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Separate versus joint continuity: A tale of four topologies

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Abstract

Several naturally occurring topologies on the product $X \times Y$ of the Tychonoff spaces X and Y are studied; each is stronger than the product topology τ. These include the cross topology γ consisting of sets meeting each horizontal and vertical fiber in a set open in the subspace topology induced by τ; the weak topology σ determined by the separately continuous real-valued functions with domain $X \times Y$; and the weak topology determined by certain special separately continuous functions. Functorial relations between γ and σ are described. Sufficient conditions for separately continuous functions to be jointly continuous on a dense subspace of $(X \times Y, \tau)$ are given. The topological structure of $(X \times Y, \sigma)$ is studied in detail. © 1999 Elsevier Science B.V. All rights reserved.

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Dedicated to Wistar Comfort

1. Introduction

All calculus students learn that a function of two real variables (x, y) can be continuous for each fixed x and for each fixed y without being continuous as a function of two real variables. The standard example illustrating this phenomenon is the function *sp* given by:

$$sp(x, y) = \frac{2xy}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0),$$

while $sp(0, 0) = 0$.

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It is clear that *sp* is continuous as a function from the plane $\mathbb{R}^2$ to the real line $\mathbb{R}$ everywhere except at the origin, but $sp(x,x)=1$ for all $x\neq 0$, while $sp(0,y)=0$ for all y. So $sp:\mathbb{R}^2\to\mathbb{R}$ is not continuous at $(0,0)$. On the other hand, $sp(0,y)=sp(x,0)=0$ for all x and y, so *sp* is continuous in each variable separately. Indeed, whenever $(a,b)\in\mathbb{R}^2$, the function $sp_{(a,b)}:\mathbb{R}^2\to\mathbb{R}$ given by $sp_{(a,b)}(x,y)=sp(x-a,y-b)$ translates this difficulty from the origin to (a,b). In fact, since $|sp(x,y)|\leqslant 1$ for all $(x,y)\in\mathbb{R}^2$, it is clear that if $S=\{(a_n,b_n)\colon n\in\omega\}$ is countable then the function $F:\mathbb{R}^2\to\mathbb{R}$ defined by

$$F(x,y)=\sum_{n=0}^{\infty}2^{-n}sp_{(a_n,b_n)}(x,y)$$

is separately continuous, but discontinuous precisely at the points of S (and hence on a dense subset of $\mathbb{R}^2$ if S is dense). See [24] for generalizations of this.

More generally, if X,Y,Z are topological spaces, and $g:X\times Y\to Z$ is a function, and $x\in X$, the function $g_x:Y\to Z$ given by $g_x(y)=g(x,y)$ is called a *vertical section of* g, and $\{x\}\times Y$ is called a *vertical fiber of* $X\times Y$. A *horizontal section* and a *horizontal fiber* are defined similarly. If each horizontal and each vertical section of g are continuous, then g is said to be *separately continuous*. The functions $sp_{(a,b)}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined above are separately continuous but not (jointly) continuous. That is, *sp* fails to be continuous if the product topology is imposed on $\mathbb{R}\times\mathbb{R}$.

There has been intensive study of the relationship between separately and jointly continuous functions for over a century. A thorough but still incomplete survey may be found in [29,30], and the genesis of this subject is discussed in [31]. Almost all of it has been concerned with the behavior of separately continuous function and their sets of points of (joint) continuity or how they may be obtained from separately continuous functions. This paper concentrates on the weak topology on the product space determined by the separately continuous functions and its relation to the usual product topology and two other topologies that arise in a natural way in the course of studying it. Apart from the papers [18,19], little has been done in the past along these lines. We confine our attention to real-valued functions defined on the product of two Hausdorff spaces.

Two of the major results on separately continuous functions are due to I. Namioka, who gave in [25] sufficient conditions on X and Y for each separately continuous function to be jointly continuous on a large dense subspace, and one due to W. Moran [22] giving sufficient conditions for each separately continuous functions to be the pointwise limit of a sequence of jointly continuous functions. These results are also discussed and examined from this point of view.

In Section 2, our four topologies on a product are introduced. In Section 3, the topology σ of separate continuity is contrasted with the cross topology γ in which a subset of the product is open if and only if its trace on each vertical and horizontal section is open. In Section 4, a duality between the k-space coreflection of σ and the complete regularization of γ is described. In Section 5, use is made of cellular families to exhibit large families of separately continuous functions. These are used to show that such functions need not be limits of sequences of jointly continuous functions, and to show that very few subspaces of separately continuous products are pseudocompact. Section 6 discusses a miscellany of properties of the separately continuous topology. Section 7 is devoted to studying when Stone–Čech remainders of separately continuous products are connected. Finally, in Section 8, the question of when a separately continuous product is realcompact is studied, and some unsolved problems are stated.

The kind of general topology that proves to be relevant for this purpose is considered exotic or pathological by many mathematicians.

2. Four topologies on a product

If every member of a topology α on a set X is a union of finite intersections of members of a family $\mathbb{S}$ of subsets of X, then $\mathbb{S}$ is said to *generate* or to be a *subbase for* α. All topological spaces considered below are assumed to be Hausdorff spaces. If $\mathbb{F}$ is a family of real-valued functions defined on a set X, recall that the smallest topology on X making each member of $\mathbb{F}$ continuous is called the *weak topology on X determined by* $\mathbb{F}$, and is denoted in this paper by $\lambda_{\mathbb{F}}$. Let $C(X) = C(X, \alpha)$ denote the family of continuous real-valued functions defined on (X, α), and let $C^*(X)$ denote the set of bounded elements of $C(X)$. The (Hausdorff) space (X, α) is called a *Tychonoff space* if $\alpha = \lambda_{C(X,\alpha)}$; that is, if $\{f^{\leftarrow}[V]\colon\ f \in C(X, \alpha)$ and V open in $\mathbb{R}\}$ generates α. Clearly the weak topology generated by a nonempty collection of real-valued functions is Tychonoff if it is Hausdorff. If $f \in C(X)$ then $Z(f) = f^{\leftarrow}(0)$ is called the *zeroset* of f and $\operatorname{coz}(f) = X \setminus Z(f)$ is called its *cozeroset*. The cozerosets form a base for the topology of a Tychonoff space. See [9, Chapters 1 and 3].

If (X, α) and (X', α') are topological spaces, then the following four topologies are considered on their product $X \times X'$:

The *product topology* τ is generated by $\{U \times U'\colon\ U \in \alpha,\ U' \in \alpha'\}$.

The *cross topology* γ on $X \times X'$ is generated by

$$\bigl\{V \subset X \times X'\colon\ V \cap \bigl(\{x\} \times X'\bigr) \in \tau|\{x\} \times X' \text{ and } V \cap \bigl(X \times \{x'\}\bigr) \in \tau|X \times \{x'\} \\ \text{for all } x \in X \text{ and for all } x' \in X'\bigr\}.$$

That is, a subset of $X \times X'$ is open in the cross topology if its intersection with each vertical fiber and each horizontal fiber is open in the subspace topology induced on these fibers by τ. In [18] the space $(X \times Y, \gamma)$ is denoted by $X \otimes Y$.

The weak topology σ on $X \times Y$ generated by the family $S(X \times Y)$ of separately continuous functions is called the *topology of separate continuity*, or the *separately continuous topology*.

If $f: X \to \mathbb{R}$ and $g: Y \to \mathbb{R}$, define $f \times g: X \times Y \to \mathbb{R}^2$ by letting $(f \times g)(x, y) = (f(x), g(y))$. Note that if f and g are continuous with respect to some topologies, then $f \times g$ is continuous with respect to the corresponding product topology.

We will be concerned also with the weak topology $\lambda_{\mathbb{H}}$ on $X \times Y$ determined by

$$\mathbb{H} = C^*(X \times Y, \tau) \cup \bigl\{sp \circ (f \times g)\colon\ f \in C(X),\ g \in C(Y)\bigr\}.$$

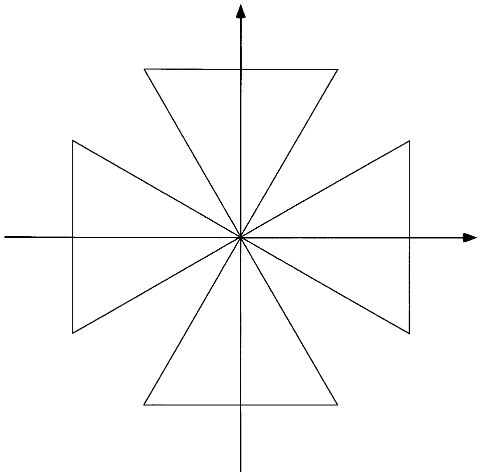

Fig. 1.

Both the cross topology γ and the topology σ of separate continuity contains the product topology τ, and it is not hard to see that σ is contained in γ.

In case $X = X'$ is the real line $\mathbb{R}$ with its usual topology, the function *sp* defined above is not continuous in the product topology, but is continuous as a map from $(\mathbb{R} \times \mathbb{R}, \sigma)$ to $\mathbb{R}$ (and hence also with respect to the stronger cross topology). Note that the τ-interior of the "Maltese cross" in Fig. 1, together with $(0, 0)$, is open in the cross topology on $[-1, 1]$, but not in the product topology.

The cross topology was studied first by J. Novák in [26] where it is called the *inductive product topology*. (A additional paper in this area was published by him in 1971; see [27].) It is called the *tensor product topology* in [18,19]. We encountered it first in [1] in the special case when $X = Y = \mathbb{R}$, where it is called the *Archimedean plane*. It was studied more recently by S. Popvassilev in [33], but this author seems to have been unaware of the Knight, Moran and Pym papers or its connection with the topology of separate continuity. This will be discussed again below. Paul Meyer and a number of other authors have used the terminology "cross topology" for purposes unrelated to ours.

The following simple fact will be needed often in what follows.

Proposition 2.1. *If X and Y are Tychonoff spaces, then $C(X \times Y, \sigma) = S(X \times Y)$.*

Proof. That $S(X \times Y) \subset C(X \times Y, \sigma)$ is immediate from the definition of σ. If $f \in C(X \times Y, \sigma)$ and U is open in $\mathbb{R}$, then $f^{\leftarrow}[U] \in \sigma$. So if $(x, y) \in X \times Y$, then $f^{\leftarrow}[U] \cap [\{x\} \times Y]$ is in $\tau|\{x\} \times Y$ and $f^{\leftarrow}[U] \cap [X \times \{y\}]$ is in $\tau|X \times \{y\}$. Thus, $f \in S(X \times Y)$. $\square$

Since $sp : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is bounded and separately continuous, clearly:

$$\tau \subset \lambda_{\mathbb{H}} \subset \sigma \subset \gamma. \tag{\#}$$

A major purpose of the balance of this paper is to investigate more closely the relationship among these four topologies (as well as a fifth one, defined only when $X = Y$, introduced in Section 6).

If X and Y are discrete, it is clear that these four topologies coincide. So some restrictions on the factor spaces are needed to show that the inclusions in (#) can be proper. Below, we find sufficient conditions on X and Y for them to be distinct, and study their differences when they are distinct.

It is shown in [18, 3.2] that if X and Y are metrizable, X contains a separable completely metrizable subspace, and Y is not discrete; in particular, if X and Y are separable complete metric spaces without isolated points then $(X \times Y, \gamma)$ fails to be regular. Independently, it was shown recently in [33] not to be regular when $X = \mathbb{R}$. Because it is a weak topology determined by a collection of functions, σ is always a Tychonoff topology, so this provides many examples when $\sigma \neq \gamma$. In the next section, it is shown that in such cases, these topologies are very different indeed.

3. The topology of separate continuity versus the cross topology

In this section, we find sufficient conditions on the Tychonoff spaces X and Y to ensure that $(X \times Y, \tau)$ and $(X \times Y, \sigma)$ share the same collection of dense subsets. In this case, σ-continuous functions are determined by their values on a τ-dense subspace. Perhaps the earliest such result is due to W. Sierpínski where this is shown when $X = Y = \mathbb{R}$ [41]. We then apply these conditions to find conditions when each $f \in S(X \times Y)$ must be τ-continuous at each of a dense set of points. We contrast these results with earlier ones due to W. Comfort [3] and I. Namioka [25], and provide some limiting examples. There have been many other generalizations of the Sierpínski theorem many of which are summarized in [35].

