Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Computational and Applied Mathematics 170 (2004) 59–77
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

An arc spline approximation to a clothoid

D.S. Meek*, D.J. Walton

Department of Computer Science, University of Manitoba, Winnipeg, Canada, R3T 2N2

Received 17 January 2003; received in revised form 3 December 2003

Abstract

The clothoid is a spiral used in highway and railway route design. Clothoids are transcendental functions and so have been approximated by polynomials, by power series and continued fractions, and by rational functions. Here the clothoid is approximated by an arc spline. The chief advantage in doing so is that arc splines are very easy to lay out and to offset. Examples show that the approximation is of extremely high accuracy. It is proved that if the arc spline has n arcs, then the error in the approximation is of order $O(1/n^2)$.

MSC: 65S05

Keywords: Arc spline approximation; Clothoid

1. Introduction

The clothoid is a spiral used in highway and railway route design [1]. This curve is also called Euler's spiral and Cornu's spiral [3, p. 207]; [7, p. 36]; a diagram of the clothoid appears in Fig. 1. The clothoid has the property that its curvature varies linearly with its arc length. Clothoids are transcendental functions and so have been approximated by polynomials, by power series and continued fractions, and by rational functions. Here the clothoid is approximated by an arc spline. The chief advantage in doing so is that arc splines are very easy to lay out and to offset. Examples show that the approximation is of extremely high accuracy. It is proved that if the arc spline has n arcs, then the error in the approximation is of order $O(h^2)$, where $h = 1/n$. This asymptotic result means that if the number of arcs is doubled, the error will be about 0.25 of the previous error. Since it is fairly easy to measure the accuracy of a given arc spline approximation, this result gives the user an idea of how many (or few) arcs are required for a desired accuracy.

* Corresponding author. Tel.: 204-474-8681; fax: 204-474-7609.
E-mail address: dereck_meek@umanitoba.ca (D.S. Meek).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2003.12.038

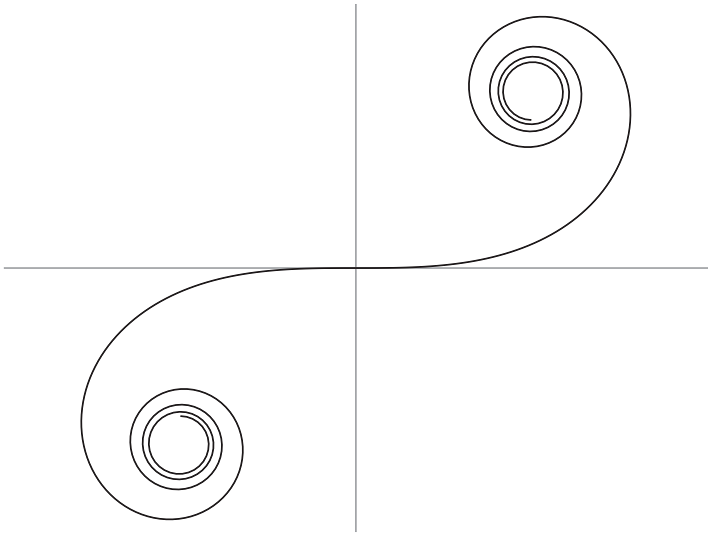

Fig. 1. The clothoid curve.

Recent works that study approximations of clothoids are as follows. Polynomials that interpolate the clothoid are given in [15,14]. The first reference gives a Bézier polynomial that is a Hermite interpolant of the clothoid, while the second gives an s-power series (polynomial) that is a Hermite interpolant of part of the clothoid. Power series and continued fraction approximations to the clothoid are given in [13]. Rational function approximations to the clothoid, which are very convenient in computer programs, are given in [4]. The last two mentioned approximations are not specifically designed to interpolate the clothoid.

An arc spline is a collection of circular arcs joined with continuous tangent vectors. The new direction in this paper is to approximate the clothoid by a special arc spline. Some advantages of the arc spline are:

(i) it is composed of circular arcs, which are familiar curves,
(ii) it is a NURBS (nonuniform rational B-spline), so it can be used in standard graphics packages,
(iii) it is easy to lay out and to offset, which is important in highway and railway route design, and
(iv) it is fairly easy to find the shortest distance from any point in the plane to an arc spline.

The arc spline has the disadvantage that it is not curvature continuous (G^2); it is only tangent vector continuous (G^1). It is anticipated that this low degree of continuity is not a difficulty in highway route design as the jumps in curvature can be made as small as one desires by taking enough arcs. In this paper, the special arc spline approximation to a clothoid is called a *discrete clothoid*.

The discrete clothoid is an arc spline with several restrictions: the arcs are all equal in length (except the first and last) and the curvatures of the arcs are an arithmetic progression. The first and last arcs are half the length of the others to give a more "balanced" partition into arcs; this is more fully described in Section 2.2. The above properties mean the discrete clothoid approximates the clothoid in the sense that the curvature varies linearly with respect to arc length.

Arc splines have appeared previously in the literature on highway design. For example, Hickerson used biarcs, triarcs [5, p. 132], and quadarcs [5, p. 139], but did not arrange that the curvatures vary linearly with arc length. Also of interest are papers by Horn and Mehlum. Horn [6] considered a multi-arc curve as a curve of least energy. In [6, p. 447], it is reported that the arcs tend to have equal length and curvature tends to have a linear relation with arc length as the number of arcs is increased. However, Horn's goal was not specifically to approximate a clothoid. Mehlum [11] also suggested using curves made of circular arcs to approximate a curve of least energy. Arc splines have been used to fit discrete data [9] and approximate smooth curves [10,12].

The plan of this paper is to study the use of the discrete clothoid in place of the clothoid in four of the common situations in highway route design where clothoids are used [1,8].

These four situations will be treated in Sections 3–6, respectively.

2. Common notation

Define the two vectors $\mathbf{r}(\theta)$ and $\boldsymbol{\rho}(\theta)$ as

$$\mathbf{r}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\rho}(\theta) = \begin{pmatrix} \sin\theta \\ 1-\cos\theta \end{pmatrix} = 2\sin\left(\frac{\theta}{2}\right)\mathbf{r}\left(\frac{\theta}{2}\right).$$

2.1. Clothoid

Let $0 \leqslant k_A < k_B$, $S > 0$, and define $s_A = Sk_A/(k_B - k_A)$, $s_B = Sk_B/(k_B - k_A)$. The clothoid

$$\mathbf{E}(s, S, k_A, k_B) = \int_0^s \mathbf{r}\left(\frac{k_B - k_A}{2S}u^2\right) \mathrm{d}u, \quad s_A \leqslant s \leqslant s_B \tag{2.1}$$

has the following properties:

- the angle of tangent vector with respect to the X-axis is $(k_B - k_A)s^2/(2S)$,
- the total rotation of the tangent vector is $S(k_A + k_B)/2$,
- the length of tangent vector is 1 , so s is an arc length parameter,
- the total arc length is $s_B - s_A = S$,
- the curvature is $k(s) = (k_B - k_A)s/S$, so the curvature varies linearly with arc length,
- the curvature is k_A at the beginning point $\mathbf{A} = \mathbf{E}(s_A, S, k_A, k_B)$ and is k_B at the ending point $\mathbf{B} = \mathbf{E}(s_B, S, k_A, k_B)$.

