Nonlinear biharmonic equations with Hardy potential and critical parameter

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ABSTRACT

Some embedding inequalities in Hardy–Sobolev spaces with weighted function $|x|^q$ are proved. The procedure is based on decomposition into spherical harmonics, where in addition various new inequalities are obtained. Next, we study the existence of nontrivial solutions of biharmonic equations with Hardy potential and critical parameter.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$, $N \geq 5$ and consider the following nonlinear elliptic equation with Dirichlet boundary condition

$$(P) \begin{cases} \Delta^2 u - \mu \frac{q(x)u}{|x|^4} = f(x,u), & x \in \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial \Omega, \end{cases}$$

where $1/|x|^4$ is called the Hardy potential or critical potential, when $q(x) \equiv 1$, $\mu = \mu^* = N^2 (N - 4)^2 / 16$ is called the critical parameter. Problem $(P)$ with $\mu < \mu^*$ has been discussed extensively (e.g. [8,14] and references therein). When $\mu > \mu^*$, the operator $\Delta^2 - \mu/|x|^4$ is unbounded. Problem $(P)$ has no nontrivial solutions in $H^2_0(\Omega)$ if $q(x) \equiv 1$, $\mu = \mu^*$ and $f(x,u) = \lambda u$, that is, the best constant

$$\lambda = \inf_{u \in H^2_0(\Omega)} \left\{ \int_\Omega \left| \Delta u \right|^2 \, dx - \left( \frac{N(N - 4)}{4} \right)^2 \int_\Omega \frac{u^2}{|x|^4} \, dx : \int_\Omega u^2 \, dx = 1 \right\}$$

is never attained in any domain $\Omega$ (see [6,7]). Hence in [6] the authors considered the perturbed problem

$$\lambda(q) = \inf_{u \in H^2_0(\Omega)} \left\{ \int_\Omega \left| \Delta u \right|^2 \, dx - \left( \frac{N(N - 4)}{4} \right)^2 \int_\Omega \frac{q u^2}{|x|^4} \, dx : \int_\Omega u^2 \, dx = 1 \right\}.$$
where \( q \in C^0(\Omega) \) be such that \( 0 \leq q(x) \leq 1 \). Denote \( \lambda(q) \) by \( \lambda_D(q) \) for the Dirichlet boundary condition, they have the following results:

(i) If \( q \) satisfies

\[
\liminf_{x \to 0} \left( \frac{1}{|x|} \right) (1 - q(x)) > \frac{6(N^2 - 4N + 8)}{N^2(N - 4)^2}
\]

then \( \lambda_D(q) \) is achieved by \( u \);

and

(ii) Let \( 0 < R < 1 \). Assume that \( q \) satisfies

\[
\sup_{0 < x < R} \left( \frac{1}{|x|} \right) (1 - q(x)) \leq \frac{6(N^2 - 4N + 8)}{N^2(N - 4)^2}
\]

if \( \Omega = B \), then \( \lambda_D(q) \) is not achieved by any non-negative function.

In this paper, we will introduce some Sobolev spaces which are larger than \( H^2_0(\Omega) \) to study the minimizing problem. In contrast to the paper [6], we do not consider the perturbed problem. Now, we introduce the Hardy–Sobolev spaces with general weight function in the following.

**Definition 1.** We denote by \( H^2_{0,1}(\Omega, |x|^{2m}) \) the Sobolev space obtained by the completion of \( C_0^\infty(\Omega) \) with respect to the norm

\[
\|u\|_{0,1,2m} = \left( \int_{\Omega} \frac{|\Delta u|^2}{|x|^{2m}} \, dx - \frac{(N + 2m)(N - 4 - 2m)}{4} \right)^2 \int_{\Omega} \frac{u^2}{|x|^{2m+4}} \, dx \right)^{1/2}
\]

associated with inner product

\[
(u, v)_{0,1,2m} = \int_{\Omega} \frac{\Delta u \Delta v}{|x|^{2m}} \, dx - \frac{(N + 2m)(N - 4 - 2m)}{4} \int_{\Omega} \frac{uv}{|x|^{2m+4}} \, dx.
\]

In particular, we denoted \( H^2_{0,1}(\Omega, |x|^0) \) and \( \|u\|_{0,1,0} \) by \( H^2_{0,1}(\Omega) \) and \( \|u\|_{0,1} \).

In Section 2, we get an improvement of the following inequality established in [7,14]: Assume that \( N \geq 5 \) and \( 0 \leq m < (N - 4)/2 \), then for \( u \in C_0^\infty(\Omega) \)

\[
\int_{\Omega} \frac{|\Delta u|^2}{|x|^{2m}} \, dx \geq \left( \frac{N + 2m)(N - 4 - 2m)}{4} \right)^2 \int_{\Omega} \frac{u^2}{|x|^{2m+4}} \, dx. \tag{1.2}
\]

On the lines of improving Hardy–Rellich inequality (see [1–7,10]) there has been a considerable interest in improving (1.2) with \( m = 0 \). For \( 0 \leq m < (N - 4)/2 \), we have the following result:

**Theorem 1.1.** Suppose that \( N \geq 5 \), \( 0 \leq m < (N - 4)/2 \) and \( D \geq \sup_{x \in \Omega} |x| \). Then for \( u \in H^2_{0,1}(\Omega, |x|^{2m}) \) there exists some constant \( c > 0 \) such that

\[
\int_{\Omega} \frac{|\Delta u|^2}{|x|^{2m}} \, dx - \frac{(N + 2m)(N - 4 - 2m)}{4} \int_{\Omega} \frac{u^2}{|x|^{2m+4}} \, dx \geq c \left( \int_{\Omega} \frac{|u|^{2m}}{|x|^m} \, dx \right)^{2m} X_1^{\frac{2N - 2m}{N - m}} \right)^{\frac{N - d}{N}}.
\]

where \( X_1 = [1 - \ln(|x|/D)]^{-1} \).