If κ is an infinite cardinal number, then a topological space X is called a *κ-Baire space* if an intersection of fewer than κ dense open subspaces of X is dense. An ω_1-Baire space is usually called a *Baire space*. (See [4, p. 134].) X is called *Čech complete* if it is a G_δ in one (and hence all) of its compactifications. It is known that every Čech complete space (and thus every locally compact and every completely metrizable space) is a Baire space. The (usual) topological product of countable many Čech complete spaces is Čech complete, but there is a Baire space X such that (X^2, τ) is not Baire. See [6, Sections 3.9, 3.10].

If Υ is a collection of sets, then Υ^+ abbreviates $\Upsilon \setminus \{\emptyset\}$.

Recall that a *π-base* $\mathcal{B}$ for a topology α on a set X is a subfamily of α^+ such that every member of α^+ contains some element of $\mathcal{B}$. The maximum of ω and the smallest cardinal number of a π-base for (X, α) is called its *π-weight* and is abbreviated by $\pi w(X, \alpha)$ or $\pi w(X)$. If Y is a dense subspace of a Tychonoff space X, then $\pi w(Y) = \pi w(X)$; see [16, 2.3]. A *π-base of* (X, α) *at* x is a subfamily $\mathcal{C}$ of α^+ such that each neighborhood of x contains a member of $\mathcal{C}$. By the *π-weight* $\pi w(X, x)$ *of* X *at a point* x is meant $\omega + \min\{\delta$: there is a π-base of X at x of cardinality $\delta\}$. By the *local π-weight* $l\pi w(X)$ *of* X is meant $\omega + \sup\{\pi w(X, x)\colon x \in X\}$.

Recall from [9] that a subspace Y of a Tychonoff space X is said to be C-embedded (respectively C^*-embedded) in X if the homomorphism $f \to f|_Y$ is a surjection of $C(X)$ (respectively $C^*(X)$) onto $C(Y)$ (respectively $C^*(Y)$). Every Tychonoff space X is a dense subspace of a compact space βX in which it is C^*-embedded. It is called the *Stone–Čech compactification* of X and is unique to within a homeomorphism keeping X pointwise fixed (see [9, Chapter 6] or [37, 4.6(g)]).

Clearly the π-weight of a space is less than or equal to its weight (i.e., $\omega +$ the minimal cardinal number of a base), but this inequality may be strict. For example, if ω is regarded as the (discrete) space of finite ordinals, then $\pi w(\beta\omega) = \omega$, while the weight of $\beta\omega$ is the cardinal number $\mathfrak{c}$ of the continuum. Similarly for any space X, $l\pi w(X) \leqslant \pi w(X)$, but if D is an uncountable discrete space, then $l\pi w(D) = \omega < \pi w(D)$.

Definition 3.1. Suppose ζ and η are topologies on a set T. It will be said that η is *Π-related* to ζ if ζ^+ is a π-base for η and η^+ is a π-base for ς.

(A) Todd uses the terminology *S-related* for our *Π-related* in Section 3 of [42] and shows two things that will be needed below:
  (1) being Π-related is an equivalence relation on the set of topologies on a fixed set, and
  (2) if ζ and η are Π-related topologies on a set T, then (T, ζ) is a Baire space if and only if (T, η) is a Baire space.

Todd uses (2) to show that the Sorgenfrey line $\mathbb{R}$ with the base of all half-open intervals $[a, b)$ as a base for its topology is a Baire space. This concept is also used in [14]. It will be used below only in case one of the two topologies contains the other, in which case the set of nonempty members of the larger one is automatically a π-base for the smaller one.

Proposition 3.2. *Consider the following properties of Hausdorff topologies $\zeta \subset \eta$ on a set X:*
(a) *η is Π-related to ς.*
(b) *Every ζ-dense subspace is η-dense.*
(c) *η-continuous real-valued functions are determined uniquely by their values on a ζ-dense subspace; i.e., if $D \subset X$ is a ζ-dense subspace, $f, g \in C(T, \eta)$, and $f|D = g|D$, then $f = g$.*

Then (a) *and* (b) *are equivalent,* (b) *implies* (c)*, and if η is Tychonoff, then* (c) *implies* (b)*. So, if η is Tychonoff, then* (a), (b)*, and* (c) *are equivalent.*

Proof. It is clear that (a) implies (b) and (b) implies (c).

If (a) fails to hold, then there is an $A \in \eta^+$ whose ς-interior is empty. Then $T \setminus A$ is ς-dense but not η-dense. So (b) fails to hold. Hence (b) implies (a).

Suppose there is a subset M of T that is ς-dense but not η-dense. Then there is a point $p \in T \setminus \mathrm{cl}_\eta M$. If η is a Tychonoff topology, there is an $f \in C(T, \eta)$ such that $f(p) = 1$ and $f[\mathrm{cl}_\eta M] = \{0\}$. Thus f and $\mathbf{0}$ agree on the ς-dense set M without being equal, so (c) implies (b).

If (T,α) is a topological space and $f \in \mathbb{R}^T$, then $C(f,\alpha)$ will denote the set $\{t \in T\colon f$ is α-continuous at $t\}$. $\square$

In [25, p. 517 ff.], I. Namioka establishes his famous result that if X is Čech complete and Y is locally compact and σ-compact, and $f \in S(X \times Y)$, then there is a dense G_δ-set $A \subset X$ such that $C(f,\tau) \supset A \times Y$. (Note that under these hypotheses, $X \times Y$ is a Baire space.) It is noted in [31] that the theorem credited to Namioka appears in Section 39 of H. Hahn's book [11] in a more general form in the special case when the spaces are metrizable.

A similar result will be established in Theorem 3.5 below using a simpler argument than the one in [25].

Proposition 3.3. *Suppose (T,ζ) is a regular Baire space and η is a finer Tychonoff topology on T that is Π-related to ζ. If $f \in C(T,\eta)$, then $C(f,\zeta)$ is ζ-dense.*

Proof. Because ζ is a regular topology, if $C(f,\zeta)$ fails to be ζ-dense, then there is a $W \in \zeta^+$ such that $\mathrm{cl}_\zeta W \cap C(f,\zeta) = \emptyset$. If $x \in \mathrm{cl}_\zeta W$, then f fails to be continuous at x, and hence there is a positive integer n such that if V is a ζ-neighborhood of x then diam $f[V] > 2/n$. Let n_x denote the least such integer. Thus, $f[V] \not\subset (f(x) - 1/n_x, f(x) + 1/n_x)$. Let $T_0 = \emptyset$, and for each positive $n < \omega$, let $T_n = \{x \in \mathrm{cl}_\zeta W\colon n_x \geqslant n\}$. Then each T_n is ζ-closed; for suppose that $y \in \mathrm{cl}_\zeta T_n$. If $y \in V \in \zeta$, there is a $z \in V \cap T_n$. Thus $n_z = n$ and V is a ζ-neighborhood of z. So diam $f[V] > 2/n$, and we may conclude that $y \in T_n$. Thus T_n is ζ-closed.

Because

$$\mathrm{cl}_\zeta W = \bigcup_{n<\omega} T_n$$

and (T,ζ) is a Baire space, there is a $k < \omega$ such that $\mathrm{int}_\zeta T_k$ contains a point p. Since f is η-continuous at p, there is an open η-neighborhood M of p such that diam $f[M] < 1/4k$. Now $\zeta \subset \eta$, so $M \cap \mathrm{int}_\zeta T_k$ is an open η-neighborhood of p. Because ζ^+ is a π-base for η, $M \cap \mathrm{int}_\zeta T_k$ contains some $J \in \zeta^+$. But diam $f[J] > 2/k$ since $J \subset \mathrm{int}_\zeta T_k$, and diam $f[J] < 1/4k$ since $J \subset M$. This contradiction completes the proof. $\square$

Lemma 3.4. *Suppose X and Y are Tychonoff spaces such that Y is a $(l\pi w(X))^+$-Baire space. Then in $X \times Y$:*

(a) *If $V \in \gamma^+$, then* $\mathrm{int}_\tau \mathrm{cl}_\gamma V \neq \emptyset$.
(b) *If $V \in \sigma^+$, then* $\mathrm{int}_\tau V \neq \emptyset$. *That is, every τ-dense subspace is σ-dense.*
(c) *σ is Π-related to τ.*

Proof. (a) Let $\kappa = l\pi w(X)$ and let $(p,q) \in V \in \gamma$. Since Y is regular, there is an open neighborhood W of q such that $\{p\} \times \mathrm{cl}_Y W \subset V$. Suppose $\{B(\alpha)\colon \alpha < \kappa\}$ is a local π-base at p in X. Because X is regular, for each $y \in \mathrm{cl}_Y W$, there is an $\alpha(y) < \kappa$ such that $\mathrm{cl}_X B(\alpha(y)) \times \{y\} \subset V$. For all $\alpha < \kappa$, let $T(\alpha) = \{y \in \mathrm{cl}_Y W\colon \alpha(y) = \alpha\}$, and note that

$\mathrm{cl}_Y W = \bigcup_{\alpha<\kappa} T(\alpha)$. The regular closed set $\mathrm{cl}_Y W$ of the κ^+-Baire space Y is also κ^+-Baire, so there is an $\alpha_0 < \kappa$ such that $\mathrm{int}_Y \mathrm{cl}_Y T(\alpha_0) \neq \emptyset$. Because $T(\alpha_0) \subset \mathrm{cl}_Y W$,

$$S = W \cap \mathrm{int}_Y \mathrm{cl}_Y T(\alpha_0)$$

is a nonempty open subset of Y. Observe that the restrictions of τ and γ to any (vertical) section are the same, so for all $x \in B(\alpha_0)$:

$$\mathrm{cl}_\gamma\big(\{x\} \times T(\alpha_0)\big) = \mathrm{cl}_\tau\big(\{x\} \times T(\alpha_0)\big) = \{x\} \times \mathrm{cl}_Y T(\alpha_0). \tag{$*$}$$

Since $B(\alpha_0) \times T(\alpha_0) \subset V$, by the definitions of S and α_0, and $(*)$ we have:

$$\begin{aligned}\emptyset \neq B(\alpha_0) \times S \subset B(\alpha_0) &\times \mathrm{cl}_Y T(\alpha_0)\\ &= \bigcup_{x\in B(\alpha_0)} \mathrm{cl}_\gamma\big(\{x\} \times T(\alpha_0)\big) \subset \mathrm{cl}_\gamma\Bigg[\bigcup_{x\in B(\alpha_0)} \{x\} \times T(\alpha_0)\Bigg]\\ &= \mathrm{cl}_\gamma\big(B(\alpha_0) \times T(\alpha_0)\big) \subset \mathrm{cl}_\gamma V.\end{aligned}$$

Because $B(\alpha_0) \times S \in \tau^+$, it follows that $\mathrm{int}_\tau \mathrm{cl}_\gamma V \neq \emptyset$.

(b) If $\emptyset \neq V \in \sigma$, then since σ is a regular topology and $\tau \subset \sigma \subset \gamma$, it follows immediately that $\mathrm{int}_\tau V \neq \emptyset$.

(c) follows immediately from (b) by Proposition 3.2. □

In [3], W. Comfort shows that if both X and Y are Baire spaces, and one of them has a countable π-base, then (in the language of this paper), σ is π-related to τ on $X \times Y$. Theorem 3.4 generalizes this as it requires only one factor to be Baire, and relates the Baire number of that factor to the local π-weight of the other factor. (Comfort's result is the special case when $\ell\pi w(X) = \omega$.) However, if neither factor is Baire, the conclusion of Lemma 3.4 and Comfort's result can fail, even if both factors have countable π-weight; see Theorem 3.6(c) below.

Theorem 3.5. *Suppose $X \times Y$ is a Tychonoff Baire space and Y is $(l\pi w(X))^+$-Baire. If $f \in S(X \times Y)$, then $C(f,\tau)$ is a dense subspace of $(X \times Y, \tau)$.*

Proof. This follows immediately from Proposition 3.3 and Lemma 3.4. □

In Theorem 2 of [36], the authors reach the above conclusion under the weaker assumption that each vertical section of f is continuous while each horizontal section is continuous on a dense subspace of its horizontal fiber, provided that Y has countable local π-weight. The referee has observed that it is possible to reach this conclusion with this weaker restriction on f in case Y is $(l\pi w(X))^+$-Baire.