To make the theory easier, assume the total rotation of the tangent vector over the parameter range is less than $\pi/2$, or

$$\frac{S(k_A + k_B)}{2} < \frac{\pi}{2}. \tag{2.2}$$

With "scaling factor" $a = \sqrt{\pi S/(k_B - k_A)}$, clothoid (2.1) can be written in terms of standard Fresnel integrals as

$$\mathbf{E}(s, S, k_A, k_B) = a\begin{pmatrix} C(s/a) \\ S(s/a) \end{pmatrix}, \quad \text{where} \begin{pmatrix} C(t) \\ S(t) \end{pmatrix} = \int_0^t \mathbf{r}\left(\frac{\pi}{2}u^2\right) \mathrm{d}u. \tag{2.3}$$

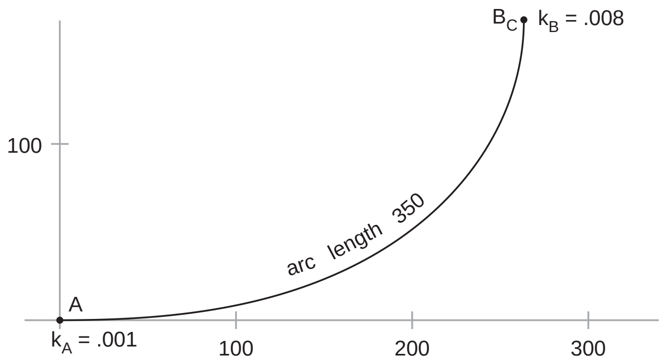

Fig. 2. Clothoid with curvatures $k_A = 0.001$, $k_B = 0.008$, and $S = 350$.

A useful standard position is the above clothoid (2.1) with the point **A** translated to the origin and the plane rotated so that the initial tangent vector is along the positive X-axis. Endpoint **B** in its new position will be labelled $\mathbf{B}_C$ (see Fig. 2). Further, the variable of integration u is replaced by $Su/(k_B - k_A)$ in (2.1), which changes the variable of integration from arc length to curvature, and the variable $k = (k_B - k_A)s/S$ replaces variable s

$$\mathbf{C}(k, S, k_A, k_B) = R\left(-\frac{Sk_A^2}{2(k_B - k_A)}\right)\frac{S}{k_B - k_A}\int_{k_A}^{k} \mathbf{r}\left(\frac{Su^2}{2(k_B - k_A)}\right) du, \quad k_A \leqslant k \leqslant k_B$$

$$= \frac{S}{k_B - k_A}\int_{k_A}^{k} \mathbf{r}\left(\frac{S(u^2 - k_A^2)}{2(k_B - k_A)}\right) du, \quad k_A \leqslant k \leqslant k_B, \tag{2.4}$$

where $R(\theta)$ is the rotation matrix. To get the final form, integrate (2.4) by parts to give

$$\mathbf{C}(k, S, k_A, k_B) = \int_{k_A}^{k} \frac{1}{u^2}\boldsymbol{\rho}\left(\frac{S(u^2 - k_A^2)}{2(k_B - k_A)}\right) du + \frac{1}{k}\boldsymbol{\rho}\left(\frac{S(k^2 - k_A^2)}{2(k_B - k_A)}\right), \quad k_A \leqslant k \leqslant k_B. \tag{2.5}$$

Note that if $k_A = 0$, limits must be used to evaluate some expressions. The vector from the starting point **A** to the ending point $\mathbf{B}_C$ of clothoid (2.5) is

$$\mathbf{C}(k_B, S, k_A, k_B) = \int_{k_A}^{k_B} \frac{1}{u^2}\boldsymbol{\rho}\left(\frac{S(u^2 - k_A^2)}{2(k_B - k_A)}\right) du + \frac{1}{k_B}\boldsymbol{\rho}\left(\frac{S(k_A + k_B)}{2}\right). \tag{2.6}$$

The vector from **A** (now at the origin) to the centre of curvature at $\mathbf{B}_C$ is

$$\mathbf{M}(S, k_A, k_B) = \mathbf{C}(k_B, S, k_A, k_B) + \frac{1}{k_B}\mathbf{r}\left(\frac{\pi}{2} + \frac{S(k_A + k_B)}{2}\right)$$

$$= \int_{k_A}^{k_B} \frac{1}{u^2}\boldsymbol{\rho}\left(\frac{S(u^2 - k_A^2)}{2(k_B - k_A)}\right) du + \frac{1}{k_B}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.7}$$

If $k_A > 0$, the vector from the centre of curvature at $\mathbf{A}$ to the centre of curvature at $\mathbf{B}_C$ is

$$\mathbf{M}^+(S,k_A,k_B) = \mathbf{M}(S,k_A,k_B) - \frac{1}{k_A}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \int_{k_A}^{k_B} \frac{1}{u^2}\boldsymbol{\rho}\left(\frac{S(u^2-k_A^2)}{2(k_B-k_A)}\right)\mathrm{d}u + \left(\frac{1}{k_B}-\frac{1}{k_A}\right)\begin{pmatrix}0\\1\end{pmatrix}. \tag{2.8}$$

Notice that

$$\frac{\partial}{\partial S}\mathbf{M}(S,k_A,k_B) = \frac{1}{2(k_B-k_A)}\int_{k_A}^{k_B} \frac{u^2-k_A^2}{u^2}\mathbf{r}\left(\frac{S(u^2-k_A^2)}{2(k_B-k_A)}\right)\mathrm{d}u. \tag{2.9}$$

Since both x- and y-components of this partial are positive for $k_B > k_A$, both x- and y-components of $\mathbf{M}(S,k_A,k_B)$ are monotone increasing with increasing S.

2.2. Discrete clothoid

With $n \geqslant 1$, the discrete clothoid of arc length S is formed of $n+1$ arcs, numbered 0 to n. Arc 0 and arc n have length $S/(2n)$, while arcs 1 to $n-1$ have length S/n. The curvatures of the $n+1$ arcs are

$$k_0 = k_A,\ \ k_j = k_0 + jh,\ \ j = 0,1,\dots,n, \quad \text{and} \quad k_n = k_B,\ \text{where } h = \frac{k_B-k_A}{n}. \tag{2.10}$$

The curvatures can be thought of as acting at the midpoint of each arc so the first and last arcs have been chosen to be half the length of the others. If two discrete clothoids meet at a point with G^2 continuity, the neighbouring arcs at the join point will be part of the same circle, and each will contribute a half-length arc to the circular arc at the join.

In standard position, the discrete clothoid starts at the origin $\mathbf{A}$ with curvature k_A, has its tangent vector at $\mathbf{A}$ along the positive X-axis and extends to point $\mathbf{B}_D$ where the curvature is k_B.

Define w_j

$$w_j = \frac{Sk_j}{2n}, \quad 0 \leqslant j \leqslant n, \tag{2.11}$$

then the angle spanned by arc j is

$$\begin{cases} w_0, & j=0,\\ 2w_j, & 1 \leqslant j \leqslant n-1,\\ w_n, & j=n. \end{cases} \tag{2.12}$$

Let θ_j be the angle of the tangent vector with respect to the positive X-axis at the centre of arc j, $1 \leqslant j \leqslant n-1$,

$$\theta_j = w_0 + 2\sum_{i=1}^{j-1} w_i + w_j = \frac{S}{2n}(2jk_0 + j^2h) = \frac{Sh}{2(k_B-k_A)}(2jk_0+j^2h). \tag{2.13}$$

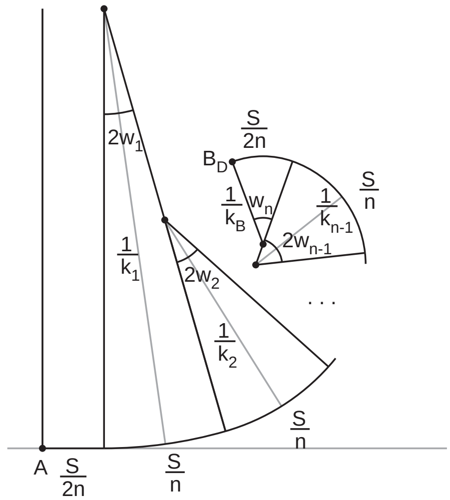

Fig. 3. Discrete clothoid with $k_A = 0$.

