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Mullineux involution and twisted affine Lie algebras

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Abstract

We use Naito and Sagaki's work [S. Naito, D. Sagaki, Lakshmibai–Seshadri paths fixed by a diagram automorphism, J. Algebra 245 (2001) 395–412; S. Naito, D. Sagaki, Standard paths and standard monomials fixed by a diagram automorphism, J. Algebra 251 (2002) 461–474] on Lakshmibai–Seshadri paths fixed by diagram automorphisms to study the partitions fixed by Mullineux involution. We characterize the set of Mullineux-fixed partitions in terms of crystal graphs of basic representations of twisted affine Lie algebras of type $A_{2\ell}^{(2)}$ and of type $D_{\ell+1}^{(2)}$. We set up bijections between the set of symmetric partitions and the set of partitions into distinct parts. We propose a notion of double restricted strict partitions. Bijections between the set of restricted strict partitions (respectively, the set of double restricted strict partitions) and the set of Mullineux-fixed partitions in the odd case (respectively, in the even case) are obtained.

Keywords: Lakshmibai–Seshadri paths; Orbit Lie algebras; Mullineux involution

1. Introduction

Let $n, e \in \mathbb{N}$. Let k be a field and $0 \neq q \in k$. Suppose that either $e > 1$ and q is a primitive eth root of unity; or $q = 1$ and $\operatorname{char} k = e$.[1] Let $\mathcal{H}_k(\mathfrak{S}_n)$ be the Iwahori–Hecke algebra associated to the symmetric group $\mathfrak{S}_n$ with parameter q and defined over k. The Mullineux involution M is a bijection defined on the set of all e-regular partitions of n, which arises naturally when one twists irreducible modules (labeled by e-regular partitions) over $\mathcal{H}_k(\mathfrak{S}_n)$ by a k-algebra automorphism # (see Section 2 for definition of #). If $q = 1$ and e is an odd prime number, the

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[1] In the latter case, e is necessarily to be a prime number.

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involution M determines which simple module splits and which remains simple when restricting to the alternating subgroup A_n. In that case, the set of partitions which are fixed by the involution M parameterizes the irreducible modules of $k\mathfrak{S}_n$ which split on restriction to A_n. In [21], Kleshchev gave a remarkable algorithm for computing the involution M. A crystal bases approach to Kleshchev's algorithm of the involution M was given in [24, (7.1)].

The purpose of this paper is to study the partitions fixed by Mullineux involution for arbitrary e. We find that the set of Mullineux-fixed partitions is related to the twisted affine Lie algebras of type $A_{2\ell}^{(2)}$ and of type $D_{\ell+1}^{(2)}$, which reveals new connection between the theory of affine Lie algebra and the theory of modular representations. Our main tool are Naito and Sagaki's work [29,30] on Lakshmibai–Seshadri paths fixed by diagram automorphisms, which was also used in [16,17] to derive explicit formulas for the number of modular irreducible representations of the cyclotomic Hecke algebras of type $G(r, p, n)$, see [13–15] for related work. We characterize the set of Mullineux-fixed partitions in terms of crystal graph of basic representations of twisted affine Lie algebras of type $A_{2\ell}^{(2)}$ and of type $D_{\ell+1}^{(2)}$ (Theorem 3.7). We set up bijections (Theorems 3.15 and 3.17) between the set of Mullineux-fixed partitions in the odd case (respectively, the set of symmetric partitions) and the set of restricted strict partitions (respectively, the set of partitions into distinct parts). As an application, we obtain new identities on the cardinality of the set of Mullineux-fixed partitions in terms of the principal specialized characters of the basic representations of these twisted affine Lie algebras (Theorems 3.13 and 3.20). Furthermore, we propose a notion of double restricted strict partitions (Definition 3.21), which is a direct explicit characterization of Kang's reduced proper Young wall of type $D_{\ell+1}^{(2)}$ [19]. We obtain a bijection (Theorem 3.24) between the set of Mullineux-fixed partitions in the even case and the set of double restricted strict partitions. Our main results shed some new insight on the modular representations of the alternating group and of Hecke–Clifford superalgebras as well as of the spin symmetric group (see Remarks 3.25 and 3.18), which clearly deserves further study.

2. Preliminaries

In this section, we shall first review some basic facts about the representation of the Iwahori–Hecke algebras associated to symmetric groups. Then we shall introduce the notion of Mullineux involution, Kleshchev's e-good lattice as well as Kleshchev's algorithm of Mullineux involution.

Let $\mathfrak{S}_n$ be the symmetric group on $\{1, 2, \dots, n\}$, acting from the right. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate. The Iwahori–Hecke algebra $\mathcal{H}_{\mathcal{A}}(\mathfrak{S}_n)$ associated to $\mathfrak{S}_n$ is the associative unital $\mathcal{A}$-algebra with generators $T_1, \dots, T_{n-1}$ subject to the following relations

$$\begin{aligned}
&(T_i - v)(T_i + 1) = 0, \quad \text{for } 1 \leqslant i \leqslant n-1,\\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leqslant i \leqslant n-2,\\
&T_i T_j = T_j T_i, \quad \text{for } 1 \leqslant i < j-1 \leqslant n-2.
\end{aligned}$$

For each integer i with $1 \leqslant i \leqslant n-1$, we define $s_i = (i, i+1)$. Then $S := \{s_1, s_2, \dots, s_{n-1}\}$ is the set of all the simple reflections in $\mathfrak{S}_n$. A word $w = s_{i_1} \cdots s_{i_k}$ for $w \in \mathfrak{S}_n$ is a reduced expression if k is minimal; in this case we say that w has length k and we write $\ell(w) = k$. Given a reduced expression $s_{i_1} \cdots s_{i_k}$ for $w \in \mathfrak{S}_n$, we write $T_w = T_{i_1} \cdots T_{i_k}$. The braid relations for generators $T_1, \dots, T_{n-1}$ ensure that T_w is independent of the choice of reduced expression. It is well known that $\mathcal{H}_{\mathcal{A}}(\mathfrak{S}_n)$ is a free $\mathcal{A}$-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$. For any field k which is an $\mathcal{A}$-algebra, we define $\mathcal{H}_k(\mathfrak{S}_n) := \mathcal{H}_{\mathcal{A}}(\mathfrak{S}_n) \otimes_{\mathcal{A}} k$. Then $\mathcal{H}_k(\mathfrak{S}_n)$ can be naturally identified with the k-algebra defined by the same generators and relations as $\mathcal{H}_{\mathcal{A}}(\mathfrak{S}_n)$ above. Specializing v to $1 \in k$, one recovers the group algebra $k\mathfrak{S}_n$ of $\mathfrak{S}_n$ over k.

We recall some combinatorics. A partition of n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, the conjugate of λ is defined to be a partition $\lambda^t = (\lambda_1^t, \lambda_2^t, \ldots)$, where $\lambda_j^t := \#\{i \mid \lambda_i \geqslant j\}$ for $j = 1, 2, \ldots$. We define $\ell(\lambda) := \max\{i \mid \lambda_i \neq 0\}$. For any partition λ of n, we denote by $\mathfrak{t}^\lambda$ (respectively, $\mathfrak{t}_\lambda$) the standard λ-tableau in which the numbers $1, 2, \ldots, n$ appear in order along successive rows (respectively, columns). The row stabilizer of $\mathfrak{t}^\lambda$, denoted by $\mathfrak{S}_\lambda$, is the standard Young subgroup of $\mathfrak{S}_n$ corresponding to λ. Let

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w, \qquad y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-v)^{-\ell(w)} T_w.$$

Let $w_\lambda \in \mathfrak{S}_n$ be such that $\mathfrak{t}^\lambda w_\lambda = \mathfrak{t}_\lambda$. Following [6, Section 4], we define $z_\lambda = x_\lambda T_{w_\lambda} y_{\lambda^t}$.

Definition 2.1. The right ideal $z_\lambda \mathcal{H}$ is called the right Specht module of $\mathcal{H} = \mathcal{H}_{\mathcal{A}}(\mathfrak{S}_n)$ corresponding to λ. We denote it by S^λ.

For any field k which is an $\mathcal{A}$-algebra, let $S_k^\lambda := S^\lambda \otimes_{\mathcal{A}} k$. There is a natural bilinear form $\langle\, , \rangle$ on each S^λ (and hence on each S_k^λ). Let $D_k^\lambda := S_k^\lambda / \operatorname{rad}\langle\, , \rangle$. Let "$\trianglelefteq$" be the dominance order on the set of all partitions as defined in [28, (3.1)].

Lemma 2.2. [6] *With the above notations, we have*

(1) *the set of all the non-zero D_k^λ (where λ runs over partitions of n) forms a complete set of pairwise non-isomorphic simple $\mathcal{H}_k(\mathfrak{S}_n)$-modules. Moreover, if $\mathcal{H}_k(\mathfrak{S}_n)$ is semisimple, then $D_k^\lambda = S_k^\lambda \neq 0$ for every partition λ of n;*
(2) *if $D_k^\mu \neq 0$ is a composition factor of S_k^λ then $\lambda \trianglelefteq \mu$, and every composition factor of S_k^λ is isomorphic to some D_k^μ with $\lambda \trianglelefteq \mu$. If $D_k^\lambda \neq 0$ then the composition multiplicity of D_k^λ in S_k^λ is* 1.

Henceforth, let k be a fixed field which is an $\mathcal{A}$-algebra. We assume that v is specialized to $q \in k$ such that $1 + q + q^2 + \cdots + q^{a-1} = 0$ for some positive integer a. We define

$$e = \min\bigl\{1 < a < \infty \bigm| 1 + q + q^2 + \cdots + q^{a-1} = 0 \text{ in } k\bigr\}.$$

Clearly, $e = \operatorname{char} k$ if $q = 1$; and otherwise e is the multiplicative order of q. For simplicity, we shall write $\mathcal{H}_k$ instead of $H_k(\mathfrak{S}_n)$.