Theorem 3.5 should be contrasted to the well-known result of Namioka (see [25, p. 317ff.]) that if X is Čech complete and Y is locally compact and σ-compact, and $f \in S(X \times Y)$, then there is a dense G_δ-set A of X such that $C(f,\tau) \supset A \times Y$. Because a locally compact space is Čech complete, every Čech complete space is Baire, and Čech completeness is countably productive, these hypotheses imply that $X \times Y$ is a Baire space;

see [6, Section 3.9]. Despite this, neither the hypotheses of Theorem 3.5, nor those of the Namioka result imply the other; see the examples in Examples 3.7 below.

Next, we show how different γ and σ can be.

Theorem 3.6. *Suppose X and Y have countable π-bases and have no isolated points. Then*:

(a) *There is a subset T of $X \times Y$ such that T is dense in the product topology, but is closed and has empty interior in the cross topology.*

(b) *If X and Y are also completely metrizable, then the subset T is also σ-dense.*

(c) *If the topologies of X and Y are locally countable* (*i.e., if each of their points has a neighborhood of countable cardinality*) *and Tychonoff*; *in particular, if both X and Y are the space Q of rational numbers, then σ fails to be Π-related to τ on $X \times Y$.*

Proof. (a) As X and Y have countable π-weight, so does $(X \times Y, \tau)$. Thus τ has a countable π-base $(B_n)_{n\in\omega}$. The set T will be constructed inductively. Choose $(x_0, y_0) \in B_0$. Assume that for some $k \in \omega$, we have chosen $\{(x_i, y_i)\colon 0 \leqslant i \leqslant k\}$ so that $i \neq j$ implies $x_i \neq x_j$ and $y_i \neq y_j$, and $(x_i, y_i) \in B_i$. Since $(X \times Y, \tau)$ has no isolated points, we can choose $(x_{k+1}, y_{k+1}) \in B_{k+1}$ such that $x_{k+1} \notin \{x_i\colon 0 \leqslant i \leqslant k\}$ and $y_{k+1} \notin \{y_i\colon 0 \leqslant i \leqslant k\}$. Let $T = \{(x_i, y_i)\colon i \in \omega\}$.

Clearly T is dense with respect to the topology τ because each member of a π-base of τ contains a point of T. But, since the x_i and y_j are distinct, each horizontal fiber and vertical fiber of $X \times Y$ meets T in at most one point. Thus T is closed, and $\operatorname{int}_\gamma T = \emptyset$ as X and Y have no isolated points.

(b) Because X and Y are Baire spaces, T is σ-dense by Lemma 3.4(c).

(c) Because X and Y are locally countable and Tychonoff, $\sigma = \gamma$ by Theorem 7.2 of [19]. So (c) follows immediately from (a) and Lemma 3.4. □

This section concludes with examples witnessing the effect of the differing hypotheses of our results and that due to Namioka quoted above. In [40], J. Saint-Raymond defines a space X to be a *Namioka space* if for every compact space Y and $f \in S(X \times Y)$, there is a dense G_δ-set $A \subset X$ such that f is τ-continuous on $A \times Y$, and shows that every Namioka space is Baire.

Recall from [9, Chapter 4] that a point a in a Tychonoff space X is called a *P-point* if any G_δ containing a is a neighborhood of a, and X is called a *P-space* if each $a \in X$ is a P-point. Every zeroset of a P-space is open. Recall also from [20] that a Tychonoff space is called an *almost P-space* if each of its zerosets is the closure of its interior. In [45], S. Watson gives an example of a compact almost P-space without any P-points. In [7], N. Fine and L. Gillman show that if X is realcompact and locally compact, then $\beta X \setminus X$ is a compact almost P-space.

Examples 3.7. (a) Any pair of compact spaces satisfies the hypothesis of Namioka's theorem, but need not satisfy the hypotheses of Theorem 3.5.

(b) If $X = Y = \mathbb{P}$ is the space of irrational numbers, then because it is homeomorphic to the countable product of the countable discrete space, it is Čech complete and hence is a Baire space such that $\ell\pi w(X)^+ = \omega_1$. So $\mathbb{P}^2$ satisfies the hypothesis of Theorem 3.5, but fails to satisfy that of Namioka's theorem because $\mathbb{P}$ is not locally compact.

(c) If Y is a compact almost P-space, then it is ω_2-Baire by the lemma on [20, p. 287]. It is known that if X is Baire and Y is Čech complete (in particular, compact), then $X \times Y$ is Baire; see [2, p. 273]. Hence if X is a Baire space of local π-weight ω_1, Y is a compact almost P-space, and $f \in S(X \times Y)$, then $C(f, \tau)$ is dense in $(X \times Y, \tau)$ by Theorem 3.5. There are such Baire spaces that fail to be Čech complete, which may be found as follows. Suppose X and Z are Baire spaces whose product is not Baire. Because closed subspaces of Čech complete spaces are Čech complete, and countable products of Čech complete spaces are Čech complete, neither X nor Z is Čech complete. Clearly, then, their topological sum $X \oplus Z$ is a Baire space that is not Čech complete. Metrizable examples of this kind (which will have countable local π-weight) appear in [8,32]. Consequently, Theorem 3.5 will witness that $C(f, \tau)$ is dense in $(X \oplus Z, \tau)$, while $(X \oplus Z, \tau)$ does not satisfy the hypothesis of Namioka's theorem. Meanwhile, it follows from Lemma 3.4 that if X is any Tychonoff space of local π-weight no greater than ω_1, and Y is a compact almost P-space, then σ is Π-related to τ on $X \times Y$. One could take X to be M^{ω_1}, where M is any noncompact space of local π-weight no greater than ω_1 to get nowhere locally compact spaces of this sort.

(d) In the proof of Theorem 2 of [42], M. Talagrand shows that not every Baire space is a Namioka space by exhibiting a Baire space X', a compact space Y' with a dense set I of isolated points and an $f' \in S(X' \times Y')$ such that $C(f', \tau)$ does not contain $A \times Y'$ for any dense G_δ of Y'. On the other hand, every $f \in S(X' \times Y')$ is τ-continuous on the dense open subset $X' \times I$ of $(X' \times Y', \tau)$. While Theorem 3.5 is not needed to reach this latter conclusion, this example serves to emphasize all the more the difference between Theorem 3.5 and Namioka's theorem.

We close this section with another set of sufficient conditions for the cross topology on a product to fail to be regular.

Theorem 3.8. *If each of X and Y has countable π-weight and no isolated points, and Y is a Baire space, then $(X \times Y, \gamma)$ is not regular.*

Proof. Let T denote the τ-dense subset of $X \times Y$ that is γ-closed and nowhere dense constructed in Theorem 3.6(a). Pick $(a, b) \in (X \times Y) \setminus T$. If $(X \times Y, \gamma)$ were regular, there would be an $S \in \gamma^+$ such that $(a, b) \in S \subset \mathrm{cl}_\gamma S \subset (X \times Y) \setminus T$. Since X has countable π-weight and Y is Baire, Lemma 3.4 yields $\mathrm{int}_\tau \mathrm{cl}_\gamma S \neq \emptyset$. But T is τ-dense and $(\mathrm{int}_\tau \mathrm{cl}_\gamma S) \cap T = \emptyset$. This contradiction yields the desired result. □

In the next section, a remarkable relationship between the cross topology and the topology of separate continuity is studied.

4. The topology of separate continuity versus the cross topology; k-space coreflections and complete regularizations

In this section, we prove that if X and Y are Tychonoff spaces, then $(X \times Y, \sigma)$ and $(X \times Y, \gamma)$ have the same collection of compact subsets; see Corollary 4.7 below. We use this to conclude that $(X \times Y, \sigma)$ is the complete regularization of $(X \times Y, \gamma)$, and that if X and Y are k-spaces, then $(X \times Y, \gamma)$ is the k-space coreflection of $(X \times Y, \sigma)$. (In fact, these are corollaries of more general results we obtain.)

Our work builds on that of Knight et al. [18,19] in which they characterize the compact subsets of $(X \times Y, \gamma)$. Next, we show how different γ and σ can be.

We review the construction of the *complete regularization* $X^{\sim}$ of a topological space X. For $x \in X$, let $[x] = \{x' \in X\colon\ f(x') = f(x)$ for every $f \in C(X)\}$. The underlying set of $X^{\sim}$ is $\{[x]\colon\ x \in X\}$, and its topology is the weak topology making the maps $\{[x] \to f(x) : f \in C(X)\}$ continuous. See [9, 3.3–3.9]. While the definition of $X^{\sim}$ in [19] looks different, it is easy to verify its equivalence with the one just given. Note that X and $X^{\sim}$ coincide if and only if X is Tychonoff. In [19], the notation $X \widetilde{\otimes} Y$ is used for the complete regularization of $X \otimes Y$.

While $\mathbb{R}^2$ is both locally compact and metrizable in the product topology, it will be seen below that neither $\mathbb{R} \otimes \mathbb{R}$ nor $\mathbb{R} \widetilde{\otimes} \mathbb{R}$ possesses either of these properties. There are, however, some interesting topological properties that are preserved when going to these larger topologies.

A topological space (X, α) is called a *k-space*, and its topology is said to be *compactly generated* if any set U such that $U \cap K$ is open in $\alpha|K$ for every compact subspace K of X is open in X. It is known that locally compact spaces and first countable spaces are k-spaces; see [6, 3.3]. Every (Hausdorff) topological space (X, α) has a *k-coreflection* $k(X, \alpha)$, a k-space whose underlying set is X, and whose topology α_k is $\{V \subset X\colon\ V \cap K \in \alpha|K$ whenever K is a compact subspace of $X\}$. It is easy to check that (X, α) and $k(X, \alpha)$ share the same compact subspaces and that they are the same if and only if (X, α) is a k-space. The name *k-coreflection* derives from categorical topology for reasons not given here. See [44, Chapter 10].

The next proposition is asserted implicitly in [19, Section 1]. It serves to reconcile the seemingly different definitions of $X \otimes Y$ given in [18,19].

Proposition 4.1. *Suppose μ is a topology on $X \times Y$ whose restriction to each horizontal and vertical section is the same as the restriction of the product topology. Then the following are equivalent*:

(a) *For each topological space Z, every separately continuous $f : (X \times Y, \mu) \to Z$ is continuous.*
(b) *μ is the cross topology γ.*

Proof. Assume that (a) holds, let $Z = (X \times Y, \gamma)$, and consider the identity map $i : (X \times Y, \mu) \to (X \times Y, \gamma)$. We claim that for each $x \in X$, $i|\{x\} \times Y$ is continuous. For, if W is open in the cross topology γ, then $(\{x\} \times Y) \cap W$ is open in $\{x\} \times Y$ and since the inverse image of W under this restriction map is $(\{x\} \times Y) \cap W$, it is clear that $i|\{x\} \times Y$ is continuous. Similarly, the restriction of i to each horizontal section is continuous, and hence i is separately continuous. By (a), it is continuous. Thus $\gamma \subset \mu$.

If (a) holds and $f:(X \times Y, \mu) \to Z$ is separately continuous, then f is separately continuous as a map of $(X \times Y, \tau)$ into Z. If $x \in X$, then $f|\{x\} \times Y$ is continuous, so the inverse image of V under this mapping is open in $\{x\} \times Y$ for each $x \in X$. Similarly, the inverse image of an open subset V of Z under the mapping $f|X \times \{y\}$ is open in $X \times \{y\}$ whenever $y \in Y$. So $f^{\leftarrow}[V] \in \gamma$ and hence $\mu \subset \gamma$. Thus (b) holds.

Clearly (b) implies (a) and the proposition holds. □

As is shown in [18, 3.2], in [33], and in Theorem 3.8 above, a product space equipped with the cross topology is often not regular. However, the cross topology on a product space is usually easier to visualize than the Tychonoff separately continuous topology. The next result also motivates the study of the cross topology.

Proposition 4.2. *If X and Y are Tychonoff spaces, then $C(X \times Y, \sigma) = C(X \times Y, \gamma)$.*

Proof. $C(X \times Y, \sigma) \subset C(X \times Y, \gamma)$ because $\sigma \subset \gamma$. Conversely, if $f \in C(X \times Y, \gamma)$ and $U \subset \mathbb{R}$ is open, then the trace of $f^{\leftarrow}[U]$ on each horizontal section and on each vertical section is open, so

$$f \in S(X \times Y) = C(X \times Y, \sigma).$$

(See Proposition 2.1.) □

Next, it will be shown that $(X \times Y, \gamma)$ and $(X \times Y, \sigma)$ share the same compact sets. First we make a definition and prove a lemma.