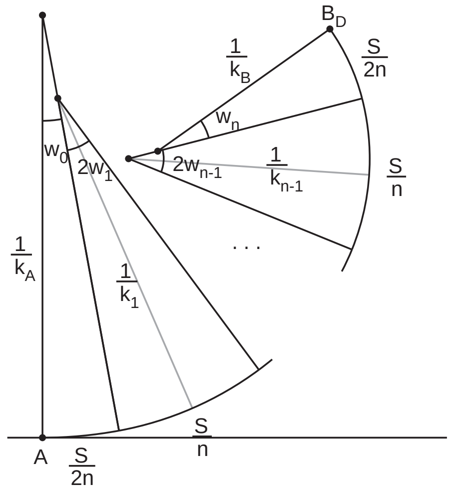

Fig. 4. Discrete clothoid with $k_A > 0$.

Note that the final formula of (2.13) gives $\theta_0 = 0$, the angle of the tangent vector at $\mathbf{A}$, and

$$\theta_n = \frac{Sh}{2(k_B - k_A)}(2nk_0 + n^2 h) = \frac{S(k_A + k_B)}{2}, \tag{2.14}$$

the angle of the tangent vector at $\mathbf{B}_D$.

Figs. 3 and 4 show discrete clothoids when $k_A = 0$ and when $k_A > 0$. Although the total rotation of the tangent vector is restricted to $\pi/2$ for the theory, in these figures a larger rotation is used. The algebraic formulas cover both $k_A = 0$ and $k_A > 0$, although sometimes the limit as k_A tends to zero must be used rather than substitution of k_A by zero.

It is convenient to express the discrete clothoid in terms of a parameter k, which is proportional to arc length. This parameter equals the curvature of the circular arc at the centre of that arc for the interior arcs and equals the curvature at the ends of the first and last arc. If k is in arc j;

$$k = k_j + \lambda h \begin{cases} 0 \leqslant \lambda \leqslant 0.5 & \text{for } j = 0, \\ -0.5 < \lambda \leqslant 0.5 & \text{for } 1 \leqslant j \leqslant n-1, \\ -0.5 < \lambda \leqslant 0 & \text{for } j = n. \end{cases} \tag{2.15}$$

Let the vector from $\mathbf{A}$ to the point with parameter k of the discrete clothoid be $\mathbf{D}(k, S, k_A, k_B, n)$.

If k is in arc 0, $j = 0$,

$$\mathbf{D}(k, S, k_A, k_B, n) = \frac{2 \sin \lambda w_0}{k_A} \mathbf{r}(\lambda w_0) = \frac{1}{k_A} \boldsymbol{\rho}(2\lambda w_0). \tag{2.16}$$

If k is in arc j, $1 \leqslant j \leqslant n$,

$$\mathbf{D}(k, S, k_A, k_B, n) = \frac{1}{k_A} \boldsymbol{\rho}(w_0) + \sum_{i=1}^{j-1} \frac{2 \sin w_i}{k_i} \mathbf{r}(\theta_i) + \frac{2 \sin(2\lambda + 1) w_j / 2}{k_j} \mathbf{r}\left(\theta_j + \frac{(2\lambda - 1) w_j}{2}\right). \tag{2.17}$$

Eq. (2.17) can be rewritten (cf. with Eq. (2.5) and keep in mind that $\theta_{j-1} + w_{j-1} = \theta_j - w_j$) as

$$\begin{aligned} \mathbf{D}(k, S, k_A, k_B, n) &= \frac{1}{k_A} \boldsymbol{\rho}(w_0) + \sum_{i=1}^{j-1} \frac{1}{k_i} (-\boldsymbol{\rho}(\theta_i - w_i) + \boldsymbol{\rho}(\theta_i + w_i)) \\ &\quad + \frac{1}{k_j} (-\boldsymbol{\rho}(\theta_j - w_j) + \boldsymbol{\rho}(\theta_j + 2\lambda w_j)) \\ &= \sum_{i=0}^{j-1} \frac{h}{k_i k_{i+1}} \boldsymbol{\rho}(\theta_i + w_i) + \frac{1}{k_j} \boldsymbol{\rho}(\theta_j + 2\lambda w_j). \end{aligned} \tag{2.18}$$

When $k = k_B = k_n$, $j = n$, and $\lambda = 0$, Eq. (2.19) gives the vector from point $\mathbf{A}$ to point $\mathbf{B}_D$ and becomes (cf. with Eq. (2.6))

$$\mathbf{D}(k_B, S, k_A, k_B, n) = \sum_{i=0}^{n-1} \frac{h}{k_i k_{i+1}} \boldsymbol{\rho}(\theta_i + w_i) + \frac{1}{k_B} \boldsymbol{\rho}(\theta_n). \tag{2.19}$$

The vector from $\mathbf{A}$ to the centre of curvature at $\mathbf{B}_D$ is (cf. with Eq. (2.7))

$$\begin{aligned} \mathbf{N}(S, k_A, k_B, n) &= \mathbf{D}(k_B, S, k_A, k_B, n) + \frac{1}{k_B} \mathbf{r}\left(\frac{\pi}{2} + \theta_n\right) \\ &= \sum_{i=0}^{n-1} \frac{h}{k_i k_{i+1}} \boldsymbol{\rho}(\theta_i + w_i) + \frac{1}{k_B} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{2.20}$$

If $k_{\mathrm{A}} > 0$, the vector from the centre of curvature at $\mathbf{A}$ to the centre of curvature at $\mathbf{B}_{\mathrm{D}}$ is (cf. with Eq. (2.8))

$$\begin{aligned}\mathbf{N}^{+}(S,k_{\mathrm{A}},k_{\mathrm{B}},n) &= \mathbf{N}(S,k_{\mathrm{A}},k_{\mathrm{B}},n) - \frac{1}{k_{\mathrm{A}}}\binom{0}{1} \\ &= \sum_{i=0}^{n-1}\frac{h}{k_i k_{i+1}}\boldsymbol{\rho}(\theta_i + w_i) + \left(\frac{1}{k_{\mathrm{B}}} - \frac{1}{k_{\mathrm{A}}}\right)\binom{0}{1}. \end{aligned} \tag{2.21}$$

Notice that using (2.13) and (2.11) (cf. with Eq. (2.9))

$$\frac{\partial}{\partial S}\mathbf{N}(S,k_{\mathrm{A}},k_{\mathrm{B}},n) = \frac{h^2}{2(k_{\mathrm{B}}-k_{\mathrm{A}})}\sum_{i=0}^{n-1}\frac{(2i+1)k_0 + i(i+1)h}{k_i k_{i+1}}\mathbf{r}(\theta_i + w_i). \tag{2.22}$$

Since both x- and y-components of this partial are positive $k_{\mathrm{B}} > k_{\mathrm{A}}$, both x- and y-components of $\mathbf{N}(S,k_{\mathrm{A}},k_{\mathrm{B}},n)$ are monotone increasing with increasing S.

Theorem 1. *If a clothoid and a discrete clothoid of $n+1$ arcs both have the same total arc lengths S, both have the same starting and ending curvatures k_{A} and k_{B}, and both are in the standard position ($\mathbf{A}$ at the origin and the tangent vector at $\mathbf{A}$ along the positive X-axis), then the distance between the two curves at any parameter k is $\mathrm{O}(h^2)$ times a vector for all $k_{\mathrm{A}} \leqslant k \leqslant k_{\mathrm{B}}$, or*

$$\mathbf{C}(k,S,k_{\mathrm{A}},k_{\mathrm{B}}) - \mathbf{D}(k,S,k_{\mathrm{A}},k_{\mathrm{B}},n) = \mathrm{O}(h^2), \quad k_{\mathrm{A}} \leqslant k \leqslant k_{\mathrm{B}},$$

where $h = 1/n$.