A partition λ is called e-regular if it contains at most $e - 1$ repeating parts, i.e., $\lambda = (1^{m_1} 2^{m_2} \cdots j^{m_j} \cdots)$ with $0 \leqslant m_i < e$ for every i. By [6], for any partition λ of n, $D_k^\lambda \neq 0$ if and only if λ is e-regular. Let $\mathcal{K}_n$ be the set of all the e-regular partitions of n. Let # (see [6], [28, (2.3)]) be the k-algebra automorphism of $\mathcal{H}_k$ which is defined on generators by $T_i^\# = -vT_i^{-1}$ for each $1 \leqslant i < n$. For each $\mathcal{H}_k(\mathfrak{S}_n)$-module V, we denote by $V^\#$ the $\mathcal{H}_k(\mathfrak{S}_n)$-module obtained by twisting V by #. That is, $V^\# = V$ as k-linear space, and $v \cdot h := vh^\#$ for any $v \in V$ and $h \in \mathcal{H}_k(\mathfrak{S}_n)$. Let $*$ be the algebra anti-automorphism on $\mathcal{H}_k$ which is defined on generators by $T_i^* = T_i$ for any $1 \leqslant i < n$.

Definition 2.3. [3,27] Let M be the unique involution defined on the set $\mathcal{K}_n$ such that $(D_k^\lambda)^\# \cong D_k^{\mathrm{M}(\lambda)}$ for any $\lambda \in \mathcal{K}_n$. We call the map M the Mullineux involution, and λ a Mullineux-fixed partition if $\mathrm{M}(\lambda) = \lambda$.

An algorithm which compute the involution M was first proposed by Mullineux in 1979, when he constructed an involution on the set of e-regular partitions and conjectured its coincidence with the above M. Mullineux worked in the setup that $q = 1$ and e being a prime number, though his combinatorial algorithm does not really depend on e being prime. In [21], Kleshchev gave a quite different remarkable algorithm of the involution M based on his work of branching rules for the modular representations of symmetric groups. In [9], Ford and Kleshchev proved that Kleshchev's algorithm is equivalent to Mullineux's original algorithm and thus proved Mullineux's conjecture. The validity of Kleshchev's algorithm of M for arbitrary e is proved in [3].

Note that the Mullineux involution M depends only on e. *Henceforth, we refer to the case when e is odd as the odd case*; *and to the case when e is even as the even case.* By [7, (3.5)] and [28, (5.2), (5.3)], $(S^\lambda)^\# \cong (S^{\lambda^t})^*$. If $\mathcal{H}_k(\mathfrak{S}_n)$ is semisimple, then $(S_k^{\lambda^t})^* \cong S_k^{\lambda^t}$, hence in that case the involution M degenerates to the map $\lambda \mapsto \lambda^t$. In this paper, we do not need Mullineux's original combinatorial algorithm [27] for defining M, but we do need Kleshchev's algorithm [21] of the involution M. To this end, we have to recall the notion of Kleshchev's e-good lattice.

Let λ be a partition of n. The Young diagram of λ is the set

$$[\lambda] = \{(a, b) \mid 1 \leqslant b \leqslant \lambda_a\}.$$

The elements of $[\lambda]$ are nodes of λ. Given any two nodes $\gamma = (a, b)$, $\gamma' = (a', b')$ of λ, say that γ is *below* γ', or γ' is *above* γ, if $a > a'$. The *residue* of $\gamma = (a, b)$ is defined to be $\operatorname{res}(\gamma) := b - a + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$, and we say that γ is a $\operatorname{res}(\gamma)$-node. Note that we can identify the set $\{0, 1, 2, \dots, e-1\}$ with $\mathbb{Z}/e\mathbb{Z}$ via $i \mapsto \bar{i}$ for each $0 \leqslant i \leqslant e-1$. Therefore, we can also think that the res(?) function takes values in $\{0, 1, 2, \dots, e-1\}$.

A removable node is a node of the boundary of the Young diagram $[\lambda]$ which can be removed, while an addable node is a concave corner on the rim of $[\lambda]$ where a node can be added. If μ is a partition of $n+1$ with $[\mu] = [\lambda] \cup \{\gamma\}$ for some removable node γ of μ, we write $\lambda \to \mu$. If in addition $\operatorname{res}(\gamma) = x$, we also write that $\lambda \xrightarrow{x} \mu$. For example, suppose $n = 42$ and $e = 3$. The nodes of $\lambda = (9^2, 8, 7, 5, 3, 1)$ have the following residues

$$\lambda = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{2} \\ \bar{2} & \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{1} \\ \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{2} & \\ \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{2} & \bar{0} & & \\ \bar{2} & \bar{0} & \bar{1} & \bar{2} & \bar{0} & & & & \\ \bar{1} & \bar{2} & \bar{0} & & & & & & \\ \bar{0} & & & & & & & & \end{pmatrix}.$$

It has six removable nodes. Fix a residue x and consider the sequence of removable and addable x-nodes obtained by reading the boundary of λ from the top down. In the above example, if we consider residue $x = \bar{0}$, then we get a sequence AARRRR, where each "A" corresponds to an addable $\bar{0}$-node and each "R" corresponds to a removable $\bar{0}$-node. Given such a sequence of letters A, R, we remove all occurrences of the string “AR” and keep on doing this until no such string “AR” is left. The “R”s that still remain are the *normal* $\bar{0}$-nodes of λ and the rightmost of these is the *good* $\bar{0}$-node. In the above example, the two removable $\bar{0}$-nodes in the last two rows survive after we delete all the string “AR.” Therefore, the removable $\bar{0}$-node in the last row is the good $\bar{0}$-node. If γ is a good x-node of μ and λ is the partition such that $[\mu] = [\lambda] \cup \gamma$, we write $\lambda \overset{x}{\twoheadrightarrow} \mu$. The *Kleshchev's e-good lattice* is, by definition, the infinite graph whose vertices are the e-regular partitions and whose arrows are given by $\lambda \overset{x}{\twoheadrightarrow} \mu \Leftrightarrow \lambda$ is obtained from μ by removing a good x-node. It is well known that, for each e-regular partition λ, there is a path (not necessary unique) from the empty partition $\emptyset$ to λ in Kleshchev's e-good lattice.

Kleshchev's e-good lattice in fact provides a combinatorial realization of the crystal graph of the basic representation of the affine Lie algebra of type $A_{e-1}^{(1)}$ (which we denote by $\widehat{\mathfrak{sl}}_e$). To be more precise, let $\{\alpha_0, \alpha_1, \dots, \alpha_{e-1}\}$ be the set of simple roots of $\widehat{\mathfrak{sl}}_e$, let $\{\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_{e-1}^\vee\}$ be set of simple coroots, let

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{e \times e} \quad \text{if } e \geqslant 3;$$

or

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}_{2 \times 2} \quad \text{if } e = 2.$$

be the corresponding affine Cartan matrix. Let d be the scaling element. Then the set $\{\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_{e-1}^\vee, d\}$ forms a basis of the Cartan subalgebra of $\widehat{\mathfrak{sl}}_e$, let $\{\Lambda_0, \Lambda_1, \dots, \Lambda_{e-1}, \delta\}$ be the corresponding dual basis, where δ denotes the null root. The integrable highest weight module of highest weight Λ_0, denoted by $L(\Lambda_0)$, is called the basic representation of $\widehat{\mathfrak{sl}}_e$. It is a remarkable fact ([26], [1, (2.11)]) that the crystal graph of $L(\Lambda_0)$ is exactly the same as the Kleshchev's e-good lattice if one use the embedding $L(\Lambda_0) \subset \mathcal{F}(\Lambda_0)$, where $\mathcal{F}(\Lambda_0)$ is the Fock space as defined in [24, §4.2]. In particular, an explicit formula for the number of irreducible $\mathcal{H}_k(\mathfrak{S}_n)$-modules, i.e., $\#\mathcal{K}_n$, is known (see [1]), which was expressed in terms of principal specialized character of the basic representation $L(\Lambda_0)$.

Now we can state Kleshchev's algorithm of the Mullineux involution M. Here we follow Lascoux–Leclerc–Thibon's reformulation in [24, (7.1)].

Lemma 2.4. [21] *Let $\lambda \in \mathcal{K}_n$ be an e-regular partition of n, and let*

$$\emptyset \overset{r_1}{\twoheadrightarrow} \cdot \overset{r_2}{\twoheadrightarrow} \cdot \ldots \cdot \overset{r_n}{\twoheadrightarrow} \lambda$$

be a path from $\emptyset$ to λ in Kleshchev's e-good lattice. Then, the sequence

$$\underline{\emptyset} \overset{e-r_1}{\twoheadrightarrow} \cdot \overset{e-r_2}{\twoheadrightarrow} \cdot \ldots \cdot \overset{e-r_n}{\twoheadrightarrow},$$

also defines a path in Kleshchev's e-good lattice, and it connects $\emptyset$ to $\mathrm{M}(\lambda)$.

Note that the Mullineux involution M gives rise to an equivalence relation on $\mathcal{K}_n$. That is, $\lambda \sim \mu$ if and only if $\lambda = \mathrm{M}(\mu)$ for any $\lambda, \mu \in \mathcal{K}_n$. Let A_n be the alternating group, which is a normal subgroup in $\mathfrak{S}_n$ of index 2. In the special case where $q = 1$ and e is an odd prime number, the involution M is closely related to the modular representation of the alternating group A_n, as can be seen from the following lemma.