**Theorem 1.2.** Let \( N \geq 5 \), \( 0 \leq m < (N - 4)/2 \) and \( D \geq \sup_{x \in \Omega} |x| \). If positive function \( g(x) \) satisfies

\[
\lim_{x \to 0} \frac{g(x)}{\theta(|x|)} = 0,
\]

then for \( u \in H^2_{0,1}(\Omega, |x|^{2m}) \)

\[
H^2_{0,1}(\Omega, |x|^{2m}) \hookrightarrow L^q(\Omega, g), \tag{1.4}
\]

where \( 2 \leq q < 2N/(N - 4) \), \( 1/q = \tau/2 + (1 - \tau)(N - 4)/2N \) and
Moreover, the constant is attained. Besides, we study the existence of nontrivial solutions of the following Dirichlet problem

\[
\begin{aligned}
\Delta^2 u - \left( \frac{N(N-4)}{4} \right)^2 \frac{u}{|x|^4} &= f(x, u), \quad x \in \Omega, \\
u &= \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

In this paper, for short, we denote $|x|$ by $r$. We use the letter $c$ indiscriminately to denote various positive constants when the exact values are irrelevant.

## 2. Improved Hardy–Rellich inequalities

Recall the following result established in [9,13].

**Theorem 2.1.** Let $\Omega$ be a bounded domain or $\mathbb{R}^N$. Assume that $\phi(r)$ is a positive continuous function and $h(r)$ satisfy $r^{N-1} \phi(r) (h^2(r))^\prime = c_0$ for some constant $c_0$. If $h^{-1}(0) = 0$, then for any $u \in C_0^\infty(\Omega)$,

\[
\int_\Omega r^{N-2} \phi(r) \frac{h^2(r)}{h(r)} u^2 dx \leq \int_\Omega \phi(r) |\nabla u|^2 dx.
\]  

Moreover, the constant 1 is the best, that is

\[
1 = \inf_{u \in C_0^\infty(\Omega), \ u \neq 0} \frac{\int_\Omega \phi(r) |\nabla u|^2 dx}{\int_\Omega \phi(r) \left( \frac{h(r)}{h^2(r)} \right)^\frac{1}{2} u^2 dx}.
\]

Denote

\[
a = \theta \sup_{x \in \Omega} |x|,
\]

where $\theta > 1$ is a given constant. Then $\Omega \subset B_a(0)$, where $B_a(0)$ denotes the ball in $\mathbb{R}^N$ with radius centered at 0.

**Example 1.** Let $N \geq 2$ and $\phi(r) = r^{2\alpha}$. Then

\[
h(r) = \begin{cases} 
    r^{1-N/2-\alpha}, & \alpha \neq 1 - N/2, \\
    (\ln R/r)^{1/2}, & \alpha = 1 - N/2,
\end{cases}
\]

where if $\alpha \neq 1 - N/2$, $\Omega$ is allowed to be $\mathbb{R}^N$, and if $\alpha = 1 - N/2$, it is assumed that $\Omega$ is bounded and $R \geq a$.

**Lemma 2.2.** Suppose that $N \geq 5$ and $m < (N-4)/2$. Then for $u \in C_0^\infty(\Omega)$,

\[
\left( \int_\Omega r^{\frac{2Nm}{N-2}} |u|^{\frac{2N}{N-2}} dx \right)^{(N-2)/N} \leq c \int_\Omega r^{-2m} |\nabla u|^2 dx.
\]  

(2.2)
Proof. By (2.1), we have
\[
\int_{\Omega} r^{-2m-2} |u|^2 \, dx \leq \frac{4}{(N - 2m - 2)^2} \int_{\Omega} r^{-2m} |\nabla u|^2 \, dx.
\]

Then by the Poincaré inequality,
\[
\left( \int_{\Omega} r^{-\frac{2Nm}{N-2}} |u|^{\frac{2N}{N-2}} \, dx \right)^\frac{(N-2)/N}{N-2} \leq c \int_{\Omega} |\nabla(r^m u)|^2 \, dx \leq c \int_{\Omega} r^{-2m} |\nabla u|^2 \, dx + c m^2 \int_{\Omega} r^{-2m-2} |u|^2 \, dx
\]
\[
\leq c \left[ 1 + \frac{4m^2}{(N - 2m - 2)^2} \right] \int_{\Omega} r^{-2m} |\nabla u|^2 \, dx. \quad \Box
\]

Lemma 2.3. Suppose that \( N \geq 5 \) and \( m < (N - 4)/2 \). Then for \( u \in C^\infty_0(\Omega) \),
\[
\left( \int_{\Omega} r^{-\frac{2Nm}{N-2}} |u|^{\frac{2N}{N-2}} \, dx \right)^\frac{N-2}{N} \leq c \int_{\Omega} r^{-\frac{2Nm}{N-2}} |u|^{\frac{2N}{N-2}} \, dx.
\] (2.3)

Proof. By (2.1) and the Hölder inequality, we have
\[
\int_{\Omega} r^{\frac{2N(m-1)}{N-2}} |u|^{\frac{2N}{N-2}} \, dx \leq c \int_{\Omega} r^{\frac{2(1-Nm-2)}{N-2}} |u|^{\frac{4}{N-2}} |\nabla u|^2 \, dx
\]
\[
\leq c \left( \int_{\Omega} r^{\frac{2N(m-1)}{N-2}} |u|^{\frac{2N}{N-2}} \, dx \right)^\frac{2}{N} \left( \int_{\Omega} r^{\frac{2Nm}{N-2}} |\nabla u|^{\frac{2N}{N-2}} \, dx \right)^\frac{N-2}{N}.
\]