Proof. From (2.5) and (2.19) using (2.15),

$$\begin{aligned}&\mathbf{C}(k,S,k_{\mathrm{A}},k_{\mathrm{B}}) - \mathbf{D}(k,S,k_{\mathrm{A}},k_{\mathrm{B}},n) \\ &\quad= \sum_{i=0}^{i-1}\left\{\int_{k_i}^{k_{i+1}}\frac{1}{u^2}\boldsymbol{\rho}\left(\frac{S(u^2-k_{\mathrm{A}}^2)}{2(k_{\mathrm{B}}-k_{\mathrm{A}})}\right)\mathrm{d}u - \frac{h}{k_i k_{i+1}}\boldsymbol{\rho}(\theta_i + w_i)\right\} + \int_{k_j}^{k_j+\lambda h}\frac{1}{u^2}\boldsymbol{\rho}\left(\frac{S(u^2-k_{\mathrm{A}}^2)}{2(k_{\mathrm{B}}-k_{\mathrm{A}})}\right)\mathrm{d}u \\ &\qquad + \frac{1}{k_j+\lambda h}\boldsymbol{\rho}\left(\frac{S((k_j+\lambda h)^2-k_{\mathrm{A}}^2)}{2(k_{\mathrm{B}}-k_{\mathrm{A}})}\right) - \frac{1}{k_j}\boldsymbol{\rho}(\theta_j + 2\lambda w_j). \end{aligned} \tag{2.23}$$

The midpoint rule [2, p. 170] is a numerical approximation to an integral that has an error of $\mathrm{O}(h^3)$,

$$\int_{k_j}^{k_{j+1}}\mathrm{f}(u)\,\mathrm{d}u = hf\left(\frac{k_j+k_{j+1}}{2}\right) + \mathrm{O}(h^3). \tag{2.24}$$

The arguments of the sine and cosine at $u = k_{\mathrm{A}} + (i+\frac{1}{2})h$ in the integral are from (2.10), (2.13), and (2.11),

$$\frac{S([k_{\mathrm{A}} + (i+\frac{1}{2})h]^2 - k_{\mathrm{A}}^2)}{2(k_{\mathrm{B}}-k_{\mathrm{A}})} = \frac{S[(2i+1)k_{\mathrm{A}} + (i^2+i+\frac{1}{4})h]h}{2(k_{\mathrm{B}}-k_{\mathrm{A}})} = \theta_i + w_i + \mathrm{O}(h^2).$$

Also note that

$$k_i k_{i+1} = \left[k_A + \left(i + \frac{1}{2}\right) h\right]^2 + O(h^2). \tag{2.25}$$

Now continuing from (2.23) and using the midpoint rule for the integral (2.24), the term of the sum with index i is

$$\frac{h}{[k_A + (i+\frac{1}{2})h]^2}\,\boldsymbol{\rho}(\theta_i + w_i + O(h^2)) + O(h^3) - \frac{h}{k_i k_{i+1}}\,\boldsymbol{\rho}(\theta_i + w_i)$$

$$= \left(\frac{h}{k_i k_{i+1}} + O(h^3)\right)(\boldsymbol{\rho}(\theta_i + w_i) + O(h^2)) + O(h^3) - \frac{h}{k_i k_{i+1}}\,\boldsymbol{\rho}(\theta_i + w_i) = O(h^3).$$

The sum gives a factor of j, so the terms included in the sigma sign are bounded by $j\ O(h^3) \leqslant n\ O(h^3) = O(h^2)$ times a vector.

For the terms of (2.23) not in the sigma sign, the arguments of the sine and cosine can be expressed in terms of the midpoint argument ω of the first (integral) term (using (2.13) and (2.11)),

$$\omega = \frac{S[(k_j + \lambda h/2)^2 - k_A^2]}{2(k_B - k_A)} = \theta_j + 2\lambda w_j + O(h).$$

Again using the midpoint rule (2.24), the last three terms of (2.23) are now

$$\frac{\lambda h}{(k_j + \lambda h/2)^2}\,\boldsymbol{\rho}(\omega) + O(h^3) + \frac{1}{k_j + \lambda h}\,\boldsymbol{\rho}(\omega + O(h)) - \frac{1}{k_j}\,\boldsymbol{\rho}(\omega + O(h))$$

$$= \frac{\lambda h}{k_j^2}\,\boldsymbol{\rho}(\omega) + O(h^2) + \left(\frac{1}{k_j + \lambda h} - \frac{1}{k_j}\right)(\boldsymbol{\rho}(\omega) + O(h)),$$

$$= \lambda h\left(\frac{1}{k_j^2} - \frac{1}{(k_j + \lambda h)k_i}\right)\boldsymbol{\rho}(\omega) + O(h^2) = O(h^2),$$

which means that the terms not included in the sigma sign are $O(h^2)$ times a vector. This proves the approximation of the clothoid by the discrete clothoid is $O(h^2)$ times a vector.

Corollary. *From Theorem* 1, *the endpoints of the clothoid and the discrete clothoid in standard position*, $\mathbf{B}_C$ *and* $\mathbf{B}_D$, *are* $O(h^2)$ *apart. From* (2.7) *and* (2.20), *the distance between the centres of curvatures at* $\mathbf{B}_C$ *and* $\mathbf{B}_D$ *are also* $O(h^2)$ *apart.*

Note that in the above theorem, the point on a clothoid that corresponds to a point on the discrete clothoid is not necessarily the closest point on the clothoid to that point on the discrete clothoid. This means that the measure of distance used here is an upper bound on the distance that one would obtain by using closest points.

3. Straight line to circle with a single clothoid

In the first arrangement [8], a clothoid and a discrete clothoid are used as transition curves from a straight line to a circle of curvature k_B; here $k_A = 0$ and $s_A = 0$. Without loss of generality, assume the straight line is the X-axis and the circle has centre $\mathbf{C}_\mathrm{B}$. Let the arc length of the clothoid be S_C and let the arc length of the discrete clothoid be S_D. The y-components of the centres of curvature of the clothoid and of the discrete clothoid must equal the y-component of the centre of the given circle. From (2.7) and (2.20),

$$\mathbf{M}(S_\mathrm{C},0,k_\mathrm{B})_y=\mathbf{C}_{\mathrm{B}y},\quad \mathbf{N}(S_\mathrm{D},0,k_\mathrm{B},n)_y=\mathbf{C}_{\mathrm{B}y}. \tag{3.1}$$

Solutions to these equations exist if $\mathbf{M}(S_\mathrm{C},0,k_\mathrm{B})_y-\mathbf{C}_{\mathrm{B}y}$ and $\mathbf{N}(S_\mathrm{D},0,k_\mathrm{B},n)_y-\mathbf{C}_{\mathrm{B}y}$ both change sign as S_C and S_D vary from s_A to s_B. The monotone behaviour of the y-components, (2.9) and (2.22), shows that if a solution exists, it is unique.

Theorem 2. *If a clothoid of arc length S_C and a discrete clothoid with $n+1$ arcs of total arc length S_D both have the same starting and ending curvatures $k_\mathrm{A}=0$ and k_B, are both in standard position ($\mathbf{A}$ at the origin and the tangent vector at $\mathbf{A}$ along the positive X-axis), and the centres of curvatures at respective endpoints $\mathbf{B}_\mathrm{C}$ and $\mathbf{B}_\mathrm{D}$ have the same y component, then $S_\mathrm{C}-S_\mathrm{D}=\mathrm{O}(h^2)$, where $h=1/n$.*

Proof. Subtracting the Eqs. (3.1) and introducing two terms,

$$\mathbf{M}(S_\mathrm{C},0,k_\mathrm{B})_y-\mathbf{N}(S_\mathrm{D},0,k_\mathrm{B},n)_y=0,$$

or

$$[\mathbf{M}(S_\mathrm{C},0,k_\mathrm{B})_y-\mathbf{M}(S_\mathrm{D},0,k_\mathrm{B})_y]+[\mathbf{M}(S_\mathrm{D},0,k_\mathrm{B})_y-\mathbf{N}(S_\mathrm{D},0,k_\mathrm{B},n)_y]=0. \tag{3.2}$$