Lemma 2.5. [8, (2.1)] *Suppose that $q = 1$ and e is an odd prime number. In particular,* $\operatorname{char} k = e$. *Assume that A_n is split over k. Then*:

(1) *for any $\lambda \in \mathcal{K}_n$ with $\mathrm{M}(\lambda) \neq \lambda$, $D^\lambda \downarrow_{A_n}$ remains irreducible*;
(2) *for any $\lambda \in \mathcal{K}_n$ with $\mathrm{M}(\lambda) = \lambda$, $D^\lambda \downarrow_{A_n}$ is a direct sum of two irreducible, non-equivalent, representations of kA_n, say D^λ_+ and D^λ_-*;
(3) *the set*

$$\big\{D^\lambda_+, D^\lambda_- \bigm| \lambda \in \mathcal{K}_n/{\sim}, \mathrm{M}(\lambda) = \lambda\big\} \sqcup \big\{D^\lambda \downarrow_{A_n} \bigm| \lambda \in \mathcal{K}_n/{\sim}, \mathrm{M}(\lambda) \neq \lambda\big\}$$

forms a complete set of pairwise non-isomorphic irreducible kA_n-modules.

As a consequence, we get that

$$\begin{aligned}\#\operatorname{Irr}(kA_n) &= \frac{1}{2}\big(\#\mathcal{K}_n - \#\big\{\lambda \in \mathcal{K}_n \bigm| \mathrm{M}(\lambda) = \lambda\big\}\big) + 2\#\big\{\lambda \in \mathcal{K}_n \bigm| \mathrm{M}(\lambda) = \lambda\big\}\\ &= \frac{1}{2}\big(\#\mathcal{K}_n + 3\#\big\{\lambda \in \mathcal{K}_n \bigm| \mathrm{M}(\lambda) = \lambda\big\}\big).\end{aligned}$$

3. The orbit Lie algebras

In this section, we shall first determine the orbit Lie algebras corresponding to the Dynkin diagram automorphisms arising from the Mullineux involution. Then we shall use Naito and Sagaki's work [29,30] to study the set of Mullineux-fixed partitions in terms of crystal graphs of basic representations of the orbit Lie algebras, which are some twisted affine Lie algebras of type $A^{(2)}_{2\ell}$ or of type $D^{(2)}_{\ell+1}$. The main results are given in Theorems 3.7, 3.13, 3.15, 3.17, 3.20 and 3.24.

Let $\mathfrak{g}$ be the Kac–Moody algebra over $\mathbb{C}$ associated to a symmetrizable generalized Cartan matrix $(a_{i,j})_{i,j\in I}$ of finite size, where $I = \{0, 1, \dots, e-1\}$. Let $\mathfrak{h}$ be its Cartan subalgebra, and W be its Weyl group. Let $\{\alpha_i^\vee\}_{0\leqslant i\leqslant e-1}$ be the set of simple coroots in $\mathfrak{h}$. Let $\mathcal{X} := \{\Lambda \in \mathfrak{h}^* \mid \Lambda(\alpha_i^\vee) \in \mathbb{Z},\ \forall 0 \leqslant i < e\}$ be the weight lattice. Let $\mathcal{X}^+ := \{\Lambda \in \mathcal{X} \mid \Lambda(\alpha_i^\vee) \geqslant 0,\ \forall 0 \leqslant i < e\}$ be the lattice of integral dominant weights. Let $\mathcal{X}_{\mathbb{R}} := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathbb{R}$ is the field of real numbers. Assume that $\Lambda \in \mathcal{X}^+$. P. Littelmann introduced [22,23] the notion of Lakshmibai–Seshadri paths (LS paths for short) of class Λ, which are piecewise linear, continuous maps $\pi : [0,1] \to \mathcal{X}_{\mathbb{R}}$ parameterized by pairs $(\underline{\nu}, \underline{a})$ of a sequence $\underline{\nu}$: $\nu_1 > \nu_2 > \cdots > \nu_s$ of elements of $W\Lambda$, where $>$ is the "relative Bruhat order" on $W\Lambda$, and a sequence $\underline{a}$: $0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers with a certain condition, called the chain condition. The set $\mathbb{B}(\Lambda)$ of all LS paths of class Λ is called the path model for the integrable highest weight module $L(\Lambda)$ of highest weight Λ over $\mathfrak{g}$. It is a remarkable fact that $\mathbb{B}(\Lambda)$ has a canonical crystal structure isomorphic to the crystal (in the sense of [20]) associated to the integrable highest weight module of highest weight Λ over the quantum algebra $U'_v(\mathfrak{g})$.

Now let $\mathfrak{g}$ be the affine Kac–Moody algebra of type $A_{e-1}^{(1)}$. Let $\omega: I \to I$ be an involution defined by $\omega(0) = 0$ and $\omega(i) = e - i$ for any $0 \neq i \in I$.

Lemma 3.1. *ω is a Dynkin diagram automorphism in the sense of* [30, §1.2]. *That is $a_{\omega(i),\omega(j)} = a_{i,j}$, $\forall i, j \in I$.*

Proof. This follows from direct verification.

By [11], ω induces a Lie algebra automorphism (which are called diagram outer automorphism) $\omega \in \mathrm{Aut}(\mathfrak{g})$ of order 2 and a linear automorphism $\omega^* \in \mathrm{GL}(\mathfrak{h}^*)$ of order 2. Following [10] and [30, §1.3] (where they work with an arbitrary Kac–Moody algebra $\mathfrak{g}$ and a Dynkin diagram automorphism ω), we set $c_{i,j} := \sum_{t=0}^{N_j-1} a_{i,\omega^t(j)}$, where $N_j := \#\{\omega^t(i) \mid t \geqslant 0\}$, $i, j \in I$. We choose a complete set $\hat{I}$ of representatives of the ω-orbits in I, and set $\check{I} := \{i \in \hat{I} \mid c_{i,i} > 0\}$. We put $\hat{a}_{i,j} := 2c_{i,j}/c_j$ for $i, j \in \hat{I}$, where $c_i := c_{ii}$ if $i \in \check{I}$, and $c_i := 2$ otherwise. Then $(\hat{a}_{i,j})_{i,j\in\hat{I}}$ is a symmetrizable Borcherds–Cartan matrix [2], and (if $\check{I} \neq \emptyset$) its submatrix $(\hat{a}_{i,j})_{i,j\in\check{I}}$ is a generalized Cartan matrix. Let $\hat{\mathfrak{g}}$ be the generalized Kac–Moody algebra over $\mathbb{C}$ associated to $(\hat{a}_{i,j})_{i,j\in\hat{I}}$, with Cartan subalgebra $\hat{\mathfrak{h}}$, Chevalley generators $\{\hat{x}_i, \hat{y}_i\}_{i\in\hat{I}}$. The orbit Lie algebra $\check{\mathfrak{g}}$ is defined to be the subalgebra of $\hat{\mathfrak{g}}$ generated by $\hat{\mathfrak{h}}$ and $\hat{x}_i, \hat{y}_i$ for $i \in \check{I}$, which is a usual Kac–Moody algebra. $\square$

Lemma 3.2. *With the above assumptions and notations, we have that in our special case, $\check{\mathfrak{g}}$ is isomorphic to the twisted affine Lie algebra of type $A_{2\ell}^{(2)}$ if $e = 2\ell + 1$; and $\check{\mathfrak{g}}$ is isomorphic the twisted affine Lie algebra of type $D_{\ell+1}^{(2)}$ if $e = 2\ell$.*

Proof. We divide the proof into two cases:

Case 1. $e = 2\ell + 1$. The involution ω is given by

$$\omega\colon \begin{cases} 0 \mapsto 0, \\ 1 \mapsto 2\ell, \\ \vdots \\ \ell-1 \mapsto \ell+2, \\ \ell \mapsto \ell+1, \end{cases} \qquad \begin{cases} \ell+1 \mapsto \ell, \\ \vdots \\ 2\ell-1 \mapsto 2, \\ 2\ell \mapsto 1. \end{cases}$$

It is easy to check that $c_{i,i} = 2$ for any $0 \leqslant i < \ell$ and $c_{\ell,\ell} = 1$. We shall take $\hat{I} = \{0, 1, \dots, l\}$. By direct verification, we get that $\check{I} = \hat{I}$ and

$$(\hat{a}_{i,j})_{i,j\in\hat{I}} = \begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{(\ell+1)\times(\ell+1)} \qquad \text{if } \ell \geqslant 2;$$

or

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}_{2\times 2} \quad \text{if } \ell = 1.$$

Clearly this is an affine Cartan matrix of type $A_{2\ell}^{(2)}$, hence in this case $\breve{\mathfrak{g}}$ is isomorphic to the twisted affine Lie algebra of type $A_{2\ell}^{(2)}$.

Case 2. $e = 2\ell$. The involution ω is given by

$$\omega\colon \begin{cases} 0 \mapsto 0, \\ 1 \mapsto 2\ell - 1, \\ \vdots \\ \ell - 1 \mapsto \ell + 1, \\ \ell \mapsto \ell, \end{cases} \qquad \begin{cases} \ell + 1 \mapsto \ell - 1, \\ \vdots \\ 2\ell - 2 \mapsto 2, \\ 2\ell - 1 \mapsto 1. \end{cases}$$

It is easy to check that $c_{i,i} = 2$ for any $0 \leqslant i \leqslant \ell$. We shall take $\hat{I} = \{0, 1, \ldots, l\}$. By direct verification, we get that $\check{I} = \hat{I}$ and

$$(\hat{a}_{i,j})_{i,j\in\hat{I}} = \begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix}_{(\ell+1)\times(\ell+1)} \quad \text{if } \ell \geqslant 2;$$

or

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}_{2\times 2} \quad \text{if } \ell = 1.$$

Clearly this is an affine Cartan matrix of type $D_{\ell+1}^{(2)}$, hence in this case $\breve{\mathfrak{g}}$ is isomorphic to the twisted affine Lie algebra of type $D_{\ell+1}^{(2)}$.