Then
\[
\int_{\Omega} r^{\frac{2N(m-1)}{N-2}} |u|^{\frac{2N}{N-2}} \, dx \leq c \int_{\Omega} r^{\frac{2Nm}{N-2}} |\nabla u|^{\frac{2N}{N-2}} \, dx.
\]

So
\[
\left( \int_{\Omega} r^{-\frac{2Nm}{N-2}} |u|^{\frac{2N}{N-2}} \, dx \right)^\frac{N-2}{N} \leq \int_{\Omega} |\nabla(r^m u)|^{\frac{2N}{N-2}} \, dx \leq c \int_{\Omega} r^{\frac{2N(m-1)}{N-2}} |u|^{\frac{2N}{N-2}} \, dx + \int_{\Omega} r^{\frac{2Nm}{N-2}} |\nabla u|^{\frac{2N}{N-2}} \, dx
\]
\[
\leq c \int_{\Omega} r^{\frac{2Nm}{N-2}} |\nabla u|^{\frac{2N}{N-2}} \, dx. \quad \Box
\]

Lemma 2.4. Suppose that \( N \geq 5 \) and \( m < (N - 4)/2 \). Then for \( u \in C^\infty_0(\Omega) \),
\[
\int_{\Omega} r^{-2m} |\nabla(|\nabla u|)|^2 \, dx \leq \int_{\Omega} r^{-2m} |\Delta u|^2 \, dx.
\] (2.4)

Proof. A direct calculation shows that
\[
\int_{\Omega} r^{-2m} |\nabla(|\nabla u|)|^2 \, dx \leq \sum_{i,j=1}^N \int_{\Omega} r^{-2m} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \, dx \leq \int_{\Omega} r^{-2m} |\Delta u|^2 \, dx. \quad \Box
\]

An immediate consequence of the inequality (2.2)–(2.4) is the following result.

Theorem 2.5. Suppose that \( N \geq 5 \) and \( m < (N - 4)/2 \). Then for \( u \in C^\infty_0(\Omega) \),
\[
\left( \int_{\Omega} r^{-\frac{2Nm}{N-2}} |u|^{\frac{2N}{N-2}} \, dx \right)^\frac{N-4}{N} \leq c \int_{\Omega} r^{-2m} |\Delta u|^2 \, dx.
\] (2.5)
Let \( u \in C_0^\infty(\Omega) \). If we extend \( u \) as zero outside \( \Omega \), we may consider \( u \in C_0^\infty(\mathbb{R}^N) \). Decomposing \( u \) into spherical harmonics we get

\[
u = \sum_{k=0}^\infty u_k := \sum_{k=0}^\infty f_k(r)\phi_k(\sigma),
\]

where \( \phi_k(\sigma) \) are the orthonormal eigenfunctions of the Laplace-Beltrami operator with responding eigenvalues \( c_k = k(N + k - 2), k \geq 0 \). The functions \( f_k(r) \) belong to \( C_0^\infty(\Omega) \), satisfying \( f_k(r) = O(r^k) \) and \( f_k'(r) = O(r^{k-1}) \) as \( r \to 0 \). In particular, \( \phi_0(\sigma) = 1 \) and \( u_0(r) = \frac{1}{|\Omega|} \int_{\partial \Omega} u \, ds \), for any \( r > 0 \). Then, for any \( k \in \mathbb{N} \), we have

\[
\Delta u_k = \left( \Delta f_k(r) - \frac{c_k f_k(r)}{r^2} \right) \phi_k(\sigma).
\]

So

\[
\int_{\mathbb{R}^N} \frac{\Delta u_k^2}{|x|^{2m}} \, dx = \int_{\mathbb{R}^N} r^{-2m} \left( \Delta f_k(r) - \frac{c_k f_k(r)}{r^2} \right) \, dx = \int_{\mathbb{R}^N} r^{-2m} \left( f_k'' \right)^2 \, dx + \left( (N-1)(2m+1) + 2c_k \right) \int_{\mathbb{R}^N} r^{-2m-2} \left( f_k' \right)^2 \, dx \\
+ c_k \left[ c_k + (N-4-2m)(2m+2) \right] \int_{\mathbb{R}^N} r^{-2m-4} (f_k)^2 \, dx.
\]

Now, we assume the function \( v \in C_0^\infty(\mathbb{R}^N \setminus \{0\}) \), such that \( v = |x|^{(N-4-2m)/2} u \). From the definition of \( u \) and \( v \), we may write that

\[
u = \sum_{k=0}^\infty u_k := \sum_{k=0}^\infty r^{(4+2m-N)/2} g_k(r)\phi_k(\sigma), \quad \forall = \sum_{k=0}^\infty v_k := \sum_{k=0}^\infty r^{k} g_k(r)\phi_k(\sigma),
\]

where \( f_k(r) = r^{(4+2m-N)/2} g_k(r) \) with \( g_k \to 0 \) and \( \lim g'_k = 0 \) at the origin. More precisely, for any \( k \in \mathbb{N} \), we have

\[
\int_{\mathbb{R}^N} \frac{\Delta u_k^2}{|x|^{2m}} \, dx = \int_{\mathbb{R}^N} r^{-2m} \left( g_k'' \right)^2 \, dx \int_{\mathbb{R}^N} \frac{C_1(k)}{r^{2m+4}} \, dx + \int_{\mathbb{R}^N} r^{-2m-2} \left( g_k' \right)^2 \, dx \frac{C_2(k)}{r^{2m+4}} \, dx \int_{\mathbb{R}^N} r^{-2m-4}(g_k)^2 \, dx,
\]

(2.6)

\[
I_\Omega[u_k] := \int_{\Omega} \frac{|\Delta u|^2}{|x|^{2m}} \, dx = \int_{\Omega} \frac{C_1(k)}{r^{2m+4}} \, dx \int_{\Omega} \frac{|u|^2}{|x|^{2m+4}} \, dx \\
\int_{\Omega} \frac{C_2(k)}{r^{2m+4}} \, dx \int_{\Omega} \frac{|u|^2}{|x|^{2m+4}} \, dx \left[ \left( \frac{N+2m)(N-4-2m)}{4} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2m+4}} \, dx \right] \int_{\Omega} \frac{r^{2N+2k}(g_k)^2}{|x|^{2m+4}} \, dx,
\]

