Operation approaches on $\alpha$-$\gamma$-$I$-open sets and $\alpha$-$\gamma$-$I$-continuous functions in topological spaces

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Abstract In this paper, the notion of $\alpha$-$\gamma$-$I$-open sets in a topological space together with its corresponding interior and closure operators are introduced. Further, the concept of $\alpha$-$\gamma$-$I$-continuous functions and $\alpha$-$\gamma$-$I$-open functions are introduced and some of their basic properties are studied.

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1. Introduction

Njastad [1] introduced $\alpha$-open sets in a topological space and studied some of their properties. The concept of semi-open sets, preopen sets and semi-preopen sets were introduced respectively by Levine [2], Corson and Michael [3] and Andrijevic [4]. The term “preopen” was introduced by Mashhour et al [5] and they studied some of their basic properties. Andrijevic [6] introduced a new class of topology generated by preopen sets and the corresponding closure and interior operators.

Kasahara [7] defined the concept of an operation on topological spaces and introduced $\alpha$-closed graphs of an operation. Ogata [8] called the operation $\alpha$ as $\gamma$ operation and introduced the notion of $\tau_\gamma$ which is the collection of all $\gamma$-open sets in a topological space $(X, \tau)$.

Kalaivani and Sai Sundara Krishnan [9] introduced $\alpha$-$\gamma$-$I$-open sets in a topological space $(X, \tau)$ and introduced the notion of $\tau_{\alpha,\gamma}$ which is the collection of all $\alpha$-$\gamma$-open sets in a topological space.

In this paper, in Section 3, we introduce the notion of $\tau_{\alpha,\gamma}$ which is the collection of all $\alpha$-$\gamma$-$I$-open sets in a topological space $(X, \tau)$. Further, we introduce the concept of $\tau_{\alpha,\gamma}$ interior and $\tau_{\alpha,\gamma}$-close operators and study some of their properties.

In Section 4, we introduce the concept of $\alpha$-$\gamma$-$I$-continuity and characterize it using the notion of $\alpha$-$\gamma$-$I$-closed or $\alpha$-$\gamma$-$I$-open sets. We investigate some of its properties and study the relationship between them.

In Section 5, we introduce the notion of $\alpha$-$\gamma$-$I$-open function and study some of its properties.
2. Preliminaries

Let \((X, \tau)\) be a topological space and \(I\) be an ideal of subsets of \(X\). An ideal is defined as a nonempty collection \(I\) of subsets of \(X\) satisfying the following two conditions: (i) If \(A \in I\) and \(B \subseteq A\) then \(B \in I\); (ii) If \(A, B \in I\), then \(A \cap B \in I\). An ideal topological space is a topological space \((X, \tau)\) with an ideal \(I\) on \(X\) and is denoted by \((X, \tau, I)\). For a subset \(A \subseteq X\), \(\tau(A) = \{ x \in X : U \cap A \neq \emptyset \text{ for each neighborhood } U \text{ of } x \}\) is called the local function of \(A\) with respect to \(I\) and \(\tau\) [10]. We simply write \(\tau^*(A)\) instead of \(\tau(A, I)\) in case there is no chance for confusion. \(X^*\) is often a proper subset of \(X\). The hypothesis \(X = X^*\) [11] is equivalent to the hypothesis \(\tau \cap \tau^* = \emptyset\) [12]. For every ideal topological space \((X, \tau, I)\), there exists a topology \(\tau^*(I)\), finer than \(\tau\), generated by \(\beta(I, \tau) = \{ U \cap \tau \text{ and } I \subseteq \tau \text{ in } \tau \}\), but in general, \(\beta(I, \tau)\) is not always a topology[13]. Additionally, \(\mathcal{C}(A) = A \cup A^*\) defines a Kuratowski closure operator for \(\tau^*(I)\).

Throughout this paper, \((X, \tau)\) and \((Y, \sigma)\) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset \(A\) of \((X, \tau)\), \(\mathcal{C}(A), \mathcal{I}(A)\) and \(\mathcal{A}\) denote the closure of \(A\), the interior of \(A\) and the complement of \(A\), respectively. We define Kuratowski closure operator as \(\tau^* - \mathcal{C}(A) = A \cup A^*\).

In this section, we recall some of the basic definitions and theorems.

**Definition 2.1.** Let \((X, \tau)\) be a topological space and \(A\) be a subset of \(X\). Then, \(A\) is said to be:

(i) [1] \(\tau\)-open set if \(A \subseteq \mathcal{I}(A)\)
(ii) [2] semi-open set if \(A \subseteq \mathcal{C}(A)\)
(iii) [4] preopen set if \(A \subseteq \mathcal{I}(A)\)
(iv) [5] semi-preopen set if \(A \subseteq \mathcal{C}(A)\)

**Definition 2.2.** Let \((X, \tau)\) be a topological space, an operation \(\gamma\) on the topology \(\tau\) is a mapping from \(\tau\) on to the power set \(\mathcal{P}(X)\) of \(X\) such that \(V \subseteq V'\) for each \(V \in \tau\), where \(V'\) denotes the value of \(\gamma\) at \(V\).