We define $(\mathfrak{h}^*)^\circ := \{\Lambda \in \mathfrak{h}^* \mid \omega^*(\Lambda) = \Lambda\}$. $\tilde{W} := \{w \in W \mid \omega^* w = w\omega^*\}$. We indicate by $\check{}$ the objects for the obit Lie algebra $\breve{\mathfrak{g}}$. For example, $\check{\mathfrak{h}}$ denotes the Cartan subalgebra of $\breve{\mathfrak{g}}$, $\check{W}$ the Weyl group of $\breve{\mathfrak{g}}$, $\{\check{\Lambda}_i\}_{0\leqslant i\leqslant\ell}$ the set of fundamental dominant weights in $\check{\mathfrak{h}}^*$. There exists a linear automorphism $P_\omega^*\colon \check{\mathfrak{h}}^* \to (\mathfrak{h}^*)^\circ$ and a group isomorphism $\Theta\colon \check{W} \to \tilde{W}$ such that $\Theta(\check{w}) = P_\omega^* \check{w} (P_\omega^*)^{-1}$ for each $w \in \check{W}$. By [11, §6.5], for each $0 \leqslant i \leqslant \ell$,

$$P_\omega^*(\check{\Lambda}_i) = \sum_{t=0}^{N_i - 1} \Lambda_{\omega^t(i)} + C\delta,$$

where N_i denotes the number of elements in the ω-orbit of i, $C \in \mathbb{Q}$ is some constant depending on ω, δ denotes the null root of $\mathfrak{g}$. It follows that $P_\omega^*(\check{\Lambda}_0) = \Lambda_0 + C'\delta$, for some $C' \in \mathbb{Q}$.

Let $\mathbb{B}(\Lambda_0)$ (respectively, $\mathbb{B}(P_\omega^*(\check{\Lambda}_0))$) be the set of all LS paths of class Λ_0 (respectively, of class $P_\omega^*(\check{\Lambda}_0)$). Let π_{Λ_0} (respectively, $\pi_{P_\omega^*(\check{\Lambda}_0)}$) be the straight path joining 0 and Λ_0 (respectively, 0 and $P_\omega^*(\check{\Lambda}_0)$). For each integer $0 \leqslant i \leqslant e-1$, let $\tilde{E}_i$, $\tilde{F}_i$ denote the raising root operator and the lowering root operator with respect to the simple root α_i. $\square$

Lemma 3.3. *The map which sends $\pi_{P_\omega^*(\check{\Lambda}_0)}$ to π_{Λ_0} extends to a bijection β from $\mathbb{B}(P_\omega^*(\check{\Lambda}_0))$ onto $\mathbb{B}(\Lambda_0)$ such that*

$$\beta\big(\tilde{F}_{i_1}\cdots\tilde{F}_{i_s}\pi_{P_\omega^*(\check{\Lambda}_0)}\big) = \tilde{F}_{i_1}\cdots\tilde{F}_{i_s}\pi_{\Lambda_0},$$

for any $i_1, \dots, i_s \in \{0, 1, \dots, e-1\}$.

Proof. This follows from the fact that $P_\omega^*(\check{\Lambda}_0) - \Lambda_0 \in \mathbb{Q}\delta$ and the definitions of $\mathbb{B}(P_\omega^*(\check{\Lambda}_0))$ and $\mathbb{B}(\Lambda_0)$ (see [22]).

Henceforth we shall identify $\mathbb{B}(P_\omega^*(\check{\Lambda}_0))$ with $\mathbb{B}(\Lambda_0)$. The action of ω^* on $\mathfrak{h}^*$ naturally extends to the set $\mathbb{B}(P_\omega^*(\check{\Lambda}_0))$ (and hence to the set $\mathbb{B}(\Lambda_0)$). By [29, (3.1.1)], if $\tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_s}\pi_{\Lambda_0} \in \mathbb{B}(\Lambda_0)$, then

$$\omega^*\big(\tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_s}\pi_{\Lambda_0}\big) = \tilde{F}_{\omega(i_1)}\tilde{F}_{\omega(i_2)}\cdots\tilde{F}_{\omega(i_s)}\pi_{\Lambda_0}. \tag{3.4}$$

We denote by $\mathbb{B}^\circ(\Lambda_0)$ the set of all LS paths of class Λ_0 that are fixed by ω^*. For $\check{\mathfrak{g}}$, for each integer $0 \leqslant i \leqslant \ell$, we denote by $\tilde{e}_i$, $\tilde{f}_i$ the raising root operator and the lowering root operator with respect to the simple root α_i. Let $\pi_{\check{\Lambda}_0}$ be the straight path joining 0 and $\check{\Lambda}_0$. By [30, (4.2)], the linear map P_ω^* naturally extends to a map from $\check{\mathbb{B}}(\check{\Lambda}_0)$ to $\mathbb{B}^\circ(\Lambda_0)$ such that if $\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_s}\pi_{\check{\Lambda}_0} \in \check{\mathbb{B}}(\check{\Lambda}_0)$, then (in the above two cases)

$$P_\omega^*\big(\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_s}\pi_{\check{\Lambda}_0}\big) = \omega\big(\tilde{F}_{i_1}\big)\omega\big(\tilde{F}_{i_2}\big)\cdots\omega\big(\tilde{F}_{i_s}\big)\pi_{\Lambda_0},$$

where

$$\omega\tilde{F}_{i_t}) := \begin{cases} \tilde{F}_{i_t}\tilde{F}_{\omega(i_t)}, & \text{if } c_{i_t,i_t} = 2 \text{ and } N_{i_t} = 2, \\ \tilde{F}_{i_t}, & \text{if } c_{i_t,i_t} = 2 \text{ and } N_{i_t} = 1, \\ \tilde{F}_{\omega(i_t)}\tilde{F}_{i_t}^2\tilde{F}_{\omega(i_t)}, & \text{if } c_{i_t,i_t} = 1. \end{cases}$$

Note that the case $c_{i_t,i_t} = 1$ only happens when $e = 2\ell + 1$ and $i_t = \ell$. $\square$

Lemma 3.5. [30, (4.2), (4.3)] $\mathbb{B}^\circ(\Lambda_0) = P_\omega^*(\check{\mathbb{B}}(\check{\Lambda}_0))$.

Note that both $\check{\mathbb{B}}(\check{\Lambda}_0)$ and $\mathbb{B}(\Lambda_0)$ have a canonical crystal structure with the raising and lowering root operators playing the role of Kashiwara operators. They are isomorphic to the crystals associated to the integrable highest weight modules $\check{L}(\check{\Lambda}_0)$ of highest weight $\check{\Lambda}_0$ over $U'_v(\check{\mathfrak{g}})$ and the integrable highest weight modules $L(\Lambda_0)$ of highest weight Λ_0 over $U'_v(\mathfrak{g})$, respectively. Henceforth, we identify them without further comments. Let $v_{\check{\Lambda}_0}$ (respectively, v_{Λ_0}) denotes the unique highest weight vector of highest weight $\check{\Lambda}_0$ (respectively, of highest weight Λ_0) in $\check{\mathbb{B}}(\check{\Lambda}_0)$ (respectively, in $\mathbb{B}(\Lambda_0)$). Therefore, by (3.4) and Lemma 3.5, we get that

Corollary 3.6. *With the above assumptions and notations, there is an injection η from the set $\check{\mathbb{B}}(\check{\Lambda}_0)$ of crystal bases to the set $\mathbb{B}(\Lambda_0)$ of crystal bases such that*

$$\eta\big(\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_s}v_{\check{\Lambda}_0}\big)\equiv\omega\big(\tilde{F}_{i_1}\big)\omega\big(\tilde{F}_{i_2}\big)\cdots\omega\big(\tilde{F}_{i_s}\big)v_{\Lambda_0}\quad\big(\mathrm{mod}\ vL(\Lambda_0)_A\big),$$

where $i_1,\dots,i_s$ are integers in $\{0,1,2,\dots,\ell\}$, and A denotes the ring of rational functions in $\mathbb{Q}(v)$ which do not have a pole at 0. Moreover, the image of η consists of all crystal basis element $\tilde{F}_{i_1}\cdots\tilde{F}_{i_s}v_{\Lambda_0}+vL(\Lambda_0)_A$ satisfying $\tilde{F}_{i_1}\cdots\tilde{F}_{i_s}v_{\Lambda_0}\equiv\tilde{F}_{\omega(i_1)}\cdots\tilde{F}_{\omega(i_s)}v_{\Lambda_0}$ $(\mathrm{mod}\ vL(\Lambda_0)_A)$.

Let $\mathcal{K}:=\bigsqcup_{n\geqslant 0}\mathcal{K}_n$. We translate the language of crystal bases into the language of partitions, we get the following combinatorial result.

Theorem 3.7. *With the above notations, there is a bijection η from the set $\check{\mathbb{B}}(\check{\Lambda}_0)$ of crystal bases onto the set $\{\lambda\in\mathcal{K}\mid \mathrm{M}(\lambda)=\lambda\}$, such that if*

$$v_{\check{\Lambda}_0}\overset{r_1}{\twoheadrightarrow}\cdot\overset{r_2}{\twoheadrightarrow}\cdot\ldots\cdot\overset{r_s}{\twoheadrightarrow}\tilde{f}_{r_s}\cdots\tilde{f}_{r_1}v_{\check{\Lambda}_0}$$

is a path from $v_{\check{\Lambda}_0}$ to $\tilde{f}_{r_s}\cdots\tilde{f}_{r_1}v_{\check{\Lambda}_0}$ in the crystal graph of $\check{L}(\check{\lambda}_0)$, then the sequence

$$\emptyset\underbrace{\overset{r_1}{\twoheadrightarrow}\cdot}_{\omega\ acts}\underbrace{\overset{r_2}{\twoheadrightarrow}\cdot}_{\omega\ acts}\ldots\cdot\underbrace{\overset{r_s}{\twoheadrightarrow}\lambda}_{\omega\ acts}:=\eta\big(\tilde{f}_{r_s}\cdots\tilde{f}_{r_1}v_{\check{\Lambda}_0}\big),$$

where

$$\underbrace{\overset{r_t}{\twoheadrightarrow}\cdot}_{\omega\ acts}:=\begin{cases}\overset{r_t}{\twoheadrightarrow}\cdot\overset{e-r_t}{\twoheadrightarrow}, & \text{if } c_{r_t,r_t}=2 \text{ and } N_{r_t}=2,\\ \overset{r_t}{\twoheadrightarrow}, & \text{if } c_{r_t,r_t}=2 \text{ and } N_{r_t}=1,\\ \overset{\ell+1}{\twoheadrightarrow}\cdot\overset{\ell}{\twoheadrightarrow}\cdot\overset{\ell}{\twoheadrightarrow}\cdot\overset{\ell+1}{\twoheadrightarrow}\cdot, & \text{if } e=2\ell+1 \text{ and } r_t=\ell,\end{cases}$$

defines a path in Kleshchev's e-good lattice which connects $\emptyset$ and e-regular partition λ satisfying $\mathrm{M}(\lambda)=\lambda$.