Definition 4.3. Suppose $\mathcal{P}$ is a topological property, X, Y are Tychonoff spaces, and α is a topology on $X \times Y$. If each subspace of $(X \times Y, \alpha)$ that satisfies $\mathcal{P}$ is a union of finitely many subspaces of horizontal or vertical fibers (with respect to the topology induced by the product topology τ) that satisfy $\mathcal{P}$, then $(X \times Y, \alpha)$ is said to satisfy $CF(\mathcal{P})$. If $\mathcal{P}$ is compactness, then $CF(\mathcal{P})$ is abbreviated by CF. That is, $(X \times Y, \alpha)$ has CF if each of its compact subspaces is a union of compact subspaces of finitely many vertical fibers and horizontal fibers.

The following is [18, 4.3].

Theorem 4.4. *Let X and Y be T_1-spaces, and let $\mathcal{P}$ be a topological property preserved by finite unions and closed subspaces, and contained in countable compactness. Then the cross product $(X \times Y, \gamma)$ has $CF(\mathcal{P})$.*

We will show that Theorem 4.4 remains true when the cross product is replaced by $(X \times Y, \lambda_{\mathbb{H}})$. This will imply that if $\mathcal{P}$ is any topological property preserved by finite unions, closed subspaces, continuous bijections, and contained in countable compactness (such as compactness and countable compactness), then $(X \times Y, \gamma)$ and $(X \times Y, \lambda_{\mathbb{H}})$ share the same subspaces that have property $\mathcal{P}$.

Lemma 4.5. *Suppose $\sum_{n=0}^{\infty} r_n$ is a convergent series of positive real numbers, X is a Tychonoff space, $S = \{x_n\}_{n\in\omega}$ is a countable discrete subspace of X, and $L = \mathrm{cl}_X S \setminus S$. Then there is an $f \in C^*(X)$ such that $f[L] = 0$, and $f(x_j) = r_j$, for any $j \in \omega$.*

Proof. Because X is Tychonoff, each x_n is isolated in $\mathrm{cl}_X S$, and L is therefore closed, there is a $g_n \in C(X)$ such that $\mathbf{0} \leqslant g_n \leqslant \mathbf{1}$, $g_n(x_n) = 1$, and $g_n[L \setminus \{x_n\}] = 0$. Let $f = \sum \boldsymbol{r}_n g_n$. Then by the Weierstrass test for uniform convergence, $f \in C^*(X)$ and has the desired properties. $\square$

In the remainder of this paper, whenever we refer to closures or interiors in a product space, the topology used will be the product topology τ unless the contrary is stated explicitly.

Theorem 4.6. *If X and Y are Tychonoff spaces and $\mathcal{P}$ is a topological property that implies countable compactness, then $(X \times Y, \lambda_{\mathbb{H}})$ satisfies CF$(\mathcal{P})$.*

Proof. Suppose the contrary. Abbreviate $(X \times Y, \lambda_{\mathbb{H}})$ by Z. Then there is a subset K of Z that satisfies $\mathcal{P}$ and contains a sequence $\{(x_n, y_n)\colon n \in \omega\}$ such that both $\{x_n\colon n \in \omega\}$ and $\{y_n\colon n \in \omega\}$ are sequences of distinct points. Indeed, because our spaces are Hausdorff, each of $\{x_n\colon n \in \omega\}$ and $\{y_n\colon n \in \omega\}$ may be replaced by sequences that are discrete. Let $M = \{(x_n, y_n)\colon n \in \omega\}$, $E = \mathrm{cl}\{x_n\colon n \in \omega\}$, $F = \mathrm{cl}\{y_n\colon n \in \omega\}$, $L_1 = E \setminus \{x_n\colon n \in \omega\}$, and $L_2 = F \setminus \{y_n\colon n \in \omega\}$. Because these sequences are discrete, L_1 is closed in X and L_2 is closed in Y. We will show next that:

$$\mathrm{cl}_Z M \setminus M \subset L_1 \times L_2. \tag{†}$$

To see this, first note that since $\tau \subset \lambda_{\mathbb{H}}$, we have $\mathrm{cl}_Z M \subset \mathrm{cl}\, M$. So to establish (†), it suffices to show that

$$(X \times Y) \setminus L_1 \times L_2 \subset \big((X \times Y) \setminus \mathrm{cl}\, M\big) \cup M.$$

Suppose $(p, q) \in (X \times Y) \setminus (L_1 \times L_2)$. We may assume $p \notin L_1$. Then either $p \notin E$ or $p = x_j$ for some $j \in \omega$. In the first case, $(X \setminus E) \times Y$ is a τ-neighborhood of (p, q) disjoint from M, and hence $(p, q) \in (X \times Y) \setminus \mathrm{cl}\, M$.

In the second case, first note that if $q = y_j$, then $(p, q) \in M$. If $q \neq y_j$, then $(p, q) \in ((X \setminus E) \cup \{x_j\}) \times (Y \setminus \{y_j\}) = W$. Because $E \setminus \{x_j\}$ is closed in X, it follows that $W \in \tau$ and $W \cap M = \emptyset$. So $(p, q) \in X \times Y \setminus \mathrm{cl}\, M$ in this case as well. Hence (†) holds.

Because $\sum_{n=0}^{\infty} 2^{-n}$ converges, there is by the previous lemma an $f \in C^*(X)$ and a $g \in C^*(Y)$ such that $f[L_1] = g[L_2] = 0$ and $f(x_n) = g(y_n) = 2^{-n}$ for each $n < \omega$.

Define $F(x, y) = 2f(x)g(y)/([f(x)]^2 + [g(y)]^2)$ if $(x, y) \notin Z(f) \times Z(g)$, and let $F(x, y) = 0$ otherwise; that is, let

$$F = sp \circ (f \times g) \in C^*(X \times Y, \lambda_{\mathbb{H}}).$$

By (†), $\mathrm{cl}_Z M = M \cup (L_1 \times L_2)$. If $(x, y) \in L_1 \times L_2 \subset Z(f) \times Z(g)$, then $F(x, y) = 0$, while if $(x_n, y_n) \in M$, then $F(x_n, y_n) = 1$. Hence the inverse image under F of the open interval $(-1, 1)$ is open in Z, contains $L_1 \times L_2$, and is disjoint from M. Thus

$$M \subset \mathrm{cl}_Z M \setminus F^{\leftarrow}\big[(-1, 1)\big]$$

is an infinite closed discrete subspace of the countably compact space K. This contradiction completes the proof of the theorem. □

As noted above, the next corollary is established in more generality in [18, 4.3] in the case of the cross topology.

Corollary 4.7. *If X and Y are Tychonoff spaces and α is a topology on $X \times Y$ that contains $\lambda_{\mathbb{H}}$ and whose restrictions to X and Y are the given topologies of X and Y, then a subspace of $(X \times Y, \alpha)$ is compact if and only if it is a finite union of compact subspaces of horizontal or vertical fibers. In particular, every compact subspace of $X \times Y$ with either the cross or the separately continuous topology has this property.*

In Theorem 5.8 below, it will be shown that if α is replaced by the topology of separate continuity in the hypothesis of Corollary 4.7, then "compact" may be replaced by "pseudocompact" in its conclusion. Recall that X is said to be *pseudocompact* if $C(X) = C^*(X)$.

The next theorem is the main result of this section and special cases of it are obtained in [19].

Theorem 4.8. *Suppose X and Y are Tychonoff spaces. Then*:

(a) *$(X \times Y, \sigma)$ is the complete regularization of $(X \times Y, \gamma)$.*

(b) *If X and Y are Tychonoff k-spaces, then the k-space coreflection of $(X \times Y, \sigma)$ is $(X \times Y, \gamma)$.*

Proof. (a) By Proposition 2.1, each of these two topologies on $X \times Y$ admit the same continuous real-valued functions. Hence the topology of separate continuity is the weak topology generated by $C(X \times Y, \gamma)$ and $(X \times Y, \sigma) = (X \times Y, \gamma)^{\sim}$.

(b) This follows immediately from Corollary 4.7 and the fact that $k(X \times Y, \gamma) = (k(X) \times k(Y), \gamma)$ (established in [18, 4.4]). □

The next corollary follows immediately from Corollary 4.7 and the fact that no space satisfying its hypothesis can contain a nonempty open set with compact closure.

Corollary 4.9. *If X and Y are Tychonoff spaces with no isolated points and α is a topology on $X \times Y$ that contains $\lambda_{\mathbb{H}}$ and whose restrictions to X and Y are the given topologies of X and Y, then $(X \times Y, \alpha)$ is nowhere locally compact.* (*That is, no point has a compact neighborhood.*)

5. The separately continuous topology versus the product topology; cellular families, Moran's theorem, and the absence of pseudocompactness

In this section, we compare the topologies τ and σ on a product under a variety of restrictions on the factor spaces. We begin by showing how to construct large families of separately continuous functions with the aid of cellular families.

Definition 5.1. Suppose X is a topological space.

(a) A family of pairwise disjoint nonempty open subsets of a space X is called a *cellular family* in X.

(b) The *cellularity* $c(X)$ of a space X is defined to be $\sup\{|\mathcal{V}|\colon \mathcal{V}$ is a cellular family in $X\} + \omega$.

(c) The *density character* $d(X)$ of X is defined to be $\min\{|D|\colon D$ is dense in $X\} + \omega$. (Clearly, $d(X) = \omega$ if and only if X is separable.)

(d) If $c(X) = \omega$, then X is said to satisfy the *countable chain condition* or to be a ccc space.

Clearly $c(X) \leqslant d(X)$, whence every separable space is ccc. For examples of when this inequality is strict and further discussion of these cardinal invariants, see [13,16].

A function $f : X \to \mathbb{R}$ that is a pointwise limit of a sequence of continuous functions is said to be of *Baire class* 1, or to be a *Baire* 1 *function*. The family of *Baire* 1 functions on X is denoted by $B_1(X)$. It is shown in [13, 10.10] that if X is Tychonoff, then $|C(X)|^\omega = |C(X)|$, and it follows that $|B_1(X)| = |C(X)|$. Hence we have:

Lemma 5.2. *If X and Y are Tychonoff spaces such that $S(X \times Y) \subset B_1(X \times Y, \tau)$, then $|C(X \times Y, \sigma)| = |C(X \times Y, \tau)|$.*

A topological space X is said to be a *dccc* space if every discrete collection of pairwise disjoint open sets is countable. Clearly any ccc space is dccc. A celebrated theorem of W. Moran as generalized by G. Vera in [43] states that:

Theorem 5.3 (Moran–Vera). *If X is a Tychonoff dccc space, and Y is compact, then every separately continuous real-valued function on $X \times Y$ is Baire* 1 (*and consequently $|C(X \times Y, \sigma)| = |C(X \times Y, \tau)|$*).

In case $X = Y = \mathbb{R}$, this was established by H. Lebesgue in his first published paper in 1898. For a particularly elegant exposition of Lebesgue's techniques and some of its more easily understood extensions making use of the fact that metrizable spaces are paracompact, see [39].

Note that if $\mathcal{W} = \{W_i\}$ is a cellular family in a Tychonoff space, then there are nonempty cozerosets U_i such that $U_i \subset W_i$ for all indices i. The next lemma describes a methods for constructing large families of separately continuous functions. Its proof is an exercise.

Lemma 5.4. *Suppose X and Y are Tychonoff spaces. Let κ denote an infinite cardinal for which there are cellular families of cozerosets $\mathcal{U} = \{U_i\}_{i<\kappa}$ in X and $\mathcal{V} = \{V_i\}_{i<\kappa}$ in Y, such that for each $i < \kappa$, $U_i = \operatorname{coz}(f_i)$ and $V_i = \operatorname{coz}(g_i)$ for some $f_i \in C(X)$, and $g_i \in C(Y)$. If $J \subset \kappa$, define $F_J : X \times Y \to \mathbb{R}$ by*

$$F_J(x, y) = \begin{cases} f_i(x)g_i(y) & \text{if } (x, y) \in U_i \times V_i \text{ and } i \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Then each $F_J \in S(X \times Y)$, $\operatorname{coz}(F_J) = \bigcup\{U_i \times V_i\colon i \in J\}$, and $F_{J_1} \neq F_{J_2}$ unless $J_1 = J_2$. In particular, $|S(X \times Y)| \geqslant 2^{\kappa}$.