The mean value theorem gives

$$\mathbf{M}(S_\mathrm{C},0,k_\mathrm{B})_y-\mathbf{M}(S_\mathrm{D},0,k_\mathrm{B})_y=(S_\mathrm{C}-S_\mathrm{D})\frac{\partial}{\partial S}\mathbf{M}(\varphi,0,k_\mathbf{B})_y,$$

where φ is between S_C and S_D; the partial of $\mathbf{M}$ is independent of h from (2.9). Theorem 1 implies

$$\mathbf{M}(S_\mathrm{D},0,k_\mathrm{B})_y-\mathbf{N}(S_\mathrm{D},0,k_\mathrm{B},n)_y=\mathrm{O}(h^2), \tag{3.3}$$

so the difference in the lengths of clothoid and discrete clothoid is

$$S_\mathrm{C}-S_\mathrm{D}=\mathrm{O}(h^2).$$

Corollary. *The distance between points on the clothoid and the discrete clothoid in Theorem 2 with the same parameter k is*

$$\begin{aligned}&\mathbf{C}(k,S_\mathrm{C},0,k_\mathrm{B})-\mathbf{D}(k,S_\mathrm{D},0,k_\mathrm{B},n)\\&\quad=[\mathbf{C}(k,S_\mathrm{C},0,k_\mathrm{B})-\mathbf{C}(k,S_\mathrm{D},0,k_\mathrm{B})]+[\mathbf{C}(k,S_\mathrm{D},0,k_\mathrm{B})-\mathbf{D}(k,S_\mathrm{D},0,k_\mathrm{B},n)]\\&\quad=\mathrm{O}(h^2)+\mathrm{O}(h^2)=\mathrm{O}(h^2).\end{aligned} \tag{3.4}$$

The first difference is $\mathrm{O}(h^2)$ using the mean value theorem and Theorem 2, and the second difference is $\mathrm{O}(h^2)$ using Theorem 1.

The Corollary of Theorem 2 and an argument as in the Corollary of Theorem 1 show that the centres of curvature at the endpoints $\mathbf{B}_\mathrm{C}$ and $\mathbf{B}_\mathrm{D}$ are $\mathrm{O}(h^2)$ apart. Eqs. (3.1) ensure that the centres of

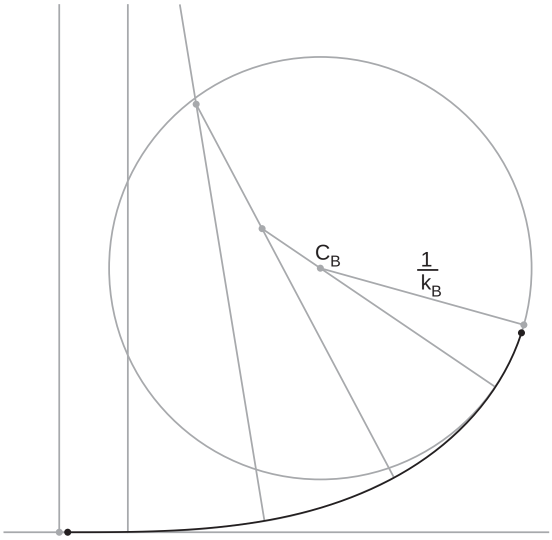

Fig. 5. Straight line to circle with a single clothoid, $n=4$, $k_A=0$, $k_B=0.00833333$, $\mathbf{C}_B=\binom{170}{150}$, $S_C=302.341$, $S_D=311.810$.

curvature at endpoints $\mathbf{B}_C$ and $\mathbf{B}_D$ have the same y-component. An $O(h^2)$ translation of the discrete clothoid is needed to make the centres of curvature at endpoints $\mathbf{B}_C$ and $\mathbf{B}_D$ coincide. With this translation, points on the clothoid and on the discrete clothoid with the same parameter k still will be $O(h^2)$ apart.

Fig. 5 shows a line and a circle between which both a clothoid and a discrete clothoid with 5 arcs form a transition. The clothoid is shown in black and the discrete clothoid in grey. The discrete clothoid is indistinguishable from the clothoid in that diagram except at the endpoints.

A better comparison between the two curves than comparing points with corresponding k values would be a comparison of points on the clothoid to the nearest points on the discrete clothoid. For any point on the clothoid, one can find which sector of the discrete clothoid corresponds to it. The nearest point on the discrete clothoid can be found from the centre of the sector and its radius. This gives a perpendicular distance from a point on the clothoid to a point on the discrete clothoid. These nearest points would not have the same parameter k, but would be closer than points with the same k. Thus, $O(h^2)$ is an upper bound on the distance between these nearest points. For Table 1, the

Table 1
Maximum scaled perpendicular distance from clothoid to discrete clothoid for various n in the straight line to circle case illustrated in Fig. 5

n	Maximum distance (scaled)
4	$1.039 \cdot 10^{-3}$
8	$2.066 \cdot 10^{-4}$
16	$4.568 \cdot 10^{-5}$
32	$1.078 \cdot 10^{-5}$

data illustrated in Fig. 5 was used and the closest distance from the clothoid to the discrete clothoid was calculated at 100 k values in the clothoid, $k_A \leqslant k \leqslant k_B$. The maximum distance divided by the length of the clothoid is shown.

The numerical results in Table 1 show how surprisingly close the discrete clothoid is to the clothoid and show that the distance between them behaves as $O(h^2)$, $h = 1/n$, as proven in this section.

4. Circle to circle with a single clothoid

In the second arrangement [8], a single clothoid and a single discrete clothoid are used as transition curves from one circle of curvature $k_A > 0$ to another circle of curvature k_B. Let the distance between the centres of those circles $\mathbf{C}_A$ and $\mathbf{C}_B$ be d. Let the arc length of the clothoid be S_C and let the arc length of the discrete clothoid be S_D. The equations to solve are from (2.8) and (2.21)

$$\|\mathbf{M}^+(S_C, k_A, k_B)\| = d, \quad \|\mathbf{N}^+(S_D, k_A, k_B, n)\| = d. \tag{4.1}$$

A solution exists if the appropriate expressions change sign as S_C and S_D vary from S_A to S_B. The monotone behaviours of $\mathbf{M}(S, k_A, k_B)$ and $\mathbf{N}(S, k_A, k_B, n)$, (2.9) and (2.22), mean that if a solution exists, it is unique.

A slight modification of Theorem 1 is needed here. From a standard inequality

$$\|\mathbf{M}^+(S, k_A, k_B) - \mathbf{N}^+(S, k_A, k_B, n)\| \geqslant \left| \|\mathbf{M}^+(S, k_A, k_B)\| - \|\mathbf{N}^+(S, k_A, k_B, n)\| \right|,$$

and from Theorem 1, (2.8) and (2.21), $\mathbf{M}^+(S, k_A, k_B) - \mathbf{N}^+(S, k_A, k_B, n) = O(h^2)$. It follows that

$$\left| \|\mathbf{M}^+(S, k_A, k_B)\| - \|\mathbf{N}^+(S, k_A, k_B, n)\| \right| = O(h^2),$$

and the lengths of $\mathbf{M}^+(S, k_A, k_B)$ and $\mathbf{N}^+(S, k_A, k_B, n)$ are within $O(h^2)$.

Theorem 3. *If a clothoid of arc length S_C and a discrete clothoid with $n+1$ arcs of total arc length S_D both have the same starting and ending curvatures k_A and k_B, are both in standard position ($\mathbf{A}$ at the origin and the tangent vector at $\mathbf{A}$ along the positive X-axis), and both have the same distance between centres of curvatures at their respective endpoints, then $S_C - S_D = O(h^2)$, where $h = 1/n$.*

Proof. Subtracting the Eqs. (4.1) and introducing two terms,

$$\|\mathbf{M}^+(S_C, k_A, k_B)\| - \|\mathbf{N}^+(S_D, k_A, k_B, n)\| = 0$$

so

$$\|\mathbf{M}^+(S_C, k_A, k_B)\| - \|\mathbf{M}^+(S_D, k_A, k_B)\| + \|\mathbf{M}^+(S_D, k_A, k_B)\| - \|\mathbf{N}^+(S_D, k_A, k_B, n)\| = 0.$$

The mean value theorem gives the first pair of terms as a multiple of $S_C - S_D$, while the above modification of Theorem 1 gives the second pair of terms as $O(h^2)$. This proves the theorem.

