomial coefficients of the second kind (combinations with repetition), denoted here as $C^R(n,k)$, or $\binom{n+k-1}{k}$, the Gaussian coefficients of the second kind, denoted $\binom{n}{k}_q$, and the Stirling numbers of the second kind, denoted $\left\{{n \atop k}\right\}$.

Besides providing a combinatorial foundation for these classical numbers, we also hope to shed some significant light on the subset–subspace problem (see [6], [9], and [3]). The traditional approach to the subset–subspace problem has been to draw the following analogy: the binomial coefficient $\binom{n}{k}$ counts k-subsets of an n-set, while the analogous Gaussian coefficient $\binom{n}{k}_q$ counts the number of k-dimensional subspaces of an n-dimensional finite vector space over the field of q elements. The implication from this analogy is that the Gaussian coefficients and related identities tend to the analogous identities for the ordinary binomial coefficients as q approaches 1. The proofs are often algebraic or mimic subset proofs. But what is the combinatorial reason for the striking parallels between the Gaussian coefficients and the binomial coefficients? We will show that interpreting the Gaussian coefficients as generalized binomial coefficients of the *second kind* (combinations with repetition) reveals the combinatorial connections between not only the binomial coefficients and the Gaussian coefficients, but the Stirling numbers as well. Thus, the ordinary Gaussian coefficient $\binom{n}{k}_q$ tends to be an *algebraic* generalization of the binomial coefficient of the first kind $\binom{n}{k}$, and a *combinatorial* generalization of the binomial coefficient of the second kind $\binom{n+k-1}{k}$.

Before defining the generalized binomial coefficients we summarize some of the fundamental properties of the combinatorial numbers under discussion for future reference (see, for example, [4], [7], [11], [14], and [10]):

Binomial coefficients:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \qquad \binom{n}{k} = \binom{n}{n-k}. \tag{1}$$

Combinations with repetition:

$$C^R(n,k) = C^R(n-1,k) + C^R(n,k-1). \tag{2}$$

Gaussian coefficients:

$$\begin{gathered}\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q, \qquad \binom{n}{k}_q = \binom{n}{n-k}_q, \\ \binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.\end{gathered} \tag{3}$$

Stirling numbers (first kind):

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \tag{4}$$

Stirling numbers (second kind):

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = k\begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}. \tag{5}$$

What is the combinatorial significance of the coefficients in these recurrence relations? The recurrence relations are strikingly similar, but the combinatorial interpretations of these numbers are diverse. What are the generalized binomial coefficients that unify these numbers and relations?

2. THE GENERALIZED BINOMIAL COEFFICIENTS

Suppose we are given n distinct labeled boxes with box i $(1 \le i \le n)$ containing w_i distinct objects, where we have $1 \le w_1 \le w_2 \le \cdots \le w_n$. The number w_i is called the *weight* of box i. We assume that collectively the objects are distinct. Also, let $\mathbf{w}$ denote the sequence of given weights: $\mathbf{w} = (w_1, w_2, \ldots, w_n)$.

The *generalized binomial coefficient of the first kind* with weight $\mathbf{w}$, denoted $C_k^n(\mathbf{w})$, is defined as the number of ways to select k objects from k of the n boxes with the following constraints: select distinct boxes (order not important) and choose one object from each selected box (i.e., we are counting *choice functions* with domain size k). Suppose we choose k boxes $i_2 < i_2 < \cdots < i_k$. Then by the fundamental counting principle there are $w_{i_1} w_{i_2} \cdots w_{i_k}$ k-selections. Thus, summing over all ways of choosing k distinct boxes we have

$$C_k^n(\mathbf{w}) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} w_{i_1} w_{i_2} \cdots w_{i_k}. \tag{6}$$

The *generalized binomial coefficient of the second kind* with weight $\mathbf{w}$, denoted $S_k^n(\mathbf{w})$, is defined as the number of ways to select k objects from k boxes (not necessarily distinct) with the following constraint: choose one object from each selected box (i.e., sampling with replacement and box repetition allowed). The formula for $S_k^n(\mathbf{w})$ is similar to (6) but this time we sum over all ways of choosing k numbers from $\{1, 2, \ldots, n\}$ with repetition allowed, that is, over $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$. Thus,

$$S_k^n(\mathbf{w}) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} w_{i_1} w_{i_2} \cdots w_{i_k}. \tag{7}$$

First we give combinatorial proofs for the general recurrence relations for $C_k^n(\mathbf{w})$ and $S_k^n(\mathbf{w})$. They are generalizations of the recurrence relations for the ordinary binomial coefficients of the first and second kind (see Eqs. (1) and (2)).