Next, examples will be presented to show that if the factor spaces have uncountable cellularity, there may be too many separately continuous functions for the conclusion of Moran's theorem to hold.

If T is a subalgebra of $\mathbb{R}^X$, let T^* denote the family of bounded elements of T. Recall that if for $f \in T^*$, we let $\|f\| = \sup\{|f(x)|\colon x \in X\}$, then T^* becomes a topological algebra and a metric space with respect to the metric ρ obtained by letting $\rho(f, g) = \|f - g\|$. The resulting topology u on T^* is called the *uniform topology*.

If $\mathbb{F}$ is a subalgebra of T^*, the *uniform closure* of $\mathbb{F}$ is defined to be the closure of $\mathbb{F}$ in (T^*, u), and is denoted by $\mathcal{U}(\mathbb{F})$. The classical Stone–Weierstrass theorem says that if X is compact, and $\mathbb{F} \subset C(X)$ is a subalgebra containing the constant functions that separates the points of X, then the closure of $\mathbb{F}$ in the uniform topology is $C(X)$. See [9].

Let X and Y denote compact spaces. Because $\{f \times g\colon f \in C(X),\ g \in C(Y)\}$ separates the points of $X \times Y$, by the Stone–Weierstrass theorem, its uniform closure is $C(X \times Y)$. Hence, using from [13, 10.10] the fact that $|C(Z)|^{\omega} = |C(Z)|$ for any Tychonoff space Z, we obtain.

Lemma 5.5. *If X and Y are infinite compact spaces, then $|C(X \times Y)| = |C(X)||C(Y)|$.*

The results about when separately continuous functions fail to be Baire 1 follow.

The next theorem improves on Proposition 5 in [34] where this result is obtained in the special case when $X = Y$ has a dense set of isolated points.

Theorem 5.6. *Suppose X and Y are Tychonoff spaces and $c(X) \leqslant c(Y)$.*

(a) *If there are cellular families in X and Y of cardinality $c(X)$, then $|S(X \times Y)| \geqslant 2^{c(X)}$.*

(b) *If X and Y are compact and have cellular families of cardinality $\mathfrak{c}$ and $|C(X)| = |C(Y)| = \mathfrak{c}$, then $|S(X \times Y)| > |C(X \times Y, \tau)|$.*

Proof. (a) follows immediately from Lemma 5.4. If the hypothesis of (b) holds, then by Lemma 5.5, $|S(X \times Y)| \geqslant 2^{\mathfrak{c}} > \mathfrak{c} = |C(X \times Y)|$. $\square$

Example 5.7. It is shown in [9, 6Q(2)] that $\beta\omega \setminus \omega$ has a cellular family of cardinality $\mathfrak{c}$. Because it is a homomorphic image of $C(\beta\omega)$, $|C(\beta\omega \setminus \omega)| = \mathfrak{c}$, and consequently

$|C(\beta\omega \setminus \omega)^2| = \mathfrak{c}$. Thus $|S([\beta\omega \setminus \omega]^2)| > |C[(\beta\omega \setminus \omega)]^2|$ by Theorem 5.6(b). So the conclusion of the Moran theorem fails to hold in this case.

Next, it shown that product spaces are usually not pseudocompact in the separately continuous topology. The next two results improve on Corollary 4.7 in case the topology α is σ.

Theorem 5.8. *If X and Y are Tychonoff spaces and $S \subset X \times Y$ is not contained in the union of finitely many horizontal or vertical fibers of $X \times Y$, then S contains a countably infinite* (*closed*) *C-embedded discrete subset D of $(X \times Y, \sigma)$.*

Proof. By assumption, there are subsets $\{x_n\}_{n<\omega}$ of X and $\{y_n\}_{n<\omega}$ of Y such that $x_i \neq x_j$ and $y_i \neq y_j$ unless $i = j$, and $\{(x_n, y_n)\colon n < \omega\} \subset S$. Because X and Y are Hausdorff spaces, subsets of $\{x_n\}_{n<\omega}$ and $\{y_n\}_{n<\omega}$ may be chosen so that the resulting sets are discrete. By a change of notation, we may assume that $\{x_n\}_{n<\omega}$ and $\{y_n\}_{n<\omega}$ are discrete. Using the fact that X and Y are Hausdorff again allows us to obtain cellular families of cozerosets $\{U_n\}_{n<\omega}$in X and $\{V_n\}_{n<\omega}$ in Y such that $x_m \in U_n$ and $y_m \in V_n$ if and only if $m = n$. Also, since these spaces are Tychonoff, there are for each $n < \omega$ an $f_n \in C(X)$ and a $g_n \in C(Y)$ such that

$$U_n = \mathrm{coz}(f_n), \quad f_n(x_n) = n, \quad V_n = \mathrm{coz}(g_n), \quad \text{and} \quad g_n(x_n) = n.$$

Letting $F(x, y) = f_n(x)g_n(y)$ if $(x, y) \in U_n \times V_n$ and $F(x, y) = 0$ if $x \in X \setminus \bigcup\{U_n \times V_n\colon n < \omega\}$, it follows from Lemma 5.4 that $F \in S(X \times Y)$ and $F(x_n, y_n) = n^2$ for each $n < \omega$. Let

$$D = \bigl\{(x_n, y_n)\colon n < \omega\bigr\}.$$

Then $F[D] = \{n^2\colon n < \omega\}$, so F carries D homeomorphically onto a closed subset of $\mathbb{R}$. Thus, by 1.19 of [9], D is C-embedded in $(X \times Y, \sigma)$. By 3B(3) of [9], the countable C-embedded subspace of the Tychonoff space $(X \times Y, \sigma)$ is closed. Since $D \subset S$, the theorem is proved. □

In [21], M. Mandelker calls a subspace Y of a space X *relatively pseudocompact* if $f|Y \in C^*(Y)$ for each $f \in C(X)$. Clearly, every pseudocompact subspace is relatively pseudocompact, but, as noted on [21, p. 74], the right hand edge of the Tychonoff plank is a copy of ω with the discrete topology that is relatively pseudocompact, so the converse does not hold.

The next corollary follow immediately from Theorem 5.8.

Corollary 5.9. *If X and Y are Tychonoff spaces, then*:

(a) *Any relatively pseudocompact subspace of $(X \times Y, \sigma)$ is contained in the union of finitely many horizontal or vertical fibers of $X \times Y$.*
(b) *If X and Y are infinite, then $(X \times Y, \sigma)$ is not pseudocompact.*
(c) *If X and Y are infinite and have no isolated points, then $(X \times Y, \sigma)$ is nowhere locally pseudocompact.*

6. More on the separately continuous topology versus the product topology; connectedness, nonnormality, the diagonal of a square, and the cardinal d

The next proposition shows that while σ is much larger than τ, it cannot be large enough to destroy connectedness.

Proposition 6.1. *If X and Y are connected, then so is $(X \times Y, \sigma)$.*

Proof. Suppose (x, y) and (x', y') are distinct points of $X \times Y$. It suffices to show that there is a connected subset of $(X \times Y, \sigma)$ containing both of them. By assumption $X \times \{y\}$ and $\{x'\} \times Y$ are connected subspaces of $(X \times Y, \sigma)$. Their union T is a connected subspace of $(X \times Y, \sigma)$ because they have the point (x', y) in common, and T contains both of these points. $\square$

In [19] it is shown that requiring that $(X \times Y, \sigma)$ be metrizable or even paracompact puts severe restrictions on the topologies of X or Y. In Theorem 6.5 below, we show that $(X \times Y, \lambda_{\mathbb{H}})$ and $(X \times Y, \sigma)$ fail to be normal even when X and Y are "nice" spaces.

The well-known theorem of Tietze and Urysohn says that X is normal if and only if every closed subspace of X is C^*-embedded; see [9, 3.12].

The projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are given by $\pi_X\ (x, y) = x$ and $\pi_Y(x, y) = y$, respectively. If $f : X \to Y$, then, as usual, its *graph* is defined to be $\{(x, f(x)){:}\ x \in X\}$. The facts in the next lemma are recorded in [6, 3.1D].

Lemma 6.2. *If X is Tychonoff and Y is compact, and if the graph of $f : X \to Y$ is closed in $(X \times Y, \tau)$, then f is continuous.*

Theorem 6.3. *Suppose X is a Tychonoff space and Y is compact and metrizable with metric d.*

(a) *If G is a closed subset of $(X \times Y, \tau)$ such that $\pi_X|G : G \to X$ is a bijection and $\pi_Y|G : G \to Y$ is injective, then G is a* (*closed*) *discrete subspace of* $(X \times Y, \lambda_{\mathbb{H}})$.

(b) *The graph of a strictly monotone continuous function on* $[0, 1]$ *into itself is a closed discrete subspace of* $([0, 1]^2, \sigma)$.

Proof. (a) Define $f : X \to Y$ by letting $f(x) = y$ if $(x, y) \in G$, and note that G is the graph of f. By hypothesis f is well-defined, and by Lemma 6.2, f is continuous. Suppose $(p, f(p)) \in G$, and define continuous functions $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$ by letting $g(x) = d(f(p), f(x))$ and $h(y) = d(f(p), y)$. If $F = sp \circ (g \times h)$, then by the definition of $\lambda_{\mathbb{H}}$ given in Section 2, $F \in C\ (X \times Y, \lambda_{\mathbb{H}})$. It is routine to verify that if $a \in X \setminus \{p\}$, then $F(a, f(a)) = 1$ and that $F(p, f(p)) = 0$. Thus, $G \cap F^{\leftarrow}[(-\frac{1}{2}, \frac{1}{2})] = \{(p, f(p))\}$, and it follows that the closed set G is discrete.

(b) follows immediately from (a). $\square$

As usual, $\Delta(X) = \{(x, x){:}\ x \in X\}$ is called the *diagonal* of the product X^2. If α is a topology on X^2, $(\Delta(X), \alpha|\Delta(X))$ will be abbreviated by $(\Delta(X), \alpha)$.

Next, the restriction of various product topologies on X^2 to the diagonal are studied. For this purpose, yet another topology on X^2 is introduced. Let

$$\mathbb{H}_\Delta(X^2) = C^*(X^2, \tau) \cup \{sp \circ (f \times f)\colon\ f \in C(X)\}$$

and let τ_Δ denote the weak topology on X^2 generated by $\mathbb{H}_\Delta(X^2)$. It should be clear from the discussion in Section 2 that on X^2:

$$\tau \subset \tau_\Delta \subset \lambda_{\mathbb{H}} \subset \sigma.$$

Every zeroset in a Tychonoff space is a closed G_δ and a singleton G_δ-set in a Tychonoff space is a zeroset; see [9, Chapters 1 and 3].

Theorem 6.4. *Suppose X is a Tychonoff space.*

(a) *If $\{a\}$ is a G_δ in X, then (a, a) is an isolated point of $(\Delta(X), \lambda_\Delta)$.*

(b) *If each point of X has a countable base of neighborhoods; in particular, if X is metrizable, then $(\Delta(X), \lambda_\Delta)$ is discrete.*

Proof. (a) As is shown in [9, 3.11], there is an $f \in C^*(X)$ such that $Z(f) = \{a\}$ and $\mathbf{0} \leqslant f \leqslant \mathbf{1}$. If $F = sp \circ (f \times f)$, then $F \in \mathbb{H}_\Delta(X^2)$ and one easily verifies that $\{(a, a)\} = F^{\leftarrow}[(-1, 1)] \cap \Delta(X)$. The result follows.

(b) follows immediately from (a). $\square$

The next result tells us that enlarging the usual product topology to the separately continuous topology can easily destroy normality.

Theorem 6.5. *If X is a separable Baire Tychonoff space that satisfies the first axiom of countability and has cardinality $\mathfrak{c}$ (in particular, if X is a compact metrizable space without isolated points), then (X^2, σ) is not normal.*

Proof. Clearly (X^2, τ) is separable. Hence by Lemma 3.4 and Proposition 3.2, so is (X^2, σ). Thus $|C(X^2, \sigma)| = \mathfrak{c}$. But the diagonal $\Delta(X)$ is the graph of the continuous identity function of X onto X, so it is closed, and hence is σ-closed. It is discrete with respect to σ by Theorem 6.4. So, if (X^2, σ) were normal, we would have $|C(X^2, \sigma)| = 2^{\mathfrak{c}}$. So (X^2, σ) is not normal. (See Jones' Lemma in [13, Section 3].) $\square$

Next, it shown that the inclusion $\lambda_{\mathbb{H}} \subset \sigma$ can be proper.