(2.7)

where positive functions \( C_1(k) \) and \( C_2(k) \) satisfy

\[
C_1(k) = (N-3)k + \frac{(N-2)^2 + 4(m+1)^2 - 2}{2},
\]

\[
C_2(k) = (N+2m)(N-2m-4)m^2 + \frac{[(N-2)^2 - 4(N-2)(m+1)^2]k}{2} + \frac{[(N+2m)(N-4-2m)]}{16}.
\]

**Theorem 2.6.** (See [7, Theorem 6.3.]) Suppose \( N \geq 5 \) and \( 0 \leq m < (N-4)/2 \). For any \( u \in C_0^\infty(\Omega) \), we set \( v = |x|^{(N-4-2m)/2} u \). Then, the following inequality holds:

\[
\int_{\Omega} \frac{\Delta u^2}{|x|^{2m}} \, dx - \left( \frac{(N+2m)(N-4-2m)}{4} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2m+4}} \, dx \int_{\Omega} \frac{|\nabla u|^2}{|x|^{2m+4}} \, dx \geq A(N, m) \int_{\Omega} |x|^{2-N} |\nabla v|^2 \, dx,
\]

(2.8)

where

\[
A(N, m) = \left\{ \begin{array}{ll}
(N-1) + \frac{1}{2}(N+2m)(N-4-2m), & m > -\frac{2+N}{2}, \\
4(1+m)^2 + \frac{1}{2}(N+2m)(N-4-2m), & m \leq -\frac{2+N}{2}.
\end{array} \right.
\]

Moreover, the constant \( 4(1+m)^2 + \frac{1}{2}(N+2m)(N-4-2m) \) for \( m < \frac{2+N}{2} \) is the best.
Theorem 2.7. Under the same conditions of Theorem 2.6, we have
\[
\int_\Omega \frac{\Delta u^2}{|x|^{2m}} dx - \left( \frac{(N + 2m)(N - 4 - 2m)}{4} \right) \int_\Omega \frac{|u|^2}{|x|^{2m+4}} dx \geq A_1(N, m) \int_\Omega |x|^{4-N} |\Delta v|^2 dx.
\]
where
\[
A_1(N, m) = \begin{cases} 
\frac{1}{2} + \frac{2(m+1)^2}{(N-2)^2}, & m > -2+\sqrt{N-1}, \\
\frac{N+2m(N-4-2m)+2(N-1)}{2(N+2m)(N-4-2m)+2(N-1)}, & m \leq -2+\sqrt{N-1}.
\end{cases}
\]

Proof. A direct calculation shows that
\[
I_\Omega [u] = \int_\Omega |x|^{4-N} |\Delta v|^2 dx - (N + 2m)(N - 4 - 2m) \int_\Omega |x|^{-N}(x \cdot \nabla v)^2 dx \\
+ \frac{(N + 2m)(N - 4 - 2m)}{2} \int_\Omega |x|^{2-N} |\nabla v|^2 dx.
\]
By (2.8) and (2.10)
\[
\int_\Omega |x|^{4-N} |\Delta v|^2 dx \geq \left[ A(N, m) - \frac{(N + 2m)(N - 4 - 2m)}{2} \right] \int_\Omega |x|^{2-N} |\nabla v|^2 dx \\
+ (N + 2m)(N - 4 - 2m) \int_\Omega |x|^{-N}(x \cdot \nabla v)^2 dx.
\]
On the other hand
\[
\int_\Omega |x|^{-N}(x \cdot \nabla v)^2 dx - \frac{1}{2} \int_\Omega |x|^{2-N} |\nabla v|^2 dx \\
\leq \frac{1}{2(N-2)^2} \left[ 4(1+m)^2 \int_\Omega |x|^{2-N} |\nabla v|^2 dx + (N + 2m)(N - 4 - 2m) \int_\Omega |x|^{-N}(x \cdot \nabla v)^2 dx \right].
\]
when \( m \leq -2+\sqrt{N-1} \), and
\[
\int_\Omega |x|^{-N}(x \cdot \nabla v)^2 dx - \frac{1}{2} \int_\Omega |x|^{2-N} |\nabla v|^2 dx \\
\leq \frac{1}{2(N+2m)(N-4-2m)+2(N-1)} \left[ (N-1) \int_\Omega |x|^{2-N} |\nabla v|^2 dx \\
+ (N + 2m)(N - 4 - 2m) \int_\Omega |x|^{-N}(x \cdot \nabla v)^2 dx \right]
\]
when \( m > -2+\sqrt{N-1} \). The result follows from (2.10)–(2.13).

Lemma 2.8. Let \( \Omega \) be a bounded domain, \( D = \sup_{x \in \Omega} |x| \) and \( u \in C_0^\infty(\Omega) \). Then
\[
I_\Omega [u] \geq I_B[u_0] + \sum_{k=1}^{\infty} I_B[u_k].
\]
Proof. Observe that \( I_\Omega [u] = I_B[u_0] + \sum_{k=1}^{\infty} I_B[u_k] \). It suffices to prove that for any \( k = 1, \ldots, \) it holds that
\[
I_B[u_k] \geq \frac{8(N - 1)(N^2 - 2N - 2) + 32m(2+m)(1-N)}{(N^2-4)^2 + 8m(2m+8m^2 - 12m + 8 - mN - 2N^2)} \int_{B_D} \frac{|\Delta (u - u_0)|^2}{|x|^{2m}} dx.
\]
Assume that the following inequality holds
\[
I_B[u_k] \geq a \int_{B_D} \frac{|\Delta u_k|^2}{|x|^{2m}} dx,
\]
for some \( 0 < a < 1 \) and any \( k = 1, 2, \ldots \). Taking into account (2.6) and (2.7) we obtain
\[ \int_{B_1}^{r^{-N+2k}} \left( g_k'' \right)^2 \, dx + C_1(k) \int_{B_0} \frac{r^{2-N+2k} \left( g_k' \right)^2}{r^{2-N+2k} \left( g_k' \right)^2} \, dx \geq \frac{1}{1-a} \left[ (a-1)C_2(k) + \left( \frac{(N+2m)(N-4-2m)}{4} \right) \right]^2 \int_{B_0} r^{-N+2k} \left( g_k' \right)^2 \, dx. \]