THEOREM 1. *Let* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ *with* $1 \le w_1 \le w_2 \le \cdots \le w_n$. *Then*

$$\text{I.} \quad C_k^n(\mathbf{w}) = C_k^{n-1}(\mathbf{w}) + w_n C_{k-1}^{n-1}(\mathbf{w}), \tag{8}$$

$$\text{II.} \quad S_k^n(\mathbf{w}) = S_k^{n-1}(\mathbf{w}) + w_n S_{k-1}^{n}(\mathbf{w}). \tag{9}$$

Proof. (I) Given a k-selection from distinct boxes, either the last box (box n) is selected or not selected. If not selected, we have a k-selection from $(n-1)$ boxes; otherwise, there are w_n objects to choose from box n

and $(k-1)$ other selections from the remaining $(n-1)$ boxes. Thus, (8) follows.

(II) Given a k-selection with box repetition allowed, either the last box selected was box n or not. If not we have a k-selection with box repetition allowed from $(n-1)$ boxes; otherwise, there are w_n objects to choose from box n and $(k-1)$ other selections with box repetition allowed on all n boxes. Thus, (9) follows. ∎

Next we derive the recurrence relations (1)–(5) as corollaries of this theorem. We will also use the notation $C_k^n(w_i)$ and $S_k^n(w_i)$ to denote the generalized binomial coefficients with weight w_i for $1 \le i \le n$. We can interpret $C_k^n(w_i)$ and $S_k^n(w_i)$ as *strict* selection and *weak* selection, respectively. Observe that if $w_i = 1$ for all i, then the generalized coefficients (6) and (7) reduce to the ordinary binomial coefficients of the first and second kind:

$$C_k^n(1) = \binom{n}{k}, \qquad S_k^n(1) = \binom{n+k-1}{k}.$$

Note. In $S_k^n(1)$ if we replace n by $n-k+1$ we obtain (see Gaussian coefficients below)

$$\binom{n}{k} = S_k^{n-k+1}(1). \tag{$*$}$$

If $w_i = i$ we obtain the Stirling numbers of the first and second kind. The Stirling number of the first kind $\left[{n \atop k}\right]$ has a well-known representation as the sum of all products of $n-k$ different integers taken from $\{1, 2, \ldots, n-1\}$ (see [4] and [8]). Thus, we have the relations

$$\left[{n \atop k}\right] = C_{n-k}^{n-1}(i), \qquad C_k^n(i) = \left[{n+1 \atop n+1-k}\right]. \tag{10}$$

Similarly, the Stirling number of the second kind $\left\{{n \atop k}\right\}$ can be expressed as the sum of all products of $n-k$ integers taken from $\{1, 2, \ldots, k\}$ with repetition allowed. Thus,

$$\left\{{n \atop k}\right\} = S_{n-k}^k(i), \qquad S_k^n(i) = \left\{{n+k \atop n}\right\}. \tag{11}$$

In terms of the generalized binomial coefficient, the Stirling number of the first kind $\left[{n \atop k}\right]$ can be interpreted as the number of ways to select $n-k$ distinct objects from $n-k$ distinct boxes (one object selected per box) chosen from a total of $n-1$ boxes with box i having weight i. Similarly, the Stirling number of the second kind $\{{n \atop k}\}$ is the number of ways to select $n-k$ objects from $n-k$ boxes with box repetition allowed (one object selected per box) chosen from a total of k boxes with box i having weight i.

Applying the theorem with $w_i = i$ and (10), (11) we obtain the recurrence relations (4) and (5):

COROLLARY 1. *If $w_i = i$ for $i \geq 1$, then*

$$\text{I.}\quad C_{n-k}^{n-1}(i) = C_{n-k}^{n-2}(i) + (n-1)C_{n-k-1}^{n-2}(i),$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix};$$

$$\text{II.}\quad S_{n-k}^{k}(i) = S_{n-k}^{k-1}(i) + kS_{n-k-1}^{k}(i),$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}.$$

Note. The combinatorial interpretation of the coefficient $(n-1)$ in (4) is now clear. It is the weight of the last box in the $(n-k)$ *strict* selection used for the Stirling numbers of the first kind. Similarly, the coefficient k in (5) is the weight of the last box in the $(n-k)$ *weak* selection used for the Stirling numbers of the second kind.