Given a topological space (X, α), the result of enlarging α to a topology α_δ by making each G_δ of X open results in a P-space $X_\delta = (X, \alpha_\delta)$ called the *P-space coreflection* of (X, α). (To recall the definition of a P-space, see [9, Chapter 4] or see the material preceding 3.7.) For any (Tychonoff) space (Z, α) and $r \in Z$, let $\mathcal{N}_\alpha(r)$ denote the family of neighborhoods of r, and let $P(Z)$ denote the set of P-points of Z.

Lemma 6.6. *If $p \in P(X)$ and $q \in P(Y)$, then $\mathcal{N}_\tau((p, q)) = \mathcal{N}_{\lambda_{\mathbb{H}}}((p, q))$ (in $X \times Y$).*

Proof. $\mathcal{N}_\tau((p,q)) \subset \mathcal{N}_{\lambda_{\mathbb{H}}}((p,q))$ because $\tau \subset \lambda_{\mathbb{H}}$. To reverse this inclusion, it is enough to show that for any open interval $(a,b) \subset \mathbb{R}$, $f \in C(X)$ and $g \in C(Y)$, if $(p,q) \in L = (sp \circ (f \times g))^{\leftarrow}[(a,b)]$, then $L \in \mathcal{N}_\tau((p,q))$. We consider two cases.

If $(p,q) \notin Z(f) \times Z(g)$, then because $sp \circ (f \times g)$ is τ-continuous at each point of $X \times Y \setminus (Z(f) \times Z(g))$, $L \cap [(X \times Y) \setminus (Z(f) \times Z(g))] \in \tau$. Thus $(p,q) \in L \cap [X \times Y \setminus (Z(f) \times Z(g))] \subset L$. Hence $(p,q) \in \operatorname{int} L$ and we know that $L \in \mathcal{N}_\tau((p,q))$.

If $(p,q) \in Z(f) \times Z(g)$,then because p and q are P-points, $(p,q) \in \operatorname{int}_X Z(f) \times \operatorname{int}_Y Z(g) \in \tau$. Further, $(sp \circ (f \times g))\ (p,q) = 0$, so $(p,q) \in \operatorname{int}_X Z(f) \times \operatorname{int}_Y Z(g) \subset (sp \circ (f \times g))^{\leftarrow}(0) \subset (sp \circ (f \times g))^{\leftarrow}[(a,b)] = L$, and again $(p,q) \in \operatorname{int}_\tau L$.

Thus, $L \in \mathcal{N}_\tau((p,q))$. Since each member of a subbase for $\lambda_{\mathbb{H}}$ that contains (p,q) also contains it in its τ-interior, we conclude that $\mathcal{N}_\tau((p,q)) = \mathcal{N}_{\lambda_{\mathbb{H}}}((p,q))$. $\square$

Lemma 6.7. *If p is a nonisolated P-point of the zero-dimensional space X, then $\mathcal{N}_\sigma((p,p)) \neq \mathcal{N}_{\lambda_{\mathbb{H}}}((p,p))$ (in X^2).*

Proof. Let $\mathcal{B} = (B_i)_{i \in I}$ be a maximal family of pairwise disjoint clopen subsets of X, each contained in $X \setminus \{p\}$. The maximality of $\mathcal{B}$ and the fact that p is nonisolated implies that $(p,p) \in \operatorname{cl}(\bigcup_{i \in I} B_i \times B_i)$. Define $F : X^2 \to \mathbb{R}$ by

$$F\left[\bigcup_{i \in I} B_i \times B_i\right] = \{1\}, \quad \text{and}$$

$$F\left[X^2 \Big\backslash \bigcup_{i \in I} B_i \times B_i\right] = \{0\}.$$

By Lemma 5.4, the fact that $\mathcal{B}$ is a cellular family implies that F is separately continuous. Clearly $(p,p) \in F^{\leftarrow}[(-1,1)] \in \sigma$. But $(p,p) \in \operatorname{cl}_\tau[\bigcup_{i \in I} B_i \times B_i]$, so $(p,p) \notin \operatorname{int}_\tau F^{\leftarrow}[(-1,1)]$. Thus, $F^{\leftarrow}[(-1,1)] \in \mathcal{N}_\sigma(p,p) \setminus \mathcal{N}_\tau((p,p))$. But $\mathcal{N}_\tau((p,p)) = \mathcal{N}_{\lambda_{\mathbb{H}}}((p,p))$ by Lemma 6.6, so we are done. $\square$

Hence we have by Lemma 6.7:

Theorem 6.8. *If X is a zero-dimensional space with a nonisolated P-point*; *in particular if X is a nondiscrete P-space, then $(X^2, \lambda_{\mathbb{H}}) \neq (X^2, \sigma)$.*

It remains an open question as to whether the topologies $\lambda_{\mathbb{H}}$ and σ must be the same on metrizable product spaces. The next example provides circumstantial evidence that this need not be the case.

Example 6.9. If $X = Y = \mathbb{R}$ and $\mathbb{H}$ is defined as above, then the uniform closure $\mathcal{U}(A(\mathbb{H}))$ of the subalgebra $A(\mathbb{H})$ of $C^*(\mathbb{R}^2, \sigma)$ generated by $\mathbb{H}$ is properly contained in $C^*(\mathbb{R}^2, \sigma)$. To establish this, note first that for any real numbers m and $x \neq 0$, $sp(x, mx) = 2m/(1+m^2)$, and hence the restriction of sp to any (nonvertical) line through the origin has a limit as $x \to 0$. Because $\mathcal{U}(A(\mathbb{H}))$ is the uniform closure of the subalgebra of $S(X,Y)$

generated by $\mathbb{H}$, every member of this latter algebra has a limit as $x \to 0$ along any of these lines.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by letting:

$$f(x, y) = \sin\left(\frac{2xy}{x^6 + y^6}\right) \quad \text{if } (x, y) \neq (0, 0), \quad \text{and}$$

$$f(0, 0) = 0.$$

Clearly, f is separately continuous and is (jointly) continuous except at the origin. For any nonzero real numbers m and x, $f(x, mx) = \sin(2m/(1 + m^6)x^4)$. So the restriction of f to the line $y = mx$ fails to have limit as $x \to 0$ if $m \neq 0$. It follows that $f \in C^*(\mathbb{R}^2, \sigma) \setminus \mathcal{U}(A(\mathbb{H}))$.

The fact that $C^*(\mathbb{R}^2, \sigma) \setminus \mathcal{U}(A(\mathbb{H})) \neq \emptyset$ is not, however, enough to let us conclude that $\lambda_{\mathbb{H}} \neq \sigma$.

We show next that the diagonal of X^2 fails to be discrete in the separately continuous topology for some compact separable spaces X that fail to be first countable; see Theorem 6.4(b).

Lemma 6.10. *If X is a compact ccc space of cardinality greater than $|C(X)|$, then $(\Delta(X), \sigma)$ is not discrete.*

Proof. As mentioned early in Section 5, $|B_1(X)| = |C(X)| = |C(X^2, \tau)|$, so by Theorem 5.3 and the hypothesis, $|S(X^2)| = |C(X)| < |X|$. If $\Delta(X)$ were discrete, then there would be for each $x \in X$ an $f_x \in S(X^2)$ such that $\operatorname{coz}(f_x) \cap \Delta(X) = \{(x, x)\}$. Because $x \neq y$ implies $f_x \neq f_y$, it follows that $|S(X^2)| \geqslant |\Delta(X)| = |X| > |C(X)|$. This contradiction shows that $\Delta(X)$ is not discrete. □

Example 6.11. We will produce a compact separable space D such that $(\Delta(D), \sigma)$ has no isolated points. Let D denote the product of $\mathfrak{c}$ copies of a two-point discrete space (with the usual product topology). D is separable because it is a product of no more than $\mathfrak{c}$ separable spaces; see [6, 2.3.15]. Because D is a topological group under addition mod 2, it is a homogeneous space. So if $p, q \in D$, there is an autohomeomorphism h of D sending p onto q, and it follows that $h \times h$ is a σ-continuous surjection of D^2 onto itself, and hence that its restriction to $\Delta(D)$ is an autohomeomorphism. Hence $(\Delta(D), \sigma)$ has no isolated points by Lemma 6.10. (Note that in this argument, the two-point discrete space may be replaced by any separable compact homogeneous space.)

Next, we show that while the restriction of σ to $\Delta(X)$ need not have any isolated points, its topology is still rather large.

Theorem 6.12. *If X is a Tychonoff space, then $(\Delta(X), \tau_\Delta)$ and the P-space coreflection X_δ of X are homeomorphic.*

Proof. It will be shown that the map $h:(\Delta(X),\tau_\Delta)\to X_\delta$ given by $h(x,x)=x$ is a homeomorphism. Clearly h is a bijection. Now it is easily seen that $\{Z(f)\colon f\in C(X)\}$ is a base for the open sets of X_δ, and a straightforward computation shows that $h^{\leftarrow}[Z(f)]=(sp\circ(f\times f))^{\leftarrow}[(-1,1)]\cap\Delta(X)$; so h is continuous.

Next, suppose $V\in\tau_\Delta$. By definition of τ_Δ, if $(x,x)\in V$, there is a cozeroset W of a function in $C(X^2,\tau)$, $f_1,\ldots,f_n$ in $C(X)$, and $U_1,\ldots,U_n$ open in $\mathbb{R}$ such that $(x,x)\in M_x\subset V$, where

$$M_x=W^2\cap\bigcap_{i=1}^{n}(sp\circ f_i\times f_i)^{\leftarrow}[U_i]\cap\Delta(X).$$

Then $x\in h[M_x]\subset h[V]$, so if we can show that each $h[M_x]$ is open in X_δ, then

$$h[V]=\bigcup_{(x,x)\in V}h[M_x]$$

is open in X_δ, and hence h is a homeomorphism.

To see that this latter holds, note that $(x,x)\in(sp\circ f_i\times f_i)^{\leftarrow}[U_i]$ if and only if $sp(f_i(x),f_i(x))\in U_i$ if and only if either $x\in Z(f_i)$ and $0\in U_1$ or $x\notin Z(f_i)$ and $1\in U_i$. Thus,

$$\bigcap_{i=1}^{n}\big((sp\circ f_i\times f_i)^{\leftarrow}[U_i]\big)\cap\Delta(X)$$
$$=\bigcap_{i=1}^{n}\big\{\big([Z(f_i)]^2\colon 0\in U_i,\ 1\notin U_i\big)\cup\big([\operatorname{coz}(f_i)]^2\colon 1\in U_i,\ 0\notin U_i\big)\big\}.$$

Recalling that h is a bijection and hence distributes over intersections that $h[A^2\cap\Delta(X)]=A$ if $A\subset X$, and the definition of M_x, we see that

$$x\in W\cap\left(\bigcap_{i=1}^{n}\big\{\big(Z(f_i)\colon 0\in U_i,\ 1\notin U_i\big)\cup\big([\operatorname{coz}(f_i)\colon 1\in U_i,\ 0\notin U_i\big)\big\}\right)=h[M_x].$$

So $h[M_x]$ is open in X_δ and we know that h is a homeomorphism. □

Corollary 6.13. *If X is a Tychonoff space, then the restriction of the separately continuous topology to $\Delta(X)$ contains a P-space topology.*

Some other results involving P-points and P-spaces follow.

Theorem 6.14. *If X and Y are Tychonoff spaces, $a\in X$ has a separable neighborhood, and b is a P-point of Y, and $f:X\times Y\to\mathbb{R}$ is separately continuous at (a,b), then f is (jointly) continuous at (a,b).*

Proof. Suppose V,W are open sets in $\mathbb{R}$ such that $f(a,b)\in W$ and $\operatorname{cl}W\subset V$. By assumption, there is an open neighborhood U of a such that $f[U\times\{b\}]\subset W$ and a countable dense subspace $T=\{t_i\}_{i\in\omega}$ of U. For each $i\in\omega$ there is an open neighborhood

M_i of b in Y such that $f[\{t_i\} \times M_i] \subset W$. Because b is a P-point of Y, $M = \bigcap_{i \in \omega} M_i$ is an open neighborhood of b. Then $f[T \times M] \subset W$ and because each vertical section of f is continuous, $f[\operatorname{cl} T \times M] \subset \operatorname{cl} W \subset V$. Hence $f[U \times M] \subset V$ and so f is continuous at (a, b). $\square$

Corollary 6.15. *If X is a Tychonoff locally separable* (*i.e., every point of X has a separable neighborhood*) *and Y is a P-space, then $S(X \times Y) = C(X \times Y)$, whence* $(X \times Y, \sigma) = (X \times Y, \tau)$.