As a consequence of the following Hardy inequalities
\[ \int_0^\infty r^{2k+3} \left( g_k'' \right)^2 \, dr \geq (k+1)^2 \int_0^\infty r^{2k+1} \left( g_k' \right)^2 \, dr, \]
\[ \int_0^\infty r^{2k+1} \left( g_k' \right)^2 \, dr \geq k^2 \int_0^\infty r^{2k-1} \left( g_k' \right)^2 \, dr, \]
we deduce that \( a \leq G(k) \), where
\[ G(k) = \frac{k^2(k+1)^2 + k^2C_1(k) + C_2(k)}{k^2(k+1)^2 + k^2C_1(k) + C_2(k)}. \]

However, \( G(k) \) is an increasing function for \( k > 1 \). Hence,
\[ a = G(1) = \frac{8(N-1)(N^2-2N-2-8m-4m^2)}{(N^2-4)^2 + 8m(2m^2 + 8m^2 - 12m + 8 - mnN - 2N^2)}. \]

**Lemma 2.9.** Let \( u_0 \in C_0^\infty([0, D]) \). Then
\[ I_{B_1}[u_0] \geq c \left( \int_{\Omega} |u|^{2N \frac{2N}{N-2}} |x|^{-2N \frac{N}{N-2}} X_1^{\frac{2N-2}{N-2}} \, dx \right)^{\frac{N-2}{N}}. \] (2.15)

**Proof.** Assume that \( D = 1 \) have
\[ I_{B_1}[u_0] \geq c \int_{B_1} |x|^{4-N} |\Delta v_0|^2 \, dx = c \int_{B_1} \left( v_0' + \frac{N-1}{r} v_0 \right)^2 \, dx = c \int_0^1 r^2 \left( v_0' \right)^2 \, dr + (N-1)(N-2) \int_0^1 r \left( v_0' \right)^2 \, dr. \]

By the Hardy inequality we have
\[ \int_{B_1} \left( v_0'' \right)^2 \, dx \geq \int_{B_1} \frac{\left( v_0' \right)^2}{|x|^2} \, dx \geq c \int_0^1 r \left( v_0' \right)^2 \, dr. \]

Then we obtain
\[ I_{B_1}[u_0] \geq c \int_0^1 r \left( v_0' \right)^2 \, dr. \]

Next, we consider the following inequality
\[ \int_0^1 r \left( v_0' \right)^2 \, dr \geq c \left( \int_0^1 |v_0|^q r^{-1} X_1^{1+q/2} \, dr \right)^{2/q}. \] (2.16)

which is implied from [15, Theorem 3] with \( X_1(t) = (1 - \ln t)^{-1}, \, dv = r \chi[0, 1], \) \( d\mu = r^{-1} X_1^\alpha \chi[0, 1] \, dr \). Setting now \( q = 2N/(N-4), \, \alpha = (2N-4)/(N-4) \) and taking into account (2.16) we conclude that
\[ I_{B_1}[u_0] \geq c \left( \int_{\Omega} |u|^{2N \frac{2N}{N-2}} |x|^{-2N \frac{N}{N-2}} X_1^{\frac{2N-2}{N-2}} \, dx \right)^{\frac{N-2}{N}}. \]
Lemma 2.10. Let $u_0 \in C_0^\infty([0, D])$. Then
\[
\int_{\Omega} \frac{\left| \Delta(u - u_0) \right|^2}{|x|^{2m}} \, dx \geq c \left( \int_{\Omega} \left| u - u_0 \right|^{\frac{2N}{N-2}} |x|^{\frac{2N}{N-2} - \frac{2m}{N-2}} x_1^{\frac{2N}{N-2} - \frac{2m}{N-2}} \, dx \right)^{\frac{N-4}{N}}. \tag{2.17}
\]

Proof. By (2.5) and the fact that $X_1$ is a bounded function imply that
\[
\int_{\Omega} \left| u - u_0 \right|^{\frac{2N}{N-2}} |x|^{\frac{2N}{N-2} - \frac{2m}{N-2}} x_1^{\frac{2N}{N-2} - \frac{2m}{N-2}} \, dx \geq c \left( \int_{\Omega} \left| u - u_0 \right|^{\frac{2N}{N-2}} |x|^{\frac{2N}{N-2} - \frac{2m}{N-2}} x_1^{\frac{2N}{N-2} - \frac{2m}{N-2}} \, dx \right)^{\frac{N-4}{N}}. \tag{2.18}
\]

Proof of Theorem 1.1. Let $u \in C_0^\infty(\Omega)$, then from (2.14)–(2.17) we conclude that
\[
\int_{\Omega} \left| u - u_0 \right|^{\frac{2N}{N-2}} |x|^{\frac{2N}{N-2} - \frac{2m}{N-2}} x_1^{\frac{2N}{N-2} - \frac{2m}{N-2}} \, dx \geq c \left( \int_{\Omega} \left| u - u_0 \right|^{\frac{2N}{N-2}} |x|^{\frac{2N}{N-2} - \frac{2m}{N-2}} x_1^{\frac{2N}{N-2} - \frac{2m}{N-2}} \, dx \right)^{\frac{N-4}{N}}. \tag{2.19}
\]
This completes the proof by a density argument. \qed