3. COMBINATORIAL PROOFS FOR GAUSSIAN COEFFICIENTS

If q is a positive integer and $w_i = q^{i-1}$ for $1 \leq i \leq n$, then the generalized binomial coefficients become the Gaussian coefficients of the first and second kind. We will study the Gaussian coefficients of the second kind in detail since these turn out to be the *ordinary* Gaussian coefficients. If q is a prime power, then the Gaussian coefficient $\binom{n}{k}_q$ counts the number of k-dimensional subspaces of an n-dimensional finite vector space over the finite field with q elements. If q is an integer $(q > 1)$, then the Gaussian coefficient can also be interpreted as the number of $k \times n$ row-reduced echelon matrices with no zero rows. However, neither one of these two interpretations seems to adequately explain why the Gaussian coefficients tend to the ordinary binomial coefficients as q approaches 1. In both cases $q = 1$ results in an ill-defined interpretation. We will try to gain some insight into this problem from a purely combinatorial perspective.

The Gaussian coefficient $\binom{n}{k}_q$ has a known representation as the sum of all products of k integers taken from $\{1, q, q^2, \ldots, q^{n-k}\}$ with *repetition* allowed. Thus, it can be expressed and interpreted in terms of the generalized binomial of the *second* kind with $w_i = q^{i-1}$ (using vector notation we let $\mathbf{w} = \mathbf{q}$):

$$\begin{aligned}\binom{n}{k}_q &= \sum_{0 \le i_1 \le i_2 \le \cdots \le i_k \le n-k} q^{i_1 + i_2 + \cdots + i_k} \\ &= \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n-k+1} q^{(i_1 - 1) + (i_2 - 1) + \cdots + (i_k - 1)} \\ &= S_k^{n-k+1}(\mathbf{q}). \end{aligned} \tag{12}$$

The Gaussian coefficients of the first kind $C_k^n(\mathbf{q})$ can be expressed in terms of $\binom{n}{k}_q$ (note the substitution $j_m = i_m - m + 1$ for $1 \le m \le k$ when going from the strict to the weak inequality):

$$\begin{aligned} C_k^n(\mathbf{q}) &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} q^{(i_1 - 1) + (i_2 - 1) + \cdots + (i_k - 1)} \\ &= \sum_{1 \le j_1 \le j_2 \le \cdots \le j_k \le n-k+1} q^{(j_1 - 1) + (j_2 - 1) + \cdots + (j_k - 1) + 1 + 2 + \cdots + (k-1)} \\ &= q^{\binom{k}{2}} \binom{n}{k}_q . \end{aligned}$$

Note. If $q = 1$ in (12) we obtain $\binom{n}{k} = S_k^{n-k+1}(1)$; see Eq. $(*)$. Thus, the combinatorial interpretation of the Gaussian coefficient as a generalized binomial coefficient of the *second* kind reduces to the ordinary binomial coefficient when q is replaced by 1.

Now we use our combinatorial interpretation to derive some classical Gaussian coefficient identities. There are typically two recurrence relations for the Gaussian coefficients of the second kind. The first is an immediate consequence of Theorem 1 and the relation from (12):

$$\binom{n}{k}_q = S_k^{n-k+1}(\mathbf{q})$$

(for other proofs see [2, p. 35], [5], [6], and [10, p. 295]).

COROLLARY 2. *If* $w_i = q^{i-1}$ *for* $i \geq 1$ (*or* $\mathbf{w} = \mathbf{q}$), *then*

$$S_k^{n-k+1}(\mathbf{q}) = S_k^{n-k}(\mathbf{q}) + q^{n-k} S_{k-1}^{n-k+1}(\mathbf{q}),$$

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.$$

The symmetry property of the binomial coefficients $\binom{n}{k} = \binom{n}{n-k}$ has a simple combinatorial proof: a selection of k objects chosen from an n-set determines a selection of $n-k$ objects not chosen. However, this proof does not seem to generalize to the Gaussian coefficients for all $q \geq 1$. The symmetry property of the Gaussian coefficients (3) can be proved *algebraically* as a generalization of the symmetry property of the binomial coefficients or by noting that the lattice of subspaces of a finite vector space is self-dual (when q is a prime power). Surprisingly, we will now show that this symmetry property can be obtained as a *combinatorial* generalization of the symmetry property of the combinations with repetition numbers (binomial coefficients of the *second* kind): $C^R(n-k+1, k) = C^R(k+1, n-k)$. Observe that algebraically this symmetric identity reduces to the symmetry property of the ordinary binomial coefficients (1).