The next example shows that the hypothesis that X is locally separable may not be dropped from Corollary 6.15.

Example 6.16. A P-space E such that $|S(E^2)| > |C(E^2)|$.

Let $D(\mathfrak{c})$ denote the discrete space of cardinality $\mathfrak{c}$, and let $E = D(\mathfrak{c}) \cup \{\infty\}$, topologized so that the intersection of a neighborhood of ∞ with $D(\mathfrak{c})$ is a co-countable set. It is easy to verify that E is a P-space, each $f \in C(E)$ is constant on a co-countable subset of E, and $|C(E^2)| = |C(E)| = \mathfrak{c}$. Because $\{\{x\}: x \in D(\mathfrak{c})\}$ is a cellular family of cardinality $\mathfrak{c}$ in E, the conclusion follows from Theorem 6.5(b).

In [19, 7.7], it is shown that if $\Omega = \omega + 1$ is the one-point compactification of the discrete space ω of finite ordinals, then the point (ω, ω) does not have a countable base of neighborhoods in (Ω^2, σ). This result will be improved below.

Let $\mathbb{F} = \mathbb{F}(\omega)$ denote the set of functions from ω to ω. If $f, g \in \mathbb{F}$, define $f \leqslant^* g$ to mean $\{n \in \omega: f(n) > g(n)\}$ is a finite set. A subset $\mathcal{G}$ of $\mathbb{F}$ is called *dominant* if it is cofinal under the quasi-order $\leqslant^*$; that is, if for each $f \in \mathbb{F}$, there is a $g \in \mathcal{G}$ such that $f \leqslant^* g$. Using the notation in [5], we denote by $\boldsymbol{d}$ the least possible cardinal of a dominant subset of $\mathbb{F}(\omega)$. This cardinal arises in a number of ways. For example, it is shown in [12] to be the smallest cardinal of a cover of the space of irrational numbers by compact sets, and to be a cardinal of uncountable cofinality no larger than c. Indeed, as noted in [12], it has been shown that it is independent in ZFC as to where in the interval $(\omega, \mathfrak{c}]$ that $\boldsymbol{d}$ lies. It is shown in [38] that if $\boldsymbol{d} = \mathfrak{c}$, then there are P-points in $\beta\omega \setminus \omega$. For illustrations of other uses of $\boldsymbol{d}$, see [5,28,47].

Recall that the *character* of a point in a topological space is the minimal cardinal number of a base of neighborhoods of the point.

Theorem 6.17. *The character of (ω, ω) in (Ω^2, σ) is $\boldsymbol{d}$.*

Proof. Because Ω is Tychonoff and countable, by [19, 7.1], $(\Omega^2, \sigma) = (\Omega^2, \gamma)$. So it suffices to show that d is the character of (ω, ω) in the cross topology. For a neighborhood V of (ω, ω) in (Ω^2, γ), let

$$k_V = \min\bigl\{j \in \omega: \{i: i \geqslant j\} \times \{\omega\} \subset V\bigr\},$$

and let

$$n_V = \min\bigl\{j \in \omega: \{\omega\} \times \{i: i \geqslant j\} \subset V\bigr\}.$$

Define $f_V \in \mathbb{F}$ by letting:

$$f_V(n) = 1 \quad \text{if } n \leqslant \max(n_V, k_V), \quad \text{and}$$
$$f_V(n) = \max\bigl[\min\bigl\{j \in \omega\colon \{i\colon i \geqslant j\} \times \{n\} \subset V \text{ properly}\bigr\},$$
$$\min\bigl\{j \in \omega\colon \{n\} \times \{i\colon i \geqslant j\} \subset V \text{ properly}\bigr\}\bigr] \quad \text{if } n > \max(n_V, k_V).$$

Clearly $V \subset W$ implies $f_W \leqslant^* f_V$.

Conversely, if $f \in \mathbb{F}$, a σ-neighborhood V_f of (ω, ω) may be defined as follows:

$$V_f = \left[\bigl(\omega \times \{\omega\}\bigr) \cup \bigcup_{n\in\omega} \{n\} \times \bigl\{j\colon j \geqslant f(n)\bigr\}\right]$$
$$\cup \left[\bigl(\{\omega\} \times \omega\bigr) \cup \bigcup_{n\in\omega} \{j\colon j \geqslant f(n)\bigr\} \times \{n\}\right] \cup \bigl\{(\omega, \omega)\bigr\}.$$

If $(p, q) \in \omega \times \omega$, let

$$V_{f,p,q} = V_f \setminus \bigl[\bigl(\Omega \times \{1, \ldots, p\}\bigr) \cup \bigl(\{1, \ldots, q\} \times \Omega\bigr)\bigr].$$

It will be shown next that if $\{f_\alpha\colon \alpha < d\}$ is a cofinal family in $(\mathbb{F}, \leqslant^*)$, then $\{V_{f_\alpha,p,q}\colon \alpha < d,\ p \in \omega,\ q \in \omega\}$ is a neighborhood base at (ω, ω) in the cross topology.

To see this, given a neighborhood V of (ω, ω), find α so that $f_V \leqslant^* f_\alpha$, and observe that $V_{f_\alpha, n_V, k_V} \subset V$.

Clearly, the cardinality of this latter base is $\boldsymbol{d} \cdot \omega \cdot \omega = \boldsymbol{d}$. Hence the character at (ω, ω) is no more than $\boldsymbol{d}$.

On the other hand, if $\{V_i\}_{i<\eta}$ is neighborhood base at (ω, ω), and if $f \in \mathbb{F}$, then there is an $i_f < \eta$ such that $V_{i_f} \subset V_f$. It follows from the above that $f \leqslant^* f_{V_{i_f}}$. This means that a set of functions indexed by a subset of η is cofinal in $(\mathbb{F}, \leqslant^*)$, so $\boldsymbol{d} \leqslant \eta$. Hence $\boldsymbol{d} = \eta$ and the proof is complete. □

7. Connectedness of Stone–Čech remainders of separately continuous products; co-absolutes

All spaces considered in this section are assumed to be Tychonoff. We will abbreviate $\beta X \setminus X$ by X^*, and $\Sigma = \Sigma(X, Y)$ will abbreviate $(X \times Y, \sigma)$. We begin by reviewing well-known properties of the Boolean algebra $R(X)$ of regular closed subsets of a space X.

Recall that $A \subset X$ is called a *regular closed set* if it is the closure of its interior. If $A, B \in R(X)$, and we let $A \vee B = A \cup B$, $A \wedge B = \mathrm{cl}(\mathrm{int}(A \cap B))$, and $A^c = \mathrm{cl}(X \setminus A)$, then $R(X)$ becomes a Boolean algebra; see [37, Chapter 3]. Recall also that if $K \subset X$, the *boundary* $\mathrm{bd}(K)$ is defined to be $\mathrm{cl}\, K \cap \mathrm{cl}(X \setminus K)$. The next proposition summarizes "well-known" facts that we will use about this Boolean algebra for what follows.

Proposition 7.1.

(a) *If X is dense in T, then the map $A \to \mathrm{cl}_T\, A$ is a Boolean algebra isomorphism g of $R(X)$ onto $R(T)$, and if $B \in R(T)$, then $g^{\leftarrow}(B) = B \cap X$.* (*See* [37, 3B(4)].) *Also,* $\mathrm{bd}(A) = A \cap A^c$.

(b) *If T is nowhere locally compact, then T^* is dense in βT, the map $A \to \mathrm{cl}_{\beta T} A \setminus A$ is a Boolean algebra isomorphism of $R(T)$ onto $R(T^*)$, and each clopen set of T^* is $\mathrm{cl}_{\beta T} A \setminus T$ for some $A \in R(T)$.*

(c) *Suppose T is nowhere locally compact. If $A \in R(T)$ and $\mathrm{cl}_{\beta T} A \setminus T$ is clopen in T^*, then $\mathrm{bd}_T A$ is compact.*

Proof. (c) If $A \in R(T)$ then $\mathrm{cl}_{\beta T} A \setminus T$ is clopen in T^* if and only if $\mathrm{bd}_{T^*}(\mathrm{cl}_{\beta T} A \setminus T) = \emptyset$ if and only if $(\mathrm{cl}_{\beta T} A \setminus T) \cap (\mathrm{cl}_{\beta T} A \setminus T)^c = \emptyset$ (by (a) using complementation in $R(T^*)$) if and only if $(\mathrm{cl}_{\beta T} A) \cap (\mathrm{cl}_{\beta T} A)^c \setminus T = \emptyset$ (by (a) using complementation in $R(\beta T)$). Because $(\mathrm{cl}_T A)^c \subset (\mathrm{cl}_{\beta T} A)^c$, this latter implies $\mathrm{bd}_T A$ is compact. $\square$

The hypothesis of the next result about product spaces implies that each factor space is connected.

Theorem 7.2. *If X and Y are spaces such the each of their nonempty proper regular closed sets has an infinite boundary, and α is a Tychonoff topology on $X \times Y$ that contains $\lambda_{\mathbb{H}}$, whose restrictions to horizontal and vertical fibers is the same as those of τ, and is such that every compact subspace is a union of finitely many compact subspaces of horizontal or vertical fibers, then $(X \times Y, \alpha)^*$ is connected. In particular, $(X \times Y, \lambda_{\mathbb{H}})^*$ and Σ^* are connected.*

Proof. Because neither X nor Y has any isolated points, $\Psi = (X \times Y, \alpha)$ is nowhere locally compact by Corollary 4.9. If Ψ^* fails to be connected, then it has a proper clopen subset, and by Proposition 7.1(b), (c), there is a proper regular closed subset E of Ψ with compact boundary. Hence there are finite subsets $F \subset X$ and $G \subset Y$ such that $\mathrm{bd}_\Psi E \subset (F \times Y) \cup (X \times G)$.

If $x \in X \setminus F$, then $\mathrm{bd}_\Psi E \cap (\{x\} \times Y) \subset (\{x\} \times G)$, and hence:

$$\{x\} \times Y = \big[\big(\{x\} \times Y\big) \cap \mathrm{int}_\Psi E\big] \cup \big[\big(\{x\} \times Y\big) \cap (\Psi \setminus E)\big] \cup \big[\big(\{x\} \times G\big)\big].$$

Thus, if $(\{x\} \times Y) \cap \mathrm{int}_\Psi E$ and $(\{x\} \times Y) \cap (\Psi \setminus E)$ are nonempty, then $\mathrm{cl}_{\{x\}\times Y}((\{x\} \times Y) \cap \mathrm{int}_\Psi E)$ is a proper member of $R(\{x\} \times Y)$ whose boundary is contained in the finite set $\{x\} \times G$, contrary to assumption. Hence if we define

$$L = \big\{x \in (X \setminus F)\colon\ \{x\} \times Y \subset \mathrm{int}_\Psi E \cup \big(\{x\} \times G\big)\big\}, \quad \text{and}$$

$$M = \big\{x \in (X \setminus F)\colon\ \{x\} \times Y \subset (\Psi \setminus E) \cup \big(\{x\} \times G\big)\big\},$$

then

$$L \times (Y \setminus G) \subset \mathrm{int}_\Psi E, \quad M \times (Y \setminus G) \subset \Psi \setminus E, \quad \text{and} \quad L \cup M \cup F = X.$$

Because F and G are finite, if M were empty, then $\mathrm{int}_\Psi E$ would be dense in Ψ and so $(\mathrm{cl}_{\beta\Psi} E) \setminus \Psi$ would fail to be a proper subspace of Ψ^*, contrary to assumption. Hence $M \neq \emptyset$. A similar argument shows that $L \neq \emptyset$.