Proof. This follows from Lemmas 2.4, 3.5 and Corollary 3.6. □

For each partition λ of n, and each integer $0\leqslant i\leqslant e-1$, we define

$$\Sigma_i(\lambda):=\big\{\gamma\in[\lambda]\ \big|\ \mathrm{res}(\gamma)=\bar{i}\big\},$$
$$N_i(\lambda):=\#\Sigma_i(\lambda).$$

Theorem 3.7 also implies that if $\tilde{f}_{r_1}\cdots\tilde{f}_{r_s}v_{\check{\Lambda}_0}\in\check{\mathbb{B}}(\check{\Lambda}_0)$, $\lambda:=\eta(\tilde{f}_{r_1}\cdots\tilde{f}_{r_s}v_{\check{\Lambda}_0})$, then

$$N_i(\lambda)=\begin{cases}\#\{1\leqslant t\leqslant s\mid r_t=i\}, & \text{if } i\in\{0,1,2,\dots,\ell-1\},\\ \#\{1\leqslant t\leqslant s\mid r_t=e-i\}, & \text{if } i\in\{\ell+2,\ell+3,\dots,e-1\},\\ \#\{1\leqslant t\leqslant s\mid r_t=\ell-1\}, & \text{if } e=2\ell \text{ and } i=\ell+1,\\ \#\{1\leqslant t\leqslant s\mid r_t=\ell\}, & \text{if } e=2\ell \text{ and } i=\ell,\\ 2\#\{1\leqslant t\leqslant s\mid r_t=\ell\}, & \text{if } e=2\ell+1 \text{ and } i\in\{\ell,\ell+1\}.\end{cases}\tag{3.8}$$

Corollary 3.9. *Let $\lambda\in\mathcal{K}_n$. Suppose that $\mathrm{M}(\lambda)=\lambda$.*

(1) *If $e=2\ell+1$, then $N_\ell(\lambda)=N_{\ell+1}(\lambda)$. Furthermore, $N_\ell(\lambda)$ and $n-N_0(\lambda)$ are both even integers.*
(2) *If $e=2\ell$, then $n-N_0(\lambda)-N_\ell(\lambda)$ is an even integer.*

For each pair of integers m,m' with $0\leqslant m+m'\leqslant n$, we define

$$\Sigma(n,m,m'):=\{\lambda\in\mathcal{K}_n\mid \mathrm{M}(\lambda)=\lambda,\ N_0(\lambda)=m,\ N_\ell(\lambda)=m'\},$$
$$N(n,m,m'):=\#\Sigma(n,m,m').$$

Note that when $e=2\ell+1$, by Corollary 3.9, $N(n,m,m')=0$ unless $m+2m'\leqslant n$.

Recall the principle graduation introduced in [18, §1.5, §10.10]. That is, the weight $\Lambda_0-\sum_{i=0}^{e-1}k_i\alpha_i$ (where $k_i\in\mathbb{Z}$ for each i) is assigned to degree $\sum_{i=0}^{e-1}k_i$. Let $\mathrm{ch}_t\,L(\Lambda_0):=\sum_{n\geqslant 0}\dim L(\Lambda_0)_n t^n$ be the principle specialized character[2] of $L(\Lambda_0)$, where $L(\Lambda_0)_n=\bigoplus_{\deg\mu=n}L(\Lambda_0)_\mu$. Similarly, let $L(\check{\Lambda}_0)$ denote the integrable highest weight module of highest weight $\check{\Lambda}_0$ over $\check{\mathfrak{g}}$. We use $\mathrm{ch}_t\,L(\check{\Lambda}_0):=\sum_{n\geqslant 0}\dim L(\check{\Lambda}_0)_n t^n$ to denote the principle specialized character of $L(\check{\Lambda}_0)$. Now applying Lemmas 2.4, 3.5 and Theorem 3.7, we get that

$$\dim L(\check{\Lambda}_0)_n=\sum_{0\leqslant m+m'\leqslant n}N(2n-m+2m',m,2m')\tag{3.10}$$

if $e=2\ell+1$; while

$$\dim L(\check{\Lambda}_0)_n=\sum_{0\leqslant m+m'\leqslant n}N(2n-m-m',m,m')\tag{3.11}$$

if $e=2\ell$.

Suppose that $e=2\ell+1$. That is, we are in the odd case. In this case, $\check{\mathfrak{g}}$ is the twisted affine Lie algebra of type $A_{2\ell}^{(2)}$. By [18, (14.5.4)], the principle specialized character of $L(\check{\Lambda}_0)$ is given by

$$\mathrm{ch}_t\,L(\check{\Lambda}_0)=\prod_{\substack{i\geqslant 1,\ i \text{ odd}\\ i\not\equiv 0\ (\mathrm{mod}\ e)}}\frac{1}{1-t^i}.\tag{3.12}$$

[2] This is called q-dimension in the book of Kac, see [18, §10.10].

Hence by (3.10) and (3.12), we get that

Theorem 3.13. *With the above notations, we have that*

$$\prod_{\substack{i\geqslant 1,\ i\ odd\\ i\not\equiv 0\ (\mathrm{mod}\ e)}} \frac{1}{1-t^i} = \sum_{n\geqslant 0}\left(\sum_{0\leqslant m+m'\leqslant n} N(2n-m+2m',m,2m')\right)t^n.$$

In [19], Kang has given a combinatorial realization of $\check{\mathbb{B}}(\check{\Lambda}_0)$ in terms of reduced proper Young walls, which are inductively defined. In our $A_{2\ell}^{(2)}$ case, a direct explicit characterization can be given in terms of restricted e-strict partitions as follows, see [4,25].

Recall that [4,5] a partition λ is called e-strict if $\lambda_i=\lambda_{i+1}\Rightarrow e\mid\lambda_i$ for each $i=1,2,\dots$. An e-strict partition λ is called restricted if in addition

$$\begin{cases}\lambda_i-\lambda_{i+1}\leqslant e, & \text{if } e\nmid\lambda_i,\\ \lambda_i-\lambda_{i+1}<e, & \text{if } e\mid\lambda_i,\end{cases}\quad \text{for each } i=1,2,\dots.$$

Let $DPR_e(n)$ denote the set of all restricted e-strict partitions of n. Let $DPR_e:=\bigsqcup_{n\geqslant 0}DPR_e(n)$.

It turns out that there is a natural 1-1 correspondence between $\check{\mathbb{B}}(\check{\Lambda}_0)$ and DPR_e. Furthermore, the crystal structure $\check{\mathbb{B}}(\check{\Lambda}_0)$ can be concretely realized via some combinatorics of DPR_e, which we now describe.

We recall some notions. Elements of $(r,s)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ are called nodes. Let λ be an e-strict partition. We label the nodes of λ with *residues*, which are the elements of $\mathbb{Z}/(\ell+1)\mathbb{Z}$. The residue of the node A is denoted res A. The labeling depends only on the column and following the repeating pattern

$$\bar{0},\bar{1},\dots,\overline{\ell-1},\bar{\ell},\overline{\ell-1},\dots,\bar{1},\bar{0},$$

starting from the first column and going to the right. For example, let $e=5$, $\ell=2$, let $\lambda=(10,10,6,1)$ be a restricted 5-strict partition of 27. Its residues are as follows:

$$\begin{array}{cccccccccc}
\bar{0}&\bar{1}&\bar{2}&\bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{2}&\bar{1}&\bar{0}\\
\bar{0}&\bar{1}&\bar{2}&\bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{2}&\bar{1}&\bar{0}\\
\bar{0}&\bar{1}&\bar{2}&\bar{1}&\bar{0}&\bar{0}&&&&\\
\bar{0}&&&&&&&&&
\end{array}$$

A node $A=(r,s)\in[\lambda]$ is called removable (for λ) if either

(R1) $\lambda_A:=\lambda-\{A\}$ is again an e-strict partition; or
(R2) the node $B=(r,s+1)$ immediately to the right of A belongs to λ, $\mathrm{res}(A)=\mathrm{res}(B)$, and both λ_B and $\lambda_{AB}:=\lambda-\{A,B\}$ are e-strict partitions.

Similarly, a node $B=(r,s)\notin[\lambda]$ is called addable (for λ) if either

(A1) $\lambda^B:=\lambda\cup\{B\}$ is again an e-strict partition; or
(A2) the node $A=(r,s-1)$ immediately to the left of B does not belong to λ, $\mathrm{res}(A)=\mathrm{res}(B)$, and both $\lambda^A:=\lambda\cup\{A\}$ and $\lambda^{AB}:=\lambda\cup\{A,B\}$ are e-strict partitions.