THEOREM 2. (*Combinatorial proof of symmetry property of Gaussian coefficients*).

$$\binom{n}{k}_q = \binom{n}{n-k}_q \qquad \textit{for all } q \geq 1.$$

Proof. We must show (by (12) above)

$$S_k^{n-k+1}(\mathbf{q}) = S_{n-k}^{k+1}(\mathbf{q}) \qquad \text{for any integer } q \geq 1,$$

that is,

$$\sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n-k+1} q^{(i_1-1)+(i_2-1)+\cdots+(i_k-1)} = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-k} \leq k+1} q^{(j_1-1)+(j_2-1)+\cdots+(j_{n-k}-1)}.$$

If we replace k on the left-hand side by $(n-k)$ we obtain the *dual* coefficient on the right-hand side. Also, when $q = 1$ we obtain the symmetry property for the combinations with repetition numbers. Now we detail the combinatorial proof. A selection of k boxes with repetition from the set $\{1, 2, \ldots, n-k+1\}$, say $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n-k+1$, can be uniquely associated with an $n-k$ selection with repetition (the *dual* or *conjugate* selection) from the set $\{1, 2, \ldots, k+1\}$, say $1 \le j_1 \le j_2 \le \cdots \le j_{n-k} \le k+1$, by a sequence of k **0**s and $n-k$ **1**s as follows. The number of **0**s sandwiched between two **1**s represents the number of times a respective box is chosen from the set $\{1, 2, \ldots, n-k+1\}$. In other words the **1**s act as separators. For example, if $n = 7$ and $k = 4$, then the sequence (read from left to right) **0110010** represents the selection $\{1, 3, 3, 4\}$. Thus, the number of **0**s to the left of the first **1** is the number of times box 1 is selected, while the number of **0**s between the first and second **1** is the number of times box 2 is selected, etc. The *dual* selection is defined by interchanging the roles of **0** and **1** and then reading the sequence from right to left. Thus, in the dual selection the **0**s act as separators. In our example the dual selection with $n - k = 3$ is $\{2, 4, 4\}$.

We will now show that a selection and its dual have the same *q-weight*. Consider our binary representation of a general k-selection with repetition from $\{1, 2, \ldots, n-k+1\}$, where we have grouped the consecutive blocks of **0**s and **1**s,

$$\underbrace{\mathbf{00\cdots0}}_{a_1}\underbrace{\mathbf{11\cdots1}}_{b_1}\underbrace{\mathbf{00\cdots0}}_{a_2}\underbrace{\mathbf{11\cdots1}}_{b_2}\cdots\underbrace{\mathbf{00\cdots0}}_{a_{r-1}}\underbrace{\mathbf{11\cdots1}}_{b_{r-1}}\underbrace{\mathbf{00\cdots0}}_{a_r},$$

and where $a_1 \ge 0$, $a_r \ge 0$ and $b_i \ge 1$, $a_j \ge 1$ for $1 \le i \le r-1$, and $2 \le j \le r-1$. We have $\sum_{i=1}^{r} a_i = k$ for the selection and $\sum_{i=1}^{r-1} b_i = n-k$ for the dual. Observe that a_1 is the number of times box 1 is chosen, while a_2 is the number of times box $b_1 + 1$ is chosen, and in general a_i is the number of times box $(b_1 + b_2 + \cdots + b_{i-1} + 1)$ is chosen (note that the weight of this box is $q^{b_1+b_2+\cdots+b_{i-1}}$ with the convention $b_0 = 0$). In the dual b_{r-1} is the number of times box $a_r + 1$ is chosen, while b_{r-2} is the number of times box $a_r + a_{r-1} + 1$ is chosen, and in general b_i is the number of times box $(a_r + a_{r-1} + \cdots + a_{i+1} + 1)$ is chosen. Let $S_1 = \sum_{i=1}^{r} a_i \sum_{j=1}^{i-1} b_j$ and $S_2 = \sum_{i=1}^{r-1} b_i \sum_{j=i+1}^{r} a_j$. But $S_1 = \sum_{1 \le j < i \le r} a_i b_j = \sum_{1 \le i < j \le r} b_i a_j = S_2$, so the *q-weights* of a selection and its dual are the same; that is, $q^{S_1} = q^{S_2}$. Hence, summing over all k-selections we have

$$\begin{aligned}
&\sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n-k+1} q^{(i_1-1)+(i_2-1)+\cdots+(i_k-1)} \\
&\quad= \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n-k+1} q^{S_1} \\
&\quad= \sum_{1 \le j_1 \le j_2 \le \cdots \le j_{n-k} \le k+1} q^{S_2} \\
&\quad= \sum_{1 \le j_1 \le j_2 \le \cdots \le j_{n-k} \le k+1} q^{(j_1-1)+(j_2-1)+\cdots+(j_{n-k}-1)}.
\end{aligned}$$