By (1.3) and the inequality established in [7, Theorem 1.6], we have

**Theorem 2.11.** Let $N \geq 5$ and $D \geq \sup_{x \in \Omega} |x|$. If $2 \leq q < \frac{2N}{N-4}$, then
\[
\left( \int_{\Omega} \theta(x) |u|^q \, dx \right)^{\frac{1}{q}} \leq c \left( \int_{\Omega} |u|^2 |x|^{-2m-4} X_1^{\frac{2N}{N-2} - \frac{2m}{N-2}} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |u|^{\frac{2N}{N-2}} |x|^{\frac{2N}{N-2} - \frac{2m}{N-2}} x_1^{\frac{2N}{N-2} - \frac{2m}{N-2}} \, dx \right)^{\frac{1}{2}}, \tag{2.20}
\]
\[
\left( \int_{\Omega} \theta(x) |u|^q \, dx \right)^{\frac{1}{q}} \leq c \|u\|_{0,1,2m}, \tag{2.21}
\]
where $2^* = 2N/(N-4)$, $1/q = 1/2 + (1/\tau)(N-4)/2N$ and
\[
\theta(x) = |x|^{q(2-r-m)} X_1^{q(N-2+2r)/N}.
\]

Proof of Theorem 1.2. Let $\{u_n\}$ be a sequence in $u \in H^2_0(\Omega \setminus \delta B_1(0))$ such that $\|u_n\|_{0,1,2m} \leq C$, then there exists a subsequence of $\{u_n\}$ weakly convergent to a function $u \in H^2_0(\Omega \setminus \delta B_1(0))$. By inequality (2.19) in $\Omega \setminus B_{\frac{1}{n}}(0)$ ($n = 1, 2, \ldots$), for some subsequence (still denoted by $\{u_n\}$) such that
\[
u \to u, \quad \text{in } L^4(\Omega \setminus B_{\frac{1}{n}}(0))
\]
for $n = 1, 2, \ldots$.
Since $\lim_{x \to 0} \frac{g(x)}{\theta(x)} = 0$, then for $\varepsilon > 0$, there exists $0 < \delta < 1/n$ such that
\[
\sup_{|x| < \delta} g(x) \leq \varepsilon \theta(|x|)
\]
and $g(x)$ is bounded in $\Omega \setminus B_{\delta}(0)$. We have
\[
\int_{\Omega} g(x) |u_n - u|^q \, dx \leq \int_{B_{\delta}(0)} g(x) |u_n - u|^q \, dx + \int_{\Omega \setminus B_{\delta}(0)} g(x) |u_n - u|^q \, dx
\]
\[
\leq \int_{B_{\delta}(0)} \theta(x) |u_n - u|^q \, dx + \max_{\Omega \setminus B_{\delta}(0)} \int_{\Omega \setminus B_{\delta}(0)} |u_n - u|^q \, dx
\]
\[
\leq c \varepsilon + \max_{\Omega \setminus B_{\delta}(0)} \int_{\Omega \setminus B_{\delta}(0)} |u_n - u|^q \, dx,
\]
where the last inequality is due to Theorem 2.11. This proves the theorem. \qed
Proof of Theorem 1.3. Analogous to the proof of Theorem 1.2, for a bounded sequence \( \{u_n\} \in H^2_{0,1}(\Omega) \), up to a subsequence, we may assume that
\[
u_n \rightharpoonup u, \quad \text{in } L^q(\Omega \setminus B_\frac{r}{2}(0)),
\]
for \( n = 1, 2, \ldots \). Then by the Hölder inequality, we have
\[
\int_\Omega |u|^q \, dx = \int_\Omega |u|^q X_{1}^{\frac{2(N-2q)}{N}} X_{1}^{\frac{-2(N-2q)}{N}} \, dx \leq \left( \int_\Omega |u|^{\frac{2N}{N-2q}} \, dx \right)^{\frac{q(N-4)}{2N}} \left( \int_\Omega X_{1}^{\frac{2(N-2q)}{N-4q}} \, dx \right)^{\frac{2N-q(N-4)}{2N}}.
\]
Hence,
\[
\int_\Omega |u_n - u|^q \, dx \leq \int_\Omega |u_n - u|^q \, dx + \int_\Omega |u_n - u|^q \, dx \leq \int_\Omega |u_n - u|^q \, dx + \left( \int_\Omega |u_n - u|^{\frac{2N}{N-2q}} \, dx \right)^{\frac{q(N-4)}{2N}} \left( \int_\Omega X_{1}^{\frac{2(N-2q)}{N-4q}} \, dx \right)^{\frac{2N-q(N-4)}{2N}}.
\]
The result then follows since \( u_n \rightharpoonup u \) in \( L^q(\Omega \setminus B_\frac{r}{2}(0)) \) and \( X_{1}^{\frac{-2(N-2q)}{N}} \in L(B_\frac{r}{2}(0)) \). \( \square \)

3. Existence of nontrivial solutions of nonlinear biharmonic equations

In order to study the existence of solutions of the nonlinear elliptic equations (1.6), first we consider the following eigenvalue problem in \( H^2_{0,1}(\Omega) \).
\[
\begin{align*}
\Delta^2 u - \left( \frac{N(N-4)}{4} \right) u & = \lambda u, \quad x \in \Omega, \\
u & = \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega.
\end{align*}
\]
(3.1)

The first eigenvalue of (3.1) is given by
\[
\lambda_1 = \inf_{u \in H^2_{0,1}(\Omega)} \left\{ \int_\Omega |\Delta u|^2 \, dx - \left( \frac{N(N-4)}{4} \right) \int_\Omega \frac{u^2}{|x|^4} \, dx : \int_\Omega u^2 \, dx = 1 \right\}.
\]
(3.2)

Lemma 3.1. The minimizing problem (3.2) has a solution \( \phi_1 \).