Note that (R2) and (A2) above are only possible for nodes with residue $\bar{0}$. Now fix a residue x and consider the sequence of removable and addable x-nodes obtained by reading the boundary of λ from the bottom left to top right. We use "A" to denote an addable x-node and use "R" to denote a removable x-node, then we get a sequence of letters A, R. Given such a sequence, we remove all occurrences of the string "AR" and keep on doing this until no such string "AR" is left. The "R"s that still remain are the *normal* x-nodes of λ and the rightmost of these is the *good* x-node, the "A"s that still remain are the *conormal* x-nodes of λ and the leftmost of these is the *cogood* x-node. Note that[3] good x-node is necessarily of type (R1), and cogood x-node is necessarily of type (A1). We define

$$\varepsilon_i(\lambda) = \#\{i\text{-normal nodes in } \lambda\},$$
$$\varphi_i(\lambda) = \#\{i\text{-conormal nodes in } \lambda\}$$

and we set

$$\tilde{e}_i(\lambda) = \begin{cases} \lambda_A, & \text{if } \varepsilon_i(\lambda) > 0 \text{ and } A \text{ is the (unique) good } i\text{-node}, \\ 0, & \text{if } \varepsilon_i(\lambda) = 0, \end{cases}$$

$$\tilde{f}_i(\lambda) = \begin{cases} \lambda^B, & \text{if } \varphi_i(\lambda) > 0 \text{ and } B \text{ is the (unique) cogood } i\text{-node}, \\ 0, & \text{if } \varphi_i(\lambda) = 0. \end{cases}$$

Then, we get an infinite colored oriented graph, whose vertices are e-strict partitions and whose arrows are given by

$$\lambda \overset{i}{\twoheadrightarrow} \mu \quad \Leftrightarrow \quad \mu = \tilde{f}_i(\lambda) \quad \Leftrightarrow \quad \lambda = \tilde{e}_i(\mu).$$

The sublattice spanned by all restricted e-strict partitions equipped with the functions ε_i, φ_i and the operators $\tilde{e}_i, \tilde{f}_i$, can be turned into a colored oriented graph which we denote by $\mathfrak{RP}_e$.

Lemma 3.14. [19] *With the above notations, the graph $\mathfrak{RP}_e$ can be identified with the crystal graph $\check{\mathbb{B}}(\check{\Lambda}_0)$ associated to the integrable highest weight $\check{\mathfrak{g}}$-module of highest weight $\check{\Lambda}_0$.*

Applying Theorem 3.7 and Lemma 3.14, we get that

Theorem 3.15. *With the above notations, there is a bijection η from the set DPR_e of restricted e-strict partitions onto the set $\{\lambda \in \mathcal{K} \mid \mathrm{M}(\lambda) = \lambda\}$, such that if*

$$\emptyset \overset{r_1}{\twoheadrightarrow} \cdot \overset{r_2}{\twoheadrightarrow} \cdot \ldots \cdot \overset{r_s}{\twoheadrightarrow} \check{\lambda}$$

[3] This is because any removable node γ of type (R2) has an adjacent neighborhood γ' in his right, which is another removable node with the same residue. If γ could survive after deleting all the string "AR," then γ' must also survive. In that case, γ' is a normal node higher than γ. So γ cannot be a good node. For cogood node of type (A2), the reason is similar.

is a path from $\emptyset$ *to* $\check{\lambda}$ *in the subgraph* $\mathfrak{RP}_e$, *then the sequence*

$$\emptyset \underbrace{\overset{r_1}{\twoheadrightarrow} \cdot \overset{2\ell+1-r_1}{\twoheadrightarrow} \cdot}_{\widetilde{N_{r_1}}\ \textit{terms}} \underbrace{\overset{r_2}{\twoheadrightarrow} \cdot \overset{2\ell+1-r_2}{\twoheadrightarrow} \cdot}_{\widetilde{N_{r_2}}\ \textit{terms}} \cdots \underbrace{\cdot \overset{r_s}{\twoheadrightarrow} \cdot \overset{2\ell+1-r_s}{\twoheadrightarrow} \lambda}_{\widetilde{N_{r_s}}\ \textit{terms}} := \eta(\check{\lambda}),$$

where

$$\underbrace{\overset{r_t}{\twoheadrightarrow} \cdot \overset{2\ell+1-r_t}{\twoheadrightarrow} \cdot}_{\widetilde{N_{r_t}}\ \textit{terms}} := \begin{cases} \overset{r_t}{\twoheadrightarrow} \cdot \overset{2\ell+1-r_t}{\twoheadrightarrow} \cdot, & \textit{if } r_t \in \{1, 2, \dots, \ell-1\}, \\ \overset{0}{\twoheadrightarrow} \cdot, & \textit{if } r_t = 0, \\ \overset{\ell+1}{\twoheadrightarrow} \cdot \overset{\ell}{\twoheadrightarrow} \cdot \overset{\ell}{\twoheadrightarrow} \cdot \overset{\ell+1}{\twoheadrightarrow}, & \textit{if } r_t = \ell, \end{cases}$$

defines a path in Kleshchev's $(2\ell+1)$*-good lattice which connects* $\emptyset$ *and* $(2\ell+1)$*-regular partition* λ *satisfying* $\mathrm{M}(\lambda) = \lambda$.

Remark 3.16. In [4,5], Brundan and Kleshchev investigated the modular representations of Hecke–Clifford superalgebras at defining parameter a primitive $(2\ell+1)$th root of unity as well as of affine Sergeev superalgebras over a field of characteristic $2\ell+1$. Their main result states that the modular socle branching rules of these superalgebras provide a realization of the crystal of the twisted affine Lie algebra of type $A_{2\ell}^{(2)}$. This applies, in particular, to the modular socle branching rules of the spin symmetric group $\hat{\mathfrak{S}}_n$, which is the double cover of the symmetric group $\mathfrak{S}_n$. It would be interesting to know if there is any connection between their results and ours, at least in the special case where $q = 1$ and $2\ell+1$ being a prime number.

Let P_n be the set of all partitions of n. Let $P := \bigsqcup_{n\geqslant 0} P_n$. Recall that when $\mathcal{H}_k(\mathfrak{S}_n)$ is semisimple, then $\mathcal{K}_n = P_n$ and M degenerates to the map $\lambda \mapsto \lambda^t$ for any $\lambda \in P_n$. Let DP_n be the set of all partitions into distinct parts (i.e., the set all 0-strict partitions). Let $DP := \bigsqcup_{n\geqslant 0} DP_n$. Let SP be the set of all symmetric partitions, i.e., $SP := \{\lambda \in P \mid \lambda = \lambda^t\}$. We shall now establish a bijection between the set DP and the set SP. Note that in the special case where $q = 1$ and $2\ell+1$ is a prime number, the set DP_n parameterizes the ordinary irreducible supermodules of the spin symmetric group $\hat{\mathfrak{S}}_n$, while the set $SP_n := \{\lambda \in P_n \mid \lambda = \lambda^t\}$ parameterizes those ordinary irreducible modules of the symmetric group which splits on restriction to the alternating group A_n.

For each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s) \in DP$ with $\ell(\lambda) = s$, let $\lambda^t = (\lambda_1^t, \lambda_2^t, \dots, \lambda_{\lambda_1}^t)$ be the conjugate of λ, we define

$$\tilde{\eta}(\lambda) = \big(\lambda_1, \lambda_2+1, \lambda_3+2, \dots, \lambda_s+s-1, \lambda_{s+1}^t, \lambda_{s+2}^t, \dots, \lambda_{\lambda_1}^t\big).$$

Theorem 3.17. *With the above notations, the map* $\tilde{\eta}$ *defines a bijection from the set DP onto the set SP.*

Proof. Let $\lambda \in DP$. By definition, $\lambda_1 > \lambda_2 > \cdots > \lambda_s$, it follows that

$$\lambda_1 \geqslant \lambda_2+1 \geqslant \lambda_3+2 \geqslant \cdots \geqslant \lambda_s+s-1 \geqslant \lambda_{s+1}^t \geqslant \lambda_{s+2}^t \geqslant \cdots \geqslant \lambda_{\lambda_1}^t.$$

That is, $\tilde{\eta}(\lambda) \in P$. We claim that $(\tilde{\eta}(\lambda))^t = \tilde{\eta}(\lambda)$.

We use induction on $\ell(\lambda)$. Suppose that $(\tilde{\eta}(\nu))^t = \tilde{\eta}(\nu)$ for any partition ν satisfying $\ell(\nu) < \ell(\lambda)$. We write $\mu = (\mu_1, \ldots, \mu_{\lambda_1}) = \tilde{\eta}(\lambda)$. Then

$$\mu_i = \begin{cases} \lambda_i + i - 1, & \text{for } 1 \leqslant i \leqslant s, \\ \lambda_i^t, & \text{for } s+1 \leqslant i \leqslant \lambda_1, \end{cases} \quad \text{for } i = 1, 2, \ldots, \lambda_1.$$

By definition, $\mu_i^t = \#\{1 \leqslant j \leqslant \lambda_1 \mid \mu_j \geqslant i\}$. It is clear that $\mu_1^t = \lambda_1 = \mu_1$. We remove away the first row as well as the first column of μ. Then we get a partition $\hat{\mu}$. It is easy to see that

$$\hat{\mu} = \bigl(\lambda_2, \lambda_3 + 1, \ldots, \lambda_s + s - 2, \lambda_{s+1}^t - 1, \lambda_{s+2}^t - 1, \ldots, \lambda_{\lambda_2}^t - 1\bigr) = \tilde{\eta}(\hat{\lambda}),$$

where $\hat{\lambda} := (\lambda_2, \lambda_3, \ldots, \lambda_s)$.

Note that $\ell(\hat{\lambda}) < \ell(\lambda)$. By induction hypothesis, we know that $(\hat{\mu})^t = \hat{\mu}$. It follows that $\mu^t = \mu$ as well. This proves our claim.