∎

From Corollary 2 and Theorem 2 we obtain the second recurrence relation for the Gaussian coefficients:

COROLLARY 3. *If* $w_i = q^{i-1}$ *for* $i \geq 1$, *then*

$$S_k^{n-k+1}(\mathbf{q}) = q^k S_k^{n-k}(\mathbf{q}) + S_{k-1}^{n-k+1}(\mathbf{q}),$$
$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q.$$

Next, we give combinatorial proofs for a couple of Gaussian coefficient identities that are q-analogs of binomial coefficient identities (see [2, p. 37]). For example, we can prove the known result

$$\sum_{j=0}^{n} q^j \binom{m+j}{m}_q = \binom{n+m+1}{m+1}_q \quad \text{or} \quad \sum_{j=1}^{n+1} q^{j-1} S_m^j(\mathbf{q}) = S_{m+1}^{n+1}(\mathbf{q})$$

as follows. The number of $(m+1)$-selections with box repetition from $\{1, 2, \ldots, n+1\}$ having j as the last box selected $(1 \leq j \leq n+1)$ is the weight of box j times the number of m-selections with box repetition from $\{1, 2, \ldots, j\}$. Thus, the identity follows.

The q-Vandermonde identity is

$$\sum_{k=0}^{h} q^{(n-k)(h-k)} \binom{n}{k}_q \binom{m}{h-k}_q = \binom{n+m}{h}_q.$$

To prove this we must show

$$\sum_{k=0}^{h} q^{(n-k)(h-k)} S_k^{n+1-k}(\mathbf{q}) S_{h-k}^{(m+1)-(h-k)}(\mathbf{q}) = S_h^{n+m+1-h}(\mathbf{q}).$$

The combinatorial proof runs as follows. Consider the h-selection with box repetition $1 \leq i_1 \leq i_2 \leq \cdots \leq i_h \leq n+m+1-h$. If $i_1 \geq n+1$, then we have an h-selection with box repetition from the set $\{n+1, n+2, \ldots, n+m+1-h\}$. Factoring in the weights this equals $q^{hn} S_h^{m+1-h}(\mathbf{q})$ (this is the term with $k = 0$ and the convention $S_0^{n+1}(\mathbf{q}) = 1$). If $i_1 \leq n \leq i_2$, then we have a 1-selection from the set $\{1, 2, \ldots, n\}$ and an $(h-1)$-selection from the set $\{n, n+1, \ldots, n+m+1-h\}$ (this is the term with $k = 1$). Factoring in the weights this equals $q^{(n-1)(h-1)} S_1^n(\mathbf{q}) S_{h-1}^{(m+1)-(h-1)}(\mathbf{q})$. Otherwise, we must have $i_1 \leq i_2 \leq n-1 \leq i_3$. In this case we have a 2-selection from the set $\{1, 2, \ldots, n-1\}$ and an $(h-2)$-selection from the set $\{n-1, n, \ldots, n+m+1-h\}$ and the term is $q^{(n-2)(h-2)} \cdot S_2^{n-1}(\mathbf{q}) S_{h-2}^{(m+1)-(h-2)}(\mathbf{q})$. Thus, continuing for $0 < k < h$ and $i_1 \leq i_2 \leq \cdots \leq i_k \leq n+1-k \leq i_{k+1}$ we have a k-selection from $\{1, 2, \ldots, n+1-k\}$ and an $(h-k)$-selection from $\{n+1-k, n+2-k, \ldots, n+m$

$+\ 1 - h\}$. The weighted result equals $q^{(n-k)(h-k)}S_k^{n+1-k}(\mathbf{q})S_{h-k}^{(m+1)-(h-k)}(\mathbf{q})$. The case $k = h$ follows similarly, and the combinatorial proof is complete.