**Definition 2.3.** Let \((X, \tau)\) be a topological space and \(A\) be a subset of \(X\) and \(\gamma\) be an operation on \(\tau\). Then, \(A\) is said to be:

(i) [8] a \(\gamma\)-open set if for each \(x \in A\) there exists an open set \(U\) such that \(x \in U\) and \(U \subseteq A\). \(\tau_\gamma\) denotes the set of all \(\gamma\)-open sets in \((X, \tau)\).
(ii) [14] \(\gamma\)-semi-open if and only if \(A \subseteq \tau_\gamma - \mathcal{C}(A)\).
(iii) [15] \(\gamma\)-preopen if and only if \(A \subseteq \tau_\gamma - \mathcal{I}(A)\).
(iv) [15] \(\gamma\)-semi-preopen if and only if \(A \subseteq \tau_\gamma - \mathcal{C}(A)\).

**Definition 2.4.**

(i) [14] Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). Then, \(\tau_\gamma\)-interior of \(A\) is defined as the union of all \(\gamma\)-open sets contained in \(A\) and is denoted \(\tau_\gamma - \mathcal{I}(A)\). That is \(\tau_\gamma - \mathcal{I}(A) = \bigcup \{ U : U \subseteq A \}\)
(ii) [9] Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). Then, \(\tau_\gamma\)-closure of \(A\) is defined as the intersection of all \(\gamma\)-closed sets containing \(A\) and it is denoted by \(\tau_\gamma - \mathcal{C}(A)\). That is \(\tau_\gamma - \mathcal{C}(A) = \cap \{ F : F \subseteq A \}\)

**Definition 2.5** [9]. Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). Then, a subset \(A\) of \(X\) is said to be a \(\tau_\gamma\)-open set if and only if \(A \subseteq \tau_\gamma - \mathcal{I}(\tau_\gamma - \mathcal{C}(A))\)

**Definition 2.6.**

(i) [9] Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\) and \(A\) be a subset of \(X\). Then, \(\tau_\gamma\)-interior of \(A\) is the union of all \(\tau_\gamma\)-open sets contained in \(A\) and it is denoted by \(\tau_\gamma\)-interior of \(A\). That is \(\tau_\gamma\)-interior of \(A\) = \(\bigcup \{ U : U \subseteq A \}\)
(ii) [9] Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). Let \(A\) be a subset of \(X\). Then, \(\tau_\gamma\)-closure of \(A\) is the intersection of all \(\tau_\gamma\)-closed sets containing \(A\) and it is denoted by \(\tau_\gamma\)-closure of \(A\). That is \(\tau_\gamma\)-closure of \(A\) = \(\bigcap \{ F : F \subseteq A \}\)

3. \(\tau_\gamma\)-I-open set

**Definition 3.1.** Let \((X, \tau, I)\) be an ideal topological space and \(\gamma\) be an operation on \(\tau\). Then, a subset \(A\) of \(X\) is said to be a \(\tau_\gamma\)-I-open set if and only if \(A \subseteq \tau_\gamma - \mathcal{I}(\tau_\gamma - \mathcal{C}(\tau_\gamma - \mathcal{I}(A)))\)

**Example 3.2.** Let \(X = \{ a, b, c, d \}\), \(\tau = \{ \emptyset, \{ a, b, c \}, \{ a, c \}, \{ a, b, d \}, \{ c \}, \{ a, b \}, \{ a, d \}, \{ a, c, d \}, \{ a, b, d \}, \{ a, b, c, d \} \}\) and \(I = \{ \{ d \} \}\). We define an operation \(\gamma : \tau \to \mathcal{P}(X)\) as follows: for every \(A \in \tau\),

\[
\mathcal{A} = \begin{cases} 
\mathcal{I}(A) & \text{if } A \neq \emptyset \\
\emptyset & \text{if } A = \emptyset
\end{cases}
\]

Then, \(\tau_\gamma = \{ \emptyset, \{ a, b, c \}, \{ a, c \}, \{ a, b, d \}, \{ c \}, \{ a, b \}, \{ a, d \}, \{ a, c, d \}, \{ a, b, d \}, \{ a, b, c, d \} \}\) and \(\tau^*_{\gamma-I} = \{ \emptyset, \{ a, b, c \}, \{ a, c \}, \{ a, d \}, \{ a, c, d \}, \{ a, b, d \} \}\)

**Definition 3.3.** A subset of an ideal topological space \((X, \tau, I)\) is said to be:

(i) \(\gamma\)-semi-I-open if and only if \(A \subseteq \tau_\gamma - \mathcal{C}(\tau_\gamma - \mathcal{I}(A))\).
(ii) \(\gamma\)-pre-I-open if and only if \(A \subseteq \tau_\gamma - \mathcal{I}(\tau_\gamma - \mathcal{C}(\tau_\gamma - \mathcal{I}(A)))\).
(iii) \(\gamma\)-semi-pre-I-open or \(\gamma\)-\(\beta\)-I-open if and only if \(A \subseteq \tau_\gamma - \mathcal{C}(\tau_\gamma - \mathcal{I}(\tau_\gamma - \mathcal{C}(\tau_\gamma - \mathcal