Second, we claim that the map $\tilde{\eta}$ is injective. In fact, suppose that

$$\begin{aligned} \tilde{\eta}(\lambda) &= \bigl(\lambda_1, \lambda_2 + 1, \lambda_3 + 2, \ldots, \lambda_s + s - 1, \lambda_{s+1}^t, \lambda_{s+2}^t, \ldots, \lambda_{\lambda_1}^t\bigr) \\ &= \bigl(\mu_1, \mu_2 + 1, \mu_3 + 2, \ldots, \mu_{s'} + s' - 1, \mu_{s'+1}^t, \mu_{s'+2}^t, \ldots, \mu_{\mu_1}^t\bigr) = \tilde{\eta}(\mu), \end{aligned}$$

where $\lambda, \mu \in DP$, $\ell(\lambda) = s$, $\ell(\mu) = s'$, $s \leqslant s'$. Then

$$\lambda_1 = \ell\bigl(\tilde{\eta}(\lambda)\bigr) = \ell\bigl(\tilde{\eta}(\mu)\bigr) = \mu_1.$$

It follows that $\lambda_i = \mu_i$ for $i = 1, 2, \ldots, s$. If $s < s'$, then $\lambda_{s+1}^t = \mu_{s+1} + s \geqslant s + 1$, which is impossible. Therefore $s = s'$, and hence $\lambda = \mu$. This proves the injectivity of $\tilde{\eta}$.

It remains to show that $\tilde{\eta}$ is surjective. Let $\mu \in P$ such that $\mu^t = \mu$. Let $A = (r, s)$ be the unique node on the boundary of $[\lambda]$ which sits on the main diagonal of $[\lambda]$. We define

$$\lambda := (\mu_1, \mu_2 - 1, \mu_3 - 2, \ldots, \mu_r - r + 1).$$

Then one sees easily that $\lambda \in DP$ and $\tilde{\eta}(\lambda) = \mu$. This proves that $\tilde{\eta}$ is surjective, hence completes the proof of the whole theorem. □

Remark 3.18. We remark that if one consider the special case where $q = 1$ and $2\ell + 1$ is a prime number, it would be interesting to know if the reduced decomposition matrices (in the sense of [25, (6.2)]) of the spin symmetric groups are embedded as submatrices into the decomposition matrices of the alternating groups in odd characteristic e via our bijections η and $\tilde{\eta}$.

Now we suppose that $e = 2\ell$. That is, we are in the even case. In this case, $\check{\mathfrak{g}}$ is the twisted affine Lie algebra of type $D_{\ell+1}^{(2)}$. By [18, (14.5.4)], the principle specialized character of $L(\check{\Lambda}_0)$ is given by

$$\mathrm{ch}_t\, L(\check{\Lambda}_0) = \prod_{i \geqslant 1,\ i \text{ odd}} \frac{1}{1 - t^i}. \tag{3.19}$$

Hence by (3.11) and (3.19), we get that

Theorem 3.20. *With the above notations, we have that*

$$\prod_{i\geqslant 1,\ i\ odd}\frac{1}{1-t^i}=\sum_{n\geqslant 0}\Biggl(\sum_{0\leqslant m+m'\leqslant n}N(2n-m-m',m,m')\Biggr)t^n.$$

We propose the following definition.

Definition 3.21. Let $f\in\mathbb{N}$ with $f>1$. An f-strict partition λ is called double restricted if

$$\begin{cases}\lambda_i-\lambda_{i+1}\leqslant 2f, & \text{if } f\nmid\lambda_i,\\ \lambda_i-\lambda_{i+1}<2f, & \text{if } f\mid\lambda_i,\end{cases}\quad\text{for each } i=1,2,\ldots.$$

Here we make the convention that $\lambda_i=0$ for any $i>\ell(\lambda)$.

Let $DDPR_f(n)$ denote the set of all double restricted f-strict partitions of n. Let $DDPR_f:=\bigsqcup_{n\geqslant 0}DDPR_f(n)$.

In [19], Kang has given a combinatorial realization of $\check{\mathbb{B}}(\check{\Lambda}_0)$ in terms of reduced proper Young walls, which are inductively defined. In our $D_{\ell+1}^{(2)}$ case, we shall give a direct explicit characterization in terms of double restricted $(\ell+1)$-strict partitions as follows.

As before, elements of $(r,s)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ are called nodes. Let λ be an $(\ell+1)$-strict partition. We label the nodes of λ with *residues*, which are the elements of $\mathbb{Z}/(\ell+1)\mathbb{Z}$. The residue of the node A is denoted $\operatorname{res}A$. The labeling depends only on the column and following the repeating pattern

$$\bar{0},\bar{1},\ldots,\overline{\ell-1},\bar{\ell},\bar{\ell},\overline{\ell-1},\ldots,\bar{1},\bar{0},$$

starting from the first column and going to the right. For example, let $e=4$, $\ell=2$, let $\lambda=(9,9,7,1)$ be a double restricted 3-strict partition of 26. Its residues are as follows:

$$\begin{matrix}\bar{0}&\bar{1}&\bar{2}&\bar{2}&\bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{2}\\ \bar{0}&\bar{1}&\bar{2}&\bar{2}&\bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{2}\\ \bar{0}&\bar{1}&\bar{2}&\bar{2}&\bar{1}&\bar{0}&\bar{0}&&\\ \bar{0}&&&&&&&&\end{matrix}$$

Let λ be an $(\ell+1)$-strict partition. A node $A=(r,s)\in[\lambda]$ is called removable (for λ) if either

(R1) $\lambda_A:=\lambda-\{A\}$ is again an $(\ell+1)$-strict partition; or
(R2) the node $B=(r,s+1)$ immediately to the right of A belongs to λ, $\operatorname{res}(A)=\operatorname{res}(B)$, and both λ_B and $\lambda_{AB}:=\lambda-\{A,B\}$ are $(\ell+1)$-strict partitions.

Similarly, a node $B=(r,s)\notin[\lambda]$ is called addable (for λ) if either

(A1) $\lambda^B:=\lambda\cup\{B\}$ is again an $(\ell+1)$-strict partition; or
(A2) the node $A=(r,s-1)$ immediately to the left of B does not belong to λ, $\operatorname{res}(A)=\operatorname{res}(B)$, and both $\lambda^A:=\lambda\cup\{A\}$ and $\lambda^{AB}:=\lambda\cup\{A,B\}$ are $(\ell+1)$-strict partitions.

Now we can define the notions of normal (respectively, conormal) nodes, good (respectively, cogood) nodes, the functions ε_i, φ_i and the operators $\tilde{e}_i, \tilde{f}_i$ in the same way as in the case where $e = 2\ell + 1$. Note that the definition of residue in the even case is different with the odd case, and in the even case we deal with $(\ell+1)$-strict partitions instead of e-strict partitions.

Lemma 3.22. *Let λ be any given double restricted $(\ell+1)$-strict partition. Then*:

(1) *there exists good* (*removable*) *node as well as cogood* (*addable*) *node for λ*;
(2) *for any good* (*removable*) *node A for λ, $\lambda - \{A\}$ is again a double restricted $(\ell+1)$-strict partition. In particular, there is a path* (*not necessary unique*) *from the empty partition $\emptyset$ to λ in the lattice spanned by double restricted $(\ell+1)$-strict partitions*;
(3) *for any cogood* (*addable*) *node A for λ, $\lambda \cup \{A\}$ is again a double restricted $(\ell+1)$-strict partition.*

Proof. We write $\lambda = (\lambda_1, \ldots, \lambda_s)$, where $\ell(\lambda) = s$. Let $B = (s, \lambda_s)$. Then, as λ is double restricted, either $\lambda_s = 1$ or $\lambda_s > 1$ and $\operatorname{res}(B) \neq \bar{0}$. In both cases, one sees easily that B must be a normal $\operatorname{res}(B)$-node (as there are no addable $\operatorname{res}(B)$-nodes below B). It follows that there must exist good (removable) $\operatorname{res}(B)$-node for λ. In a similar way, one can show that $B' = (1, \lambda_1 + 1)$ is a conormal $\operatorname{res}(B')$-node, which implies that there must exist cogood (addable) $\operatorname{res}(B')$-node for λ. This proves (1).

Now let $A = (a, \lambda_a)$ be a good (removable) node for $[\lambda]$. Then A is necessarily of type (R1). If $a = 1$, then it is easy to check that $\lambda - \{A\}$ is again double restricted $(\ell+1)$-strict. Suppose that $a > 1$. We write $\operatorname{res}(A) = i$. We claim that $\lambda_{a-1} - \lambda_a < 2(\ell+1)$. In fact, If $\lambda_{a-1} - \lambda_a = 2(\ell+1)$, then either $\lambda_a \not\equiv 0 \pmod{\ell+1}$, or $\lambda_a \equiv 0 \pmod{\ell+1}$. In the former case, one sees easily that $(a-1, \lambda_{a-1})$ is a removable i-node of type (R1) next to (the right of) A and there is no addable i-node sitting between them. Now as A survives after deleting all the string "AR," the node $(a-1, \lambda_{a-1})$ must also survive after deleting all the string "AR." In other words, it is in fact a normal i-node of λ higher than A, which is impossible (since A is the unique good i-node of λ); while in the latter case, it would follows that $\lambda_{a-1} \equiv 0 \pmod{\ell+1}$, and hence $\lambda_{a-1} - \lambda_a < 2(\ell+1)$ because λ is double restricted $(\ell+1)$-strict, which is again a contradiction. This proves our claim. Now there are only five possibilities:

Case 1. $i \notin \{\bar{0}, \bar{\ell}\}$.

Then either $\lambda_{a-1} \not\equiv 0 \pmod{\ell+1}$ or $\lambda_{a-1} \equiv 0 \pmod{\ell+1}$ and $\lambda_{a-1} - \lambda_a < 2\ell + 1$. In both cases, one checks easily that $\lambda - \{A\}$ is again a double restricted $(\ell+1)$-strict.

Case 2. $i = \bar{\ell}$ and $\lambda_a \equiv 0 \pmod{\ell+1}$.

Since $\lambda_{a-1} - \lambda_a < 2(\ell+1)$, it follows that $\lambda_{a-1} - (\lambda_a - 1) \leqslant 2(\ell+1)$. Now $\lambda_a \equiv 0 \pmod{\ell+1}$ implies that either $\lambda_{a-1} \not\equiv 0 \pmod{\ell+1}$ or $\lambda_{a-1} = \lambda_a + \ell + 1$. In both cases one sees easily that λ is double restricted $(\ell+1)$-strict must imply that $\lambda - \{A\}$ is double restricted $(\ell+1)$-strict too.