If the roles of X and Y are interchanged in this argument and we define

$$H = \big\{y \in (Y \setminus G)\colon\ X \times \{y\} \subset \mathrm{int}_\Psi E \cup (F \times \{y\})\big\}, \quad \text{and}$$

$$K = \{y \in (Y \setminus G)\colon\ X \times \{y\} \subset (\Psi \setminus E) \cup (F \times \{y\})\},$$

and then observe as above that H, K are nonempty and $H \cup K \cup G = Y$. Because $L \times (Y \setminus G) \subset \mathrm{int}_\Psi E$, and this latter is disjoint from $\Psi \setminus E$, it follows that K is empty. This contradiction shows that $(X \times Y, \alpha)^*$ is connected. By Corollary 4.9, this argument may be applied to the spaces $(X \times Y, \lambda_{\mathbb{H}})$ and to $(X \times Y, \lambda_{\mathbb{H}})$ as well. □

Corollary 7.3. *If n and m are integers no smaller than 2, then $([0,1]^n \times [0,1]^m, \lambda_{\mathbb{H}})^*$ and $([0,1]^n \times [0,1]^m, \sigma)^*$ are connected.*

It is well known and easily seen that if X, Y, Z are topological spaces, $U = (X \times Y, \tau)$, and $V = (Y \times Z, \tau)$, then the spaces obtained by imposing the product topology on $U \times Z$ and on $X \times V$ are homeomorphic. We will see below that this kind of associativity need not hold for separately continuous products.

Recall from [6, 1.5.19] that a space is *perfectly normal* if each of its closed sets is a zeroset, and note that every metrizable space is perfectly normal.

The next result shows that the requirement in the hypothesis of Theorem 7.2 that the boundaries of regular closed sets of the factor spaces be infinite cannot be discarded.

Theorem 7.4. *Suppose X is a space that has a nonempty proper regular closed set A with a finite boundary such that A and A^c are zerosets, and K is compact. If $\Phi = (X \times K, \sigma)$, then Φ^* fails to be connected.*

Proof. Let $B = A \times K$, and note that $B^c = A^c \times K$ is its Boolean algebra complement in $R(X \times K)$ and both B and B^c are zerosets in $(X \times K, \tau)$. It will be shown next that $B \in R(\Phi)$.

We need to show that each $(p, q) \in B$ is in the σ-closure of its interior. Since $\tau \subset \sigma$, $\mathrm{int}_\tau B \subset \mathrm{int}_\sigma B$, so because Φ is a Tychonoff topology, it suffices to show that each cozeroset of a function in $C(\Phi)$ that contains (p, q) meets $\mathrm{int}_\tau B$. If $(p, q) \in \mathrm{coz}(f)$ for some $f \in C(\Phi)$, then since $\mathrm{coz}(f) \cap [X \times \{q\}]$ is open in the restriction of $\tau \subset \sigma$ to $[X \times \{q\}]$, then since $p \in A$, there is an $r \in \mathrm{int}\, A$ such that $(r, q) \in \mathrm{coz}(f)$. Because $(r, q) \in \mathrm{int}_\tau B$, this shows that $B \in R(\Phi)$. Similarly, $B^c \in R(\Phi)$.

If $F = \mathrm{bd}(A)$, then by [37, 4(g)(4)],

$$\mathrm{cl}_{\beta\Phi} B \cap \mathrm{cl}_{\beta\Phi} B^c = \mathrm{cl}_{\beta\Phi}(B \cap B^c) = \mathrm{cl}_{\beta\Phi}(\{F \times K\}) = F \times K \subset \Phi.$$

It follows that $(\mathrm{cl}_{\beta\Phi} B \setminus B)$ is a nonempty proper clopen subset of Φ^*. □

So, by Corollary 7.3 and Theorem 7.4, we have:

Corollary 7.5. $([0,1] \times [0,1]^3, \sigma)^*$ *is not connected. So* $([0,1] \times [0,1]^3, \sigma)$ *and* $([0,1]^2 \times [0,1]^2, \sigma)$ *are not homeomorphic.*

Remark. The arguments to establish Theorem 7.4 and Corollary 7.5 apply equally well if the topology σ is replaced by $\lambda_{\mathbb{H}}$.

Recall that every continuous map f of a Tychonoff space X into a compact space Y has a continuous extension βf (called the Stone extension of f) over βX into Y. See [9, Chapter 6] or [37, Chapter 4].

A continuous surjection $f: X \to Y$ is called a *perfect map* if it is closed and inverse images of one point sets are compact. The Stone extension $\beta f: \beta X \to \beta Y$ of a continuous surjection f is always a perfect surjection. The continuous surjection f is called *irreducible* if it maps no proper closed subset of X onto Y. A Tychonoff space X is called *extremally disconnected* if each of its open sets has an open closure. Equivalently, X is extremally disconnected if each of its dense subspaces is C^*-embedded. If D is discrete, then βD is extremally disconnected. Moreover, extremally disconnected spaces are projective in the category of Tychonoff spaces and perfect maps, and for each Tychonoff space X, there is an (essentially unique in a sense not described in this paper) extremally disconnected space EX, called the *absolute* of X, and a perfect irreducible surjection of EX onto X. It is known that X and EX have the same π-weight. If there is a perfect irreducible map of X onto Y, then they have the same absolute—in which case they are said to be *co-absolute*. See [37, Chapter 6].

In case the Tychonoff spaces X and Y are such that X is first countable and Y is a Baire space, Lemma 3.4 tells us that the nonempty members of the usual product topology τ on $X \times Y$ is a π-base for γ and hence for the smaller topology σ. Use will be made of the following technical lemma.

Lemma 7.6. *Suppose (Z, α) and (Z', α') are Tychonoff spaces and $j: Z \to Z'$ is a continuous surjection. If $\{j^{\leftarrow}[T]: \emptyset \neq T \in \alpha'\}$ is a π-base for α, then the Stone extension $\beta j: \beta Z \to \beta Z'$ is irreducible and the spaces $\beta Z, \beta Z'$ are co-absolute.*

Proof. If βj fails to be irreducible, there is a proper closed subspace K of Z such that $\beta j[K] = \beta Z'$. If $p \in \beta Z \setminus K$ and V is a regular βZ-open neighborhood of p disjoint from K, then $A = \beta Z \setminus V$ is a proper regular closed subset (i.e., one that coincides with the closure of its interior) of βZ containing K. It follows that $A \cap Z$ is dense in A. Hence $j[A \cap Z] = \beta j[A \cap Z]$ is dense in $\beta j[A] = \beta Z'$.

By our assumption on π-bases, there is a $T \in \alpha$ such that

$$j^{\leftarrow}[T] \subset (Z \setminus A).$$

Choose W open in $\beta Z'$ so that $W \cap Z' = T$. Then $W \cap j[A \cap Z] = \emptyset$, contrary to what was just established. So βj is irreducible. By the remarks made above, the lemma holds. $\square$

Combining Lemmas 3.4 and 7.6 (with $Z = Z' = X \times Y$, $\alpha = \sigma$, $\alpha' = \tau$, and j the identity map) yields:

Theorem 7.7. *If X and Y are Tychonoff spaces such that τ^+ is a π-base for $(X \times Y, \sigma)$; in particular if Y is $(\pi w(X))^+$-Baire, then $\beta[(X \times Y), \tau]$ and $\beta[(X \times Y), \sigma]$ are co-absolute.*

Corollary 7.8. *If X and Y are compact metrizable spaces, then $\beta(X \times Y, \tau)$ and $\beta(X \times Y, \sigma)$ are co-absolute.*

8. Realcompactness of separately continuous products; open problems and remarks

Recall that a Tychonoff space is *realcompact* if it cannot be densely C-embedded in any properly larger Tychonoff space, and is called *hereditarily realcompact* if each of its subspaces is realcompact. It is well known that an arbitrary product of realcompact spaces is realcompact, and that a metrizable space of nonmeasurable power is realcompact. (See [9, Chapters 8, 11, and 12] for the definition of nonmeasurable cardinals.) In view of the scarcity of pseudocompact subspaces of a product in the separately continuous topology exhibited in Corollary 5.9, it seems natural to ask:

(A) *If X and Y are realcompact spaces, must $(X \times Y, \sigma)$ be realcompact*?

In [19, Section 9] it is shown that if K denotes the one-point compactification of an uncountable discrete space of measurable power, then $(K \times K, \sigma)$ fails to be realcompact (because its diagonal is a discrete space of measurable power). So this question cannot be answered affirmatively without some set-theoretic assumptions. It is shown in [19] also that a product of two realcompact metrizable spaces is realcompact in the separately continuous topology. The next proposition generalizes this latter result. It follow from the fact established in [9, 8.18] that if (T, α) is hereditarily realcompact, and if δ is a Tychonoff topology on T that contains α, then (T, δ) is hereditarily realcompact.

Proposition 8.1. *If $(X \times Y, \tau)$ is hereditarily realcompact, then so is $(X \times Y, \sigma)$.*

Corollary 8.2. *If X and Y are realcompact spaces in which points are G_δ's, then $(X \times Y, \sigma)$ is hereditarily realcompact.*

Proof. Clearly the hypothesis implies that $X \times Y$ is realcompact and that its points are G_δ's. Hence, as noted in [46, 8.10], every subspace of $X \times Y$ is realcompact.

In connection with Corollary 8.2 A. Bella has pointed out that *A realcompact space Z in which points are G_δ*'s *has nonmeasurable power*. For if $|Z|$ is measurable, and Z' denotes the set Z with the discrete topology, then there is a point p in $\upsilon Z' \setminus Z'$ (where $\upsilon Z'$ denotes the realcompactification of Z'). Then the identity map i of Z' onto Z has a continuous extension υi over $\upsilon Z'$ onto Z, and it is easy to see that $(\upsilon i)^{\leftarrow}(p)$ is a closed G_δ in $\upsilon Z'$ disjoint from Z', contrary to the assumption that Z' fails to be realcompact. See [9, Chapter 8].

Our final example shows that the hypothesis that a product (in the usual sense) of two hereditarily realcompact spaces need not be hereditarily realcompact.

Example 8.3. Let $D(\omega_1) = D \cup \{\omega_1\}$ denote a discrete space of power ω_1 together with the additional point whose neighborhoods include a co-countable subset of D and let

$\Omega = \omega + 1$ denote the one-point compactification of a countable discrete space as described just after Example 6.16. It is noted both in [23,15] that both $D(\omega_1)$ and Ω are hereditarily realcompact, while their product is not.

This leads us to ask:

(B) *If X and Y are hereditarily realcompact spaces, must $(X \times Y, \sigma)$ be hereditarily realcompact*?

Few of our results are best possible, so whether they can be improved produces a set of open problems that need no explicit restatement. Some that merit special attention follow.

(C) Namioka's theorem and its generalizations provide sufficient conditions on a product of two Tychonoff spaces X and Y that ensure that whenever $f \in S(X,Y)$, there is a dense G_δ-set $A \subset X$ such that $C(f,\tau)$ contains $A \times Y$, while our Theorem 3.5 gives sufficient conditions that guarantee only that $C(f,\tau)$ is dense in $(X \times Y, \tau)$.

Are there Tychonoff spaces X and Y such that $(X \times Y, \tau)$ is a Baire space and $C(f,\tau)$ is dense in $(X \times Y, \tau)$ whenever $f \in S(X,Y)$, but there is a $g \in S(X,Y)$ for which $C(g,\tau)$ fails to contain either $A \times Y$ or $X \times B$ for any dense *G_δ-set $A \subset X$ or dense G_δ-set $B \subset Y$*?

See the examples given in Example 3.7.

(D) In Theorem 6.8, an example is given of a P-space X such that $(X^2, \lambda_{\mathbb{H}}) \neq (X^2, \sigma)$.

Are there metrizable spaces X and Y such that $(X \times Y, \lambda_{\mathbb{H}}) \neq (X \times Y, \sigma)$?

That is, is the separately continuous topology obtained as a weak topology from the usual product topology by making use of the function sp and its translates in a "natural" way in case the factor spaces are metrizable? See Example 6.9.

(E) Consider the topology λ_Δ introduced just before Theorem 6.4.

Is there a (*metrizable*) *Tychonoff space X such that $(X^2, \lambda_\Delta) \neq (X^2, \lambda_{\mathbb{H}})$*?

(F) *If X is a Tychonoff space, must the restriction of the topology of separate continuity to the diagonal of X^2 make it into a P-space*?

See Corollary 6.13 ff.

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