Proof. Let \( \{u_n\} \) be a sequence, that is
\[
\|u\|_{0,1}^2 \to \lambda_1, \quad \text{with } \int_\Omega u^2 \, dx = 1.
\]
Then \( \{u_n\} \) is bounded in \( H^2_{0,1}(\Omega) \). By the compactness of imbedding \( H^2_{0,1}(\Omega) \hookrightarrow \hookrightarrow L^2(\Omega) \), up to a subsequence,
\[
u_n \rightharpoonup u \quad \text{with } \int_\Omega u^2 \, dx = 1.
\]
Note that
\[
\left\| \frac{u_n - u_m}{2} \right\|_{0,1}^2 + \left\| \frac{u_n + u_m}{2} \right\|_{0,1}^2 = \frac{1}{2} \left( \|u_m\|^2_{0,1} + \|u_n\|^2_{0,1} \right)
\]
for all \( n, m \geq 1 \), we have
\[
\left\| \frac{u_n - u_n}{2} \right\|_{0,1}^2 \leq \frac{1}{2} \left( \|u_m\|^2_{0,1} + \|u_n\|^2_{0,1} \right) - \lambda_1 \int_\Omega \left( \frac{u_n + u_m}{2} \right)^2 \, dx \to 0,
\]
as \( n, m \to \infty \). Hence, \( \{u_n\} \) is a Cauchy sequence in \( H^2_{0,1}(\Omega) \), which means \( u_n \) strongly converges to some \( \phi_1 \) in \( H^2_{0,1}(\Omega) \), and then \( \|\phi_1\|^2_{0,1} = \lambda_1 \).
We know that the second eigenvalue is given by
\[
\lambda_2 = \inf_{u \in H^1_0(\Omega), u \neq 0} \left\{ \int_\Omega |\Delta u|^2 \, dx - \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^N} \, dx \right\}.
\]
Similarly, we can characterize the nth eigenvalue \(\lambda_n\), the corresponding eigenfunction is denoted by \(\varphi_n\). We prove \(\lambda_n \to +\infty\) as \(n \to \infty\). □

Lemma 3.2. \(\lambda_n \to +\infty\) as \(n \to \infty\).

Proof. As in [11,12], if \(\lambda_n\) is bounded, then
\[
\|u_n\|_{0,1}^2 \leq \lambda_n,
\]
which shows that there is a sequence convergent subsequence of \(\{u_n\}\) (still denote by \(\{u_n\}\)) in \(H^2_0(\Omega)\). By Theorem 1.1, \(\{u_n\}\) is strongly convergent in \(L^2(\Omega)\). But for \(n \neq k\)
\[
\int_\Omega |u_k - u_n|^2 \, dx = \int_\Omega u_k^2 \, dx - 2 \int_\Omega u_k u_n \, dx + \int_\Omega u_n^2 \, dx = 2.
\]
This is a contradiction.

Now, let us state our result for the existence of solutions of the nonlinear elliptic equation (1.6). Assume that \(f(x,t)\) satisfies

\((f_1)\) There exists some constant \(c > 0\) such that
\[
|f(x,t)| \leq c(1 + |t|^{q-1}), \quad \text{a.e. } x \in \Omega, \quad \forall t \in \mathbb{R},
\]
where \(1 \leq q < \frac{2N}{N-4}\).

\((f_2)\) \(\limsup_{|t| \to 0} \frac{2F(x,t)}{t^2} < \lambda_1\) uniformly in \(\Omega\), where \(f(x,t) = \int_0^t f(x,s) \, ds\).

\((f_3)\) There exist \(\theta > 0\) and \(t_0 > 0\) such that
\[
\theta G(x,t) \geq G(x, |t|), \quad \forall x \in \Omega, \quad |t| > t_0, \quad s \in [0,1]
\]
where \(G(x,t) = f(x,t)t - 2F(x,t)\).

\((f_4)\) \(\lim_{|t| \to +\infty} \frac{F(x,t)}{t} = +\infty\) uniformly in \(\Omega\). □

Theorem 3.3. Assume \((f_1), (f_2), (f_3)\) and \((f_4)\), then (1.5) has at least one nontrivial solution in \(H^2_0(\Omega)\).

Definition 2. A functional \(J \in C^1(X, \mathbb{R})\), where \(X\) is a real Banach space, is said to satisfy the Ceramì condition in \((c_1, c_2)\)
\((-\infty \leq c_1 < c_2 \leq +\infty)\), if

(i) any bounded sequence \(\{u_n\} \subset X\) such that \(J(u_n) \to c\) and \(J'(u_n) \to 0\) possesses a convergent subsequence; and

(ii) \(\forall \varepsilon \in (c_1, c_2)\), there exist \(\delta, R, \alpha > 0\) such that \(\|J'(u)\|_X \cdot \|u\| \geq \alpha\) for any \(u \in J^{-1}[c - \delta, c + \delta]\) with \(\|u\| \geq R\).

Lemma 3.4. If \(f(x, u)\) satisfies \((f_1)-(f_4)\), then the functional
\[
J(u) = \frac{1}{2} \|u\|_{0,1}^2 - \int_\Omega F(x, u) \, dx
\]
satisfies the Ceramì condition.