Case 3. $i = \bar{\ell}$ and $\lambda_a \equiv 1 \pmod{\ell+1}$.

We know that $\lambda_{a-1}-\lambda_a<2(\ell+1)$. We claim that $\lambda_{a-1}-\lambda_a<2\ell+1$. In fact, if $\lambda_{a-1}-\lambda_a=2\ell+1$, then $(a-1,\lambda_{a-1})$ must be another normal $\bar{\ell}$-node higher than A, which is impossible. This proves our claim. Therefore, $\lambda_{a-1}-(\lambda_a-1)\leqslant 2\ell+1$, which implies that $\lambda-\{A\}$ is still double restricted $(\ell+1)$-strict.

Case 4. $i=\bar{0}$ and $\lambda_a\equiv 0 \pmod{2(\ell+1)}$.

In this case one proves that $\lambda-\{A\}$ is double restricted $(\ell+1)$-strict by using the same argument as in the proof of Case 2.

Case 5. $i=\bar{0}$ and $\lambda_a\equiv 1 \pmod{2(\ell+1)}$.

In this case one proves that $\lambda-\{A\}$ is double restricted $(\ell+1)$-strict by using the same argument as in the proof of Case 3.

This completes the proof of (2). The proof of (3) is similar and is left to the readers. □

Therefore, the lattice spanned by all double restricted $(\ell+1)$-strict partitions equipped with the functions ε_i,φ_i and the operators $\tilde{e}_i,\tilde{f}_i$, can be turned into a colored oriented graph which we denote by $\widetilde{\mathfrak{RP}}_{\ell+1}$.

Lemma 3.23. *The graph $\widetilde{\mathfrak{RP}}_{\ell+1}$ can be identified with the crystal graph $\check{\mathbb{B}}(\check{\Lambda}_0)$ associated to the integrable highest weight $\check{\mathfrak{g}}$-module of highest weight $\check{\Lambda}_0$.*

Proof. This follows from Lemma 3.22 and Kang's combinatorial construction of the proper Young wall (see [12,19]). Note that our definition of removable and addable node are in accordance with the definition given in [19, pp. 275, 278]. To translate the language of proper Young walls into the language of double restricted strict partitions, one has to think the columns of the Young walls in [19] as the rows of our double restricted strict partitions. □

Applying Theorem 3.7, we get that

Theorem 3.24. *With the above notations, there is a bijection η from the set $DDPR_{\ell+1}$ of double restricted $(\ell+1)$-strict partitions onto the set $\{\lambda\in\mathcal{K}\mid \mathrm{M}(\lambda)=\lambda\}$, such that if*

$$\emptyset \overset{r_1}{\twoheadrightarrow}\cdot\overset{r_2}{\twoheadrightarrow}\cdot\ldots\cdot\overset{r_s}{\twoheadrightarrow}\check{\lambda}$$

is a path from $\emptyset$ to $\check{\lambda}$ in the graph $\widetilde{\mathfrak{RP}}_{\ell+1}$, then the sequence

$$\emptyset\underbrace{\overset{r_1}{\twoheadrightarrow}\cdot\overset{2\ell-r_1}{\twoheadrightarrow}\cdot}_{N_{r_1}\ terms}\underbrace{\overset{r_2}{\twoheadrightarrow}\cdot\overset{2\ell-r_2}{\twoheadrightarrow}\cdot}_{N_{r_2}\ terms}\ldots\cdot\underbrace{\overset{r_s}{\twoheadrightarrow}\cdot\overset{2\ell-r_s}{\twoheadrightarrow}\lambda}_{N_{r_s}\ terms}:=\eta(\check{\lambda}),$$

where

$$\underbrace{\overset{r_t}{\twoheadrightarrow}\cdot\overset{2\ell-r_t}{\twoheadrightarrow}\cdot}_{N_{r_t}\ terms}:=\begin{cases}\overset{r_t}{\twoheadrightarrow}\cdot\overset{2\ell-r_t}{\twoheadrightarrow}\cdot, & \text{if } r_t\in\{1,2,\ldots,\ell-1\},\\ \overset{r_t}{\twoheadrightarrow}\cdot, & \text{if } r_t\in\{0,\ell\},\end{cases}$$

defines a path in Kleshchev's (2ℓ)*-good lattice which connects* $\emptyset$ *and* (2ℓ)*-regular partition* λ *satisfying* $\mathrm{M}(\lambda) = \lambda$.

Remark 3.25. In [25], Leclerc–Thibon conjectured that the decomposition matrices of Hecke–Clifford superalgebras with parameter q should related to the Fock space representation of the twisted affine Lie algebra of type $A_{2\ell}^{(2)}$ if q is a primitive $(2\ell + 1)$th root of unity; or of type $D_{\ell+1}^{(2)}$ if q is a primitive 2ℓth root of unity. In [4,5], Brundan and Kleshchev show that the modular irreducible super-representations of Hecke–Clifford superalgebras at defining parameter q a primitive $(2\ell + 1)$th root of unity as well as of affine Sergeev superalgebras over a field of characteristic $2\ell + 1$ are parameterized by the set of restricted $(2\ell + 1)$-strict partitions, which partly verified the idea of [25]. It would be interesting to know if our notion of double restricted $(\ell + 1)$-strict partitions give a natural parameterization of the modular irreducible super-representations of Hecke–Clifford superalgebras when q is a primitive (2ℓ)th root of unity.

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References

[1] S. Ariki, A. Mathas, The number of simple modules of the Hecke algebras of type $G(r, 1, n)$, Math. Z. 233 (3) (2000) 601–623.
[2] R.E. Borcherds, Generalized Kac–Moody algebras, J. Algebra 115 (2) (1988) 501–512.
[3] J. Brundan, Modular branching rules and the Mullineux map for Hecke algebras of type A, Proc. London. Math. Soc. 77 (3) (1998) 551–581.
[4] J. Brundan, A. Kleshchev, Hecke–Clifford superalgebras, crystals of type $A_{2l}^{(2)}$ and modular branching rules for $\hat{S}_n$, Represent. Theory 5 (2001) 317–403.
[5] J. Brundan, A. Kleshchev, Projective representations of symmetric groups via Sergeev duality, Math. Z. 239 (2002) 27–68.
[6] R. Dipper, G.D. James, Representations of Hecke algebras of general linear groups, Proc. London Math. Soc. 52 (3) (1986) 20–52.
[7] R. Dipper, G.D. James, Blocks and idempotents of Hecke algebras of general linear groups, Proc. London Math. Soc. 54 (3) (1987) 57–82.
[8] B. Ford, Irreducible representations of the alternating group in odd characteristic, Proc. Amer. Math. Soc. 125 (2) (1997) 375–380.
[9] B. Ford, A. Kleshchev, A proof of Mullineux conjecture, Math. Z. 226 (1997) 267–308.
[10] J. Fuchs, U. Ray, C. Schweigert, Some automorphisms of generalized Kac–Moody algebras, J. Algebra 191 (1997) 518–540.
[11] J. Fuchs, B. Schellekens, C. Schweigert, From Dynkin diagram symmetries to fixed point structures, Comm. Math. Phys. 180 (1996) 39–97.
[12] J. Hong, S.-J. Kang, Introduction to Quantum Groups and Crystal Bases, Grad. Stud. Math., vol. 42, Amer. Math. Soc., Providence, RI, 2002.
[13] J. Hu, A Morita equivalence theorem for Hecke algebras of type D_n when n is even, Manuscripta Math. 108 (2002) 409–430.
[14] J. Hu, Crystal basis and simple modules for Hecke algebra of type D_n, J. Algebra 267 (1) (2003) 7–20.
[15] J. Hu, Modular representations of Hecke algebras of type $G(p, p, n)$, J. Algebra 274 (2) (2004) 446–490.

[16] J. Hu, Crystal bases and simple modules for Hecke algebra of type $G(p, p, n)$, preprint, arXiv: math.RT/0506555, 2005.
[17] J. Hu, The number of simple modules for the Hecke algebra of type $G(r, p, n)$, preprint, arXiv: math.RT/0601572, 2006.
[18] V.G. Kac, Infinite Dimensional Lie Algebras, Cambridge Univ. Press, 1985.
[19] S.-J. Kang, Crystal bases for quantum affine algebras and combinatorics of Young walls, Proc. London Math. Soc. 86 (2003) 29–69.
[20] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke J. Math. 63 (1991) 465–516.
[21] A. Kleshchev, Branching rules for the modular representations of symmetric groups III; some corollaries and a problem of Mullineux, J. London Math. Soc. 54 (1995) 25–38.
[22] P. Littelmann, A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras, Invent. Math. 116 (1994) 329–346.
[23] P. Littelmann, Paths and root operators in representation theory, Ann. of Math. 142 (2) (1995) 499–525.
[24] A. Lascoux, B. Leclerc, J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases quantum affine algebras, Comm. Math. Phys. 181 (1996) 205–263.
[25] B. Leclerc, J.-Y. Thibon, q-Deformed Fock spaces and modular representations of spin symmetric groups, J. Phys. A 30 (1997) 6163–6176.
[26] T. Misra, K.C. Miwa, Crystal bases for the basic representations of $U_q(\widehat{\mathfrak{sl}}_n)$, Comm. Math. Phys. 134 (1990) 79–88.
[27] G. Mullineux, Bijections of p-regular partitions and p-modular irreducibles of the symmetric groups, J. London Math. Soc. 20 (1979) 60–66.
[28] E. Murphy, The representations of Hecke algebras of type A_n, J. Algebra 173 (1995) 97–121.
[29] S. Naito, D. Sagaki, Lakshmibai–Seshadri paths fixed by a diagram automorphism, J. Algebra 245 (2001) 395–412.
[30] S. Naito, D. Sagaki, Standard paths and standard monomials fixed by a diagram automorphism, J. Algebra 251 (2002) 461–474.