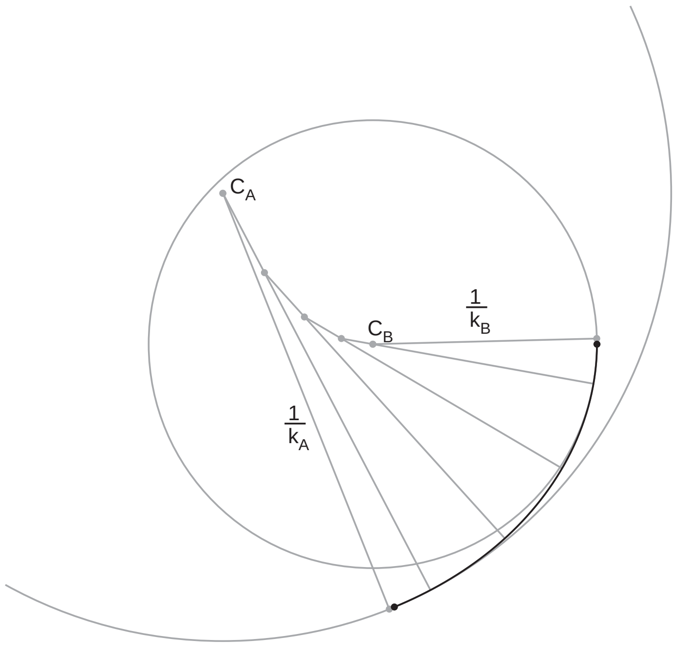

Fig. 6. Circle to circle with a single clothoid, $n=4$, $k_A=0.00392699$, $\mathbf{C}_A=\binom{99.4882}{261.157}$, $k_B=0.00785398$, $\mathbf{C}_B=\binom{184.633}{175.304}$, $S_C=200$, $S_D=206.387$.

Corollary. *The distance between points on the clothoid and on the discrete clothoid in Theorem* 3 *with the same parameter k is*

$$\begin{aligned}&\mathbf{C}(k,S_C,k_A,k_B)-\mathbf{D}(k,S_D,k_A,k_B,n)\\&\quad=[\mathbf{C}(k,S_C,k_A,k_B)-\mathbf{C}(k,S_D,k_A,k_B)]+[\mathbf{C}(k,S_D,k_A,k_B)-\mathbf{D}(k,S_D,k_A,k_B,n)]\\&\quad=\mathrm{O}(h^2)+\mathrm{O}(h^2)=\mathrm{O}(h^2).\end{aligned}$$

The first difference is $\mathrm{O}(h^2)$ *using the mean value theorem and Theorem* 3, *and the second is* $\mathrm{O}(h^2)$ *using Theorem* 1.

The Corollary of Theorem 3 and an argument as in the Corollary of Theorem 1 show that the respective endpoints $\mathbf{B}_C$ and $\mathbf{B}_D$ and the centres of curvature at those endpoints are $\mathrm{O}(h^2)$ apart. If the discrete clothoid is rotated about the centre of curvature at $\mathbf{A}$ (these centres coincide in the standard position) so that the centres of curvature at the endpoints $\mathbf{B}_C$ and $\mathbf{B}_D$ coincide, the rotation will be $\mathrm{O}(h^2)$. This means that points on the clothoid and the discrete clothoid with the same parameter k still will be $\mathrm{O}(h^2)$ apart.

Fig. 6 shows two circles between which both a clothoid and a discrete clothoid with 5 arcs form a transition. The clothoid is shown in black and the discrete clothoid in grey. The clothoid and discrete clothoid in Fig. 6 are compared as in Section 3 and the results are shown in Table 2. Again, the results are of extremely high accuracy and exhibit an $\mathrm{O}(h^2)$ asymptotic behaviour.

Table 2
Maximum scaled perpendicular distance from clothoid to discrete clothoid for various n in the circle to circle case illustrated in Fig. 6

n	Maximum distance (scaled)
4	$3.232 \cdot 10^{-4}$
8	$6.383 \cdot 10^{-5}$
16	$1.406 \cdot 10^{-5}$
32	$3.322 \cdot 10^{-6}$

5. Circle to circle with two clothoids forming an S-curve

In the third arrangement [8], a pair of clothoids forming an S-shape and a pair of discrete clothoids forming an S-shape are used as transition curves from one circle of unsigned curvature k_A to another circle of unsigned curvature k_B. Let the distance between the centres of the circles $\mathbf{C}_A$ and $\mathbf{C}_B$ be d. Let the total arc length of the pair of clothoids be S_C and let the total arc length of the pair of discrete clothoids be S_D. Let the points of contact of the clothoids on the circles be $\mathbf{T}_A$ and $\mathbf{T}_B$ and let the join point of the pair of clothoids, where the curvature is zero, be $\mathbf{Z}$.

Choose the arc lengths of the first and second clothoids in the proportion $k_B : k_A$ so that the arc length of the first clothoid is $[k_B/(k_A + k_B)]S_C$ and the arc length of the second clothoid is $[k_A/(k_A + k_B)]S_C$. This somewhat arbitrary choice simplifies formulas and results in $\mathbf{Z}$ being on the line joining the centres of the two circles (see Fig. 7). To prove the latter statement, consider the two triangles $\mathbf{T}_A\mathbf{Z}\mathbf{C}_A$ and $\mathbf{T}_B\mathbf{Z}\mathbf{C}_B$. Since the clothoids are scaled versions of each other, these triangles are similar, and since the clothoids have a common tangent at $\mathbf{Z}$, the line $\mathbf{C}_A\mathbf{Z}\mathbf{C}_B$ is straight and the point $\mathbf{Z}$ divides $\mathbf{C}_A\mathbf{C}_B$ in the proportion $k_B : k_A$.

Now assume the second clothoid of the pair is in standard position and let the endpoints of the two clothoids be $\mathbf{A}_C$ and $\mathbf{B}_C$. This alignment will cause $\mathbf{Z}$ to be at the origin, $\mathbf{T}_A = \mathbf{A}_C$, and $\mathbf{T}_B = \mathbf{B}_C$. Let $\mathbf{M}_A(S_C)$ and $\mathbf{M}_B(S_C)$ be the vectors from the origin to the centres of curvature at $\mathbf{A}_C$ and $\mathbf{B}_C$,

$$\mathbf{M}_B(S_C) = \mathbf{M}\left(\frac{k_A S_C}{k_A + k_B}, 0, k_B\right), \quad \mathbf{M}_A(S_C) = -\frac{k_B}{k_A}\mathbf{M}_B(S_C).$$

The equation to solve for S_C is

$$f_C(S_C) = \|\mathbf{M}_A(S_C)\| + \|\mathbf{M}_B(S_C)\| = \frac{k_B}{k_A}\|\mathbf{M}_B(S_C)\| + \|\mathbf{M}_B(S_C)\|$$
$$= \frac{k_A + k_B}{k_A}\|\mathbf{M}_B(S_C)\| = d.$$

If the appropriate expression changes sign as $[k_A/(k_A + k_B)]S_C$ varies from 0 to s_B, there is a solution to the equation. The function f_C is monotone increasing (2.9), so if a solution exists, it is unique.