Proof. It is easy to show that \(J(u)\) satisfies the first part (i). Now let us prove by contradiction that \(J(u)\) verifies the second part (ii). Suppose that there is a sequence \(\{u_n\} \subset H^2_{0,1}(\Omega)\) such that
\[
J(u_n) \to c \quad \text{and} \quad \{J'(u_n), u_n\} \to 0.
\]
Then
\[ \lim_{n \to \infty} \int_{\Omega} \left( \frac{1}{2} f(x, u_n)u_n - F(x, u_n) \right) dx = \lim_{n \to \infty} \left( f(u_n) - \frac{1}{2} f'(u_n), u_n \right) = c. \]

Assume that \( \|u_n\|_{0,1} \to \infty \) as \( n \to \infty \). Define
\[ \omega_n = \frac{u_n}{\|u_n\|_{0,1}} \]
then \( \|\omega_n\|_{0,1} = 1 \), up to a subsequence, we may assume that \( \omega_n \to \omega \) in \( H^2_{0,1}(\Omega) \) and \( \omega_n \to \omega \) in \( L^2(\Omega) \). Now, we claim that both \( \omega \equiv 0 \) and \( \omega \not\equiv 0 \) are impossible, and the results follows.

Suppose \( \omega \equiv 0 \). Then for any constant \( k > 0 \), it follows from \( \omega_n \to 0 \) a.e. that
\[ \lim_{n \to \infty} \int_{\Omega} F(x, k\omega_n) dx = 0. \]
As in [13], let \( t_n \) satisfy
\[ f(t_n u_n) = \max_{t \in [0,1]} f(tu_n). \]
Then we have
\[ f(t_n u_n) \geq f(k\omega_n) = \frac{1}{2} \|\omega_n\|_{0,1} - \int_{\Omega} F(x, k\omega_n) dx \geq ck^2 \]
for \( n \) large enough. This means \( \lim_{n \to \infty} f(t_n u_n) = \infty \). Note that \( 0 < t_n < 1 \), by the fact \( f(0) = 0 \) and \( \lim_{n \to \infty} f(u_n) = c \), so \( f(t_n u_n, tu_n)|_{t=t_n} = 0 \), that is
\[ \|t_n u_n\|_{0,1}^2 = \int_{\Omega} f(x, t_n u_n)t_n u_n dx. \]
Hence, we have
\[ \frac{1}{2} \int_{\Omega} G(x, t_n u_n) dx = \int \left( \frac{1}{2} f(x, t_n u_n)u_n - F(x, u_n) \right) dx = \frac{1}{2} \|t_n u_n\|_{0,1}^2 - \int_{\Omega} F(x, u_n) dx = \frac{1}{2} f(t_n u_n) \to \infty \]
as \( n \to \infty \). It follows from \( (f_3) \) that for some bounded function \( a(x) \),
\[ \theta G(x, u) \geq G(x, tu) + a(x), \quad \forall (x, u) \in \Omega \times \mathbb{R} \]
and then
\[ \int_{\Omega} \left( \frac{1}{2} f(x, t_n u_n)u_n - F(x, u_n) \right) dx = \frac{1}{2} \int_{\Omega} G(x, u_n) dx \geq \frac{1}{2\theta} \int_{\Omega} G(x, t_n u_n) dx + \int_{\Omega} a(x) dx \to \infty \]
as \( n \to \infty \).

Suppose \( \omega \not\equiv 0 \). Denote \( \Omega_0 = \{x \in \Omega \mid \omega(x) \neq 0\} \), then the Lebesgue measure \( |\Omega_0| \) is positive and \( \lim_{n \to \infty} u_n = \infty \) in \( \Omega_0 \).
By \( (f_2) \) and the Fatou’s lemma, we have
\[ \lim_{n \to \infty} \int_{\Omega} \frac{f(x, u_n)u_n}{u_n^2} \omega_n dx = +\infty. \]
Moreover, for some positive constant \( c \),
\[ \frac{f(x, u)}{u^2} \geq -c. \]
Since
\[ \lim_{n \to \infty} \int_{\Omega_0} \omega_n^2 dx = 0, \]
we have
\[ \int_{\Omega \setminus \Omega_0} \frac{f(x, u_n)u_n}{u_n^2} \omega_n^2 dx \geq -c \int_{\Omega \setminus \Omega_0} \omega_n^2 dx > -\infty. \]
However, if we divide \( \langle J' (u_n), u_n \rangle = o (1) \) by \( \| u_n \|_{0,1}^2 \), we obtain
\[
1 - o (1) = \int \frac{f (x, u_n) u_n}{\| u_n \|_{0,1}^2} \, dx = \int \frac{f (x, u_n) u_n}{u^2} \omega_n^2 \, dx + \int \frac{f (x, u_n) u_n}{u^2} \, dx \to + \infty.
\]
This is a contradiction.

**Proof of Theorem 3.3.** (i) First \( J (0) = 0 \). By \((f_1)\) and \((f_2)\), choose \( 2 < q < \frac{4N}{N-4} \), we have
\[
J (u) \geq \frac{1}{2} \| u \|_{0,1}^2 - \frac{\lambda_1 - \varepsilon}{2} \int \Omega u^2 \, dx - c \int |u|^q \, dx \geq \left( \frac{1}{2} - \frac{\lambda_1 - \varepsilon}{2\lambda_1} \right) \| u \|_{0,1}^2 - c \int \Omega |u|^q \, dx.
\]
Denote \( \| u \|_{0,1} = r \), then
\[
J (u) \geq \left( \frac{1}{2} - \frac{\lambda_1 - \varepsilon}{2\lambda_1} \right) r^2 - Cr^q \left( \frac{1}{2} - \frac{\lambda_1 - \varepsilon}{2\lambda_1} \right) - Cr^q - 2.
\]
Hence, there exists a constant \( \alpha > 0 \) such that for small \( r \),
\[
J (u) \geq \alpha, \quad \forall u \in \partial B_r (0) \subset X.
\]
Let \( u_0 \neq 0 \in H^2_{0,1} (\Omega) \), obviously, \( \lim_{t \to - \infty} J (tu_0) = - \infty \), and so there exists some \( u_1 \in \Omega \setminus B_t (0) \) such that \( J (u_1) < 0 \). Then the result follows from the mountain pass theorem with Cerami type condition.

**References**