4. GENERATING FUNCTIONS

The formal properties of the generalized binomial coefficients can be studied in terms of generating functions. The coefficients of the first kind $C_k^n(\mathbf{w})$ can be formally represented as *elementary symmetric functions*. Thus, if $\mathbf{w} = (w_1, w_2, \dots, w_n)$ and

$$f(x) = (1 - w_1x)(1 - w_2x)\cdots(1 - w_nx) = \sum_{k=0}^{n}(-1)^k\sigma_k x^k$$

then $\sigma_0 = 1$ and, for $k \geq 1$,

$$\sigma_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} w_{i_1}w_{i_2}\cdots w_{i_k}$$

denotes the kth *elementary symmetric function*. But, by (6), $C_k^n(\mathbf{w}) = \sigma_k$, so $f(x)$ is a generating function for $C_k^n(\mathbf{w})$,

$$f(x) = (1 - w_1x)(1 - w_2x)\cdots(1 - w_nx) = \sum_{k=0}^{n}(-1)^k C_k^n(\mathbf{w})x^k. \quad (13)$$

What is the corresponding generating function for the generalized binomial coefficients of the second kind? We quickly show it is the reciprocal of $f(x)$:

$$\frac{1}{f(x)} = \frac{1}{(1 - w_1x)(1 - w_2x)\cdots(1 - w_nx)} = \prod_{j=1}^{n}\sum_{i=0}^{\infty} w_j^i x^i = \sum_{k=0}^{\infty} a_k x^k,$$

where $a_k = \sum_{i_1+i_2+\cdots+i_n=k} w_1^{i_1}w_2^{i_2}\cdots w_n^{i_n}$ with $i_1 \geq 0,\ i_2 \geq 0, \dots,\ i_n \geq 0$. But a_k is just $S_k^n(\mathbf{w})$:

$$a_k = \sum_{i_1+i_2+\cdots+i_n=k} w_1^{i_1}w_2^{i_2}\cdots w_n^{i_n}$$
$$= \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} w_{i_1}w_{i_2}\cdots w_{i_k} = S_k^n(\mathbf{w}).$$

Thus, we have

$$\frac{1}{f(x)} = \frac{1}{(1 - w_1x)(1 - w_2x)\cdots(1 - w_nx)} = \sum_{k=0}^{\infty} S_k^n(\mathbf{w})x^k. \quad (14)$$

Setting $w_i = 1$, i, or q^{i-1} in (13) and (14) we obtain generating function relations for the binomial coefficients, Stirling numbers, and Gaussian coefficients, respectively:

Binomial coefficients ($w_i = 1$):

$$(1-x)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k;$$

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k.$$

Stirling numbers ($w_i = i$):

$$(1-x)(1-2x)\cdots(1-nx) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n+1 \\ n+1-k \end{bmatrix} x^k;$$

$$\frac{1}{(1-x)(1-2x)\cdots(1-nx)} = \sum_{k=0}^{\infty} \left\{ \begin{matrix} n+k \\ n \end{matrix} \right\} x^k.$$

Gaussian coefficients ($w_i = q^{i-1}$):

$$(1-x)(1-qx)\cdots(1-q^{n-1}x) = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{n}{k}_q x^k;$$

$$\frac{1}{(1-x)(1-qx)\cdots(1-q^{n-1}x)} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q x^k.$$

The generalized binomial coefficients satisfy orthogonality relations obtained by multiplying the generating functions $f(x)$ and $1/f(x)$. So by (13) and (14) we have

$$f(x)\frac{1}{f(x)} = 1 = \sum_{m=0}^{\infty} c_m x^m.$$

Thus, if $m > 0$, $\sum_{k=0}^{m} (-1)^k C_k^n(\mathbf{w}) S_{m-k}^n(\mathbf{w}) = 0$. If $m = n$, then

$$\sum_{k=0}^{n} (-1)^k C_k^n(\mathbf{w}) S_{n-k}^n(\mathbf{w}) = 0. \tag{15}$$

Replacing k by $n-k$ we get a dual to (15):

$$\sum_{k=0}^{n} (-1)^{n-k} C_{n-k}^n(\mathbf{w}) S_k^n(\mathbf{w}) = 0. \tag{16}$$

For example, if $w_i = 1$, then from (16) we obtain an orthogonality relation for the binomial coefficients of the first and second kind:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{n-k} \binom{n+k-1}{k} = 0.$$

If $w_i = q^{i-1}$, we obtain the q-orthogonality relation:

$$\sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \binom{n}{n-k}_q \binom{n+k-1}{k}_q = 0.$$

Finally, $w_i = i$ results in an orthogonality relation for the Stirling numbers:

$$\sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{Bmatrix} n+k \\ n \end{Bmatrix} = 0.$$

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