For each of the pair of discrete clothoids, use $n + 1$ arcs and arrange to have the two discrete clothoids in the proportion $k_B : k_A$. As above, the join point divides the line joining $\mathbf{C}_A$ and $\mathbf{C}_B$ in proportion $k_B : k_A$, so the join point is $\mathbf{Z}$. With the second discrete clothoid in standard position, let

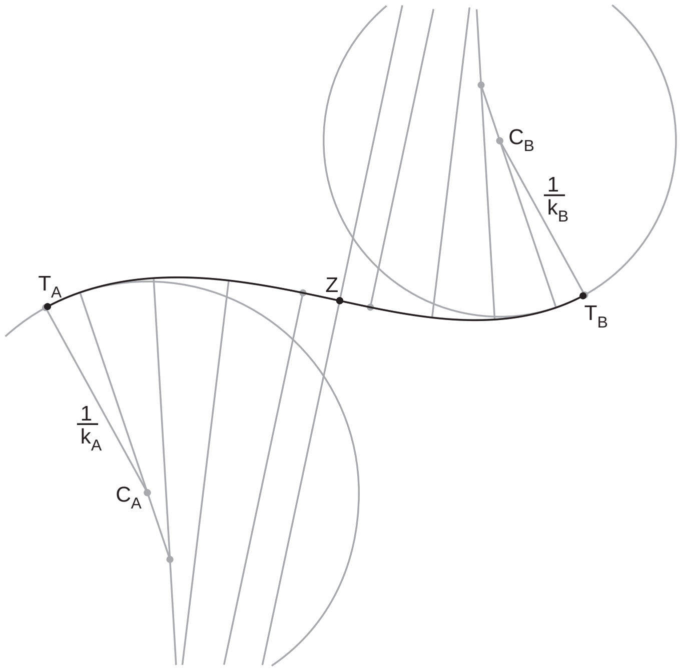

Fig. 7. Circle to circle with a pair of clothoids forming an S-curve, $n = 4$, $k_A = 0.00833333$, $\mathbf{C}_A = \binom{100}{100}$, $k_B = 0.01$, $\mathbf{C}_B = \binom{300}{300}$, $S_C = 310.829$, $S_D = 313.056$.

the endpoints of the two clothoids be $\mathbf{A}_D$ and $\mathbf{B}_D$. The equation for S_D is found using analysis similar to the above and the following theorem results.

Theorem 4. *If two* S*-shaped curves made from a pair of clothoids of total arc length S_C and from a pair of discrete clothoids with $n+1$ arcs and of total arc length S_D both have the same starting and ending curvatures, both have join points (points of zero curvature) at the origin and tangent vectors at those points along the X-axis, and both have the same distance between the centres of curvatures at their respective endpoints, then $S_C - S_D = O(h^2)$, where $h = 1/n$.*

It can be shown that the pair of points $\mathbf{A}_C, \mathbf{A}_D$ and the pair of points $\mathbf{B}_C, \mathbf{B}_D$, and the centres of curvature at those pairs of endpoints, are all $O(h^2)$ apart. If the discrete clothoid pair is translated so that the centre of curvature at $\mathbf{A}_D$ coincides with the centre of curvature at $\mathbf{A}_C$, and is rotated about the centre of curvature at $\mathbf{A}_C$ so that the centre of curvature at $\mathbf{B}_D$ coincides with the centre of curvature at $\mathbf{B}_C$, both transformations will be $O(h^2)$. That means points with the same parameter k on the clothoid pair and the discrete clothoid pair still will be $O(h^2)$ apart.

Fig. 7 shows two circles between which an S-shaped pair of clothoids and an S-shaped pair of discrete clothoids, each with 5 arcs, form a transition. The pair of clothoids is shown in black and the pair of discrete clothoids is shown in grey. The pair of clothoids and pair of discrete clothoids in Fig. 7 are compared as in Section 3 and the results are shown in Table 3. Again, the results are of extremely high accuracy and exhibit $O(h^2)$ asymptotic behaviour.

Table 3
Maximum scaled perpendicular distance from clothoid to discrete clothoid for various n in the circle to circle with an S-curve case illustrated in Fig. 7

n	Maximum distance (scaled)
4	$4.950 \cdot 10^{-4}$
8	$1.084 \cdot 10^{-4}$
16	$2.569 \cdot 10^{-5}$
32	$6.213 \cdot 10^{-6}$

6. Circle to circle with two clothoids forming a C-curve

In the fourth arrangement [8], a pair of clothoids forming a C-shape and a pair or discrete clothoids forming a C-shape are used as transition curves from one circle of unsigned curvature k_A to another circle of unsigned curvature k_B. Let the distance between the centres of the circles $\mathbf{C}_A$ and $\mathbf{C}_B$ be d. Let the total arc length of the pair of clothoids be S_C and let the total arc length of the pair of discrete clothoids be S_D.

Choose the arc lengths of the first and second clothoid in the proportion $k_B : k_A$. Now assume the second clothoid of the pair is in standard position and let the endpoints of the two clothoids be $\mathbf{A}_C$ and $\mathbf{B}_C$. Let $\mathbf{M}_A(S_C)$ and $\mathbf{M}_B(S_C)$ be the vectors from the origin to the centres of curvature at $\mathbf{A}_C$ and $\mathbf{B}_C$,

$$\mathbf{M}_B(S_C) = \mathbf{M}\left(\frac{k_A S_C}{k_A + k_B}, 0, k_B\right), \quad \mathbf{M}_A(S_C) = \frac{k_B}{k_A}\begin{pmatrix} -\mathbf{M}_B(S_C)_x \\ \mathbf{M}_B(S_C)_y \end{pmatrix}.$$

The equation to solve for S_C is

$$f_C(S_C) = \|\mathbf{M}_B(S_C) - \mathbf{M}_A(S_C)\| = \left\| \begin{pmatrix} \mathbf{M}_B(S_C)_x \\ \mathbf{M}_B(S_C)_y \end{pmatrix} - \frac{k_B}{k_A}\begin{pmatrix} -\mathbf{M}_B(S_C)_x \\ \mathbf{M}_B(S_C)_y \end{pmatrix} \right\|$$

$$= \frac{1}{k_A}\sqrt{((k_A + k_B)\mathbf{M}_B(S_C)_x)^2 + ((k_A - k_B)\mathbf{M}_B(S_C)_y)^2} = d.$$

If the appropriate expression changes sign as $[k_A/(k_A + k_B)]S_C$ varies from 0 to s_B, there is a solution to the equation. The function f_C is monotone increasing (2.9), so if a solution exists, it is unique.

For the discrete clothoid, use $n+1$ arcs in each part of the C-curve and arrange to have the two discrete clothoids in the proportion $k_B : k_A$. With the second discrete clothoid in standard position, let the endpoints of the two discrete clothoids be $\mathbf{A}_D$ and $\mathbf{B}_D$. The equation for S_D is found using analysis similar to the above and the following theorem results.

Theorem 5. *If two* C*-shaped curves made from a pair of clothoids of total length* S_C *and made from a pair of discrete clothoids with* $n+1$ *arcs and of total length* S_D *both have their join points* (*points of zero curvature*) *at the origin and the tangent vectors at the those points along the* X*-axis*, *both have the same starting and ending curvatures*, *and both have the same distance between centres of curvatures at their respective endpoints*, *then* $S_C - S_D = O(h^2)$, *where* $h = 1/n$.

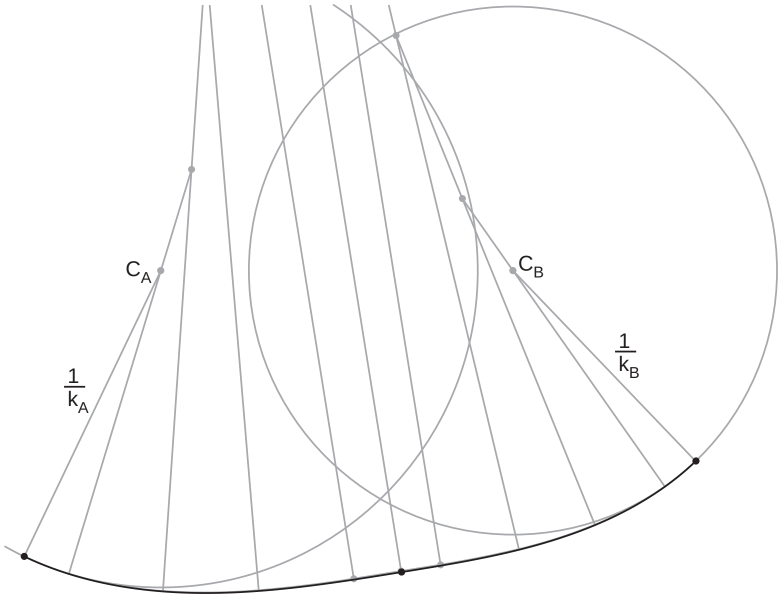

Fig. 8. Circle to circle with two clothoids forming a C-curve, $n = 4$, $k_A = 0.00555556$, $\mathbf{C}_A = \binom{100}{200}$, $k_B = 0.00666667$, $\mathbf{C}_B = \binom{300}{200}$, $S_C = 399.747$, $S_D = 399.284$.

Table 4
Maximum scaled perpendicular distance from clothoid to discrete clothoid for various n in the circle to circle with a C-curve case illustrated in Fig. 8

n	Maximum distance (scaled)
4	$1.157 \cdot 10^{-3}$
8	$3.920 \cdot 10^{-4}$
16	$9.795 \cdot 10^{-5}$
32	$2.447 \cdot 10^{-5}$

It can be shown that points with the same parameter k on the clothoid pair and the discrete clothoid pair are $O(h^2)$ apart. If the C-curve made of a pair of discrete clothoids is translated so the centre of curvature at $\mathbf{A}_D$ coincides with the centre of curvature at $\mathbf{A}_C$, and is rotated about the centre of curvature at $\mathbf{A}_C$ so that the centre of curvature at $\mathbf{B}_D$ coincides with the centre of curvature at $\mathbf{B}_C$, both transformations are $O(h^2)$. That means points with the same parameter k on the clothoid pair and the discrete clothoid pair still will be $O(h^2)$ apart.

Fig. 8 shows two circles between which both a C-shaped pair of clothoids and a C-shaped pair of discrete clothoids, each with 5 arcs, form a transition. The pair of clothoids is shown in black and the pair of discrete clothoids in grey. The pair of clothoids and the pair of discrete clothoids in Fig. 8 are compared as in Section 3 and the results are shown in Table 4. Again, the results are of extremely high accuracy and exhibit $O(h^2)$ asymptotic behaviour.

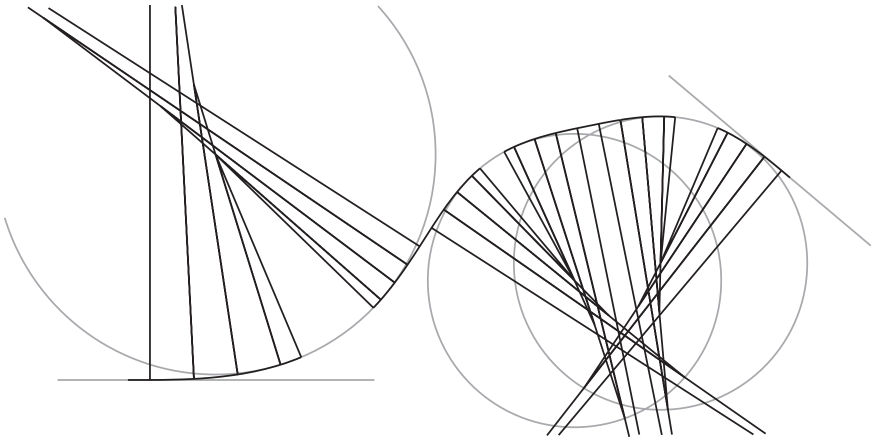

Fig. 9. An example route using discrete clothoids as transition curves.

7. Example and conclusions

Fig. 9 shows an example of using discrete clothoids as transition curves to form a highway route. Starting from the left, the route follows a straight line (grey) (along the X-axis), then a discrete clothoid (black) joins it to the first circle (grey) (curvature 0.008, centre $\binom{59.7613}{128.109}$). The route follows the first circle, then a pair of discrete clothoids in an S-shape (black) joins it to the second circle (grey) (curvature 0.012, centre $\binom{263.670}{56.4850}$). The route follows the second circle, then a pair of discrete clothoids in a C-shape (black) joins it to the third circle (grey) (curvature 0.012, centre $\binom{312.542}{66.3918}$). The route follows the third circle, then a discrete clothoid (black) joins it to the second straight line (grey) (expressed parametrically as $\binom{386.049}{114.960} + t\begin{pmatrix}\cos(-0.7)\\ \sin(-0.7)\end{pmatrix}$). The route in Fig. 9 perhaps overuses discrete clothoids as many highway routes are mainly straight lines and circular arcs joined by short transition curves.

In this paper, it is shown that the clothoid can be replaced by a discrete clothoid and a very high accuracy is achieved even when the discrete clothoid has a small number of arcs. In the examples presented here with discrete clothoids of 5 arcs, the deviation of the discrete clothoid from the clothoid it approximated is around 0.001 of the length of the clothoid. The asymptotic result allows one to estimate how many arcs would be needed for a given required accuracy.

Acknowledgements

The authors appreciate the useful comments from two anonymous referees; their advice helped improve the presentation of this paper. The authors acknowledge the financial support of the Natural Sciences and Engineering Research Council of Canada for this research.

References

[1] K.G. Baass, The use of clothoid templates in highway design, Transportation Forum 1 (1984) 47–52.

[2] R.L. Burden, J.D. Faires, Numerical Analysis, 4th Edition, PWS-Kent Publishing Company, Boston, 1989.
[3] P.J. Davis, Spirals: from Theodorus to Chaos, A.K. Peters, Wellesley, Maine, 1993.
[4] M.A. Heald, Rational approximations for the Fresnel integrals, Math. Comp. 44 (1985) 459–461; errata 46 (1986) 771.
[5] T.F. Hickerson, Route Location and Design, McGraw-Hill Book Company, New York, 1964.
[6] B.K.P. Horn, The curve of least energy, ACM Trans. Math. Software 9 (1983) 441–460.
[7] E. Jahnke, F. Emde, Tables of Functions with Formulae and Curves, 4th Edition, Dover Publications, New York, 1945.
[8] D.S. Meek, D.J. Walton, The use of Cornu spirals in drawing planar curves of controlled curvature, J. Comput. Appl. Math. 25 (1989) 69–78.
[9] D.S. Meek, D.J. Walton, Approximation of discrete data by G^1 arc splines, Comput. Aided Des. 24 (1992) 301–306.
[10] D.S. Meek, D.J. Walton, Approximating smooth planar curves by arc splines, J. Comput. Appl. Math. 59 (1995) 221–231.
[11] E. Mehlum, Nonlinear splines, in: R.E. Barnhill, R.F. Riesenfield (Eds.), Computer Aided Geometric Design, Academic Press, New York, 1974.
[12] L.A. Piegl, W. Tiller, Biarc approximation of NURBS curves, Comput. Aided Des. 34 (2002) 807–814.
[13] W.H. Press, Numerical Recipes in C++: The Art of Scientific Computing, 2nd Edition, Cambridge University Press, Cambridge, 2002.
[14] J. Sánchez-Reyes, J.M. Chacón, Polynomial approximation to clothoids via s-power series, Comput. Aided Des. 35 (2003) 1305–1313.
[15] L.Z. Wang, K.T. Miura, E. Nakamae, T. Yamamoto, T.J. Wang, An approximation approach of the clothoid curve defined in the interval $[0, \pi/2]$ and its offset by free-form curves, Comput. Aided Des. 33 (2001) 1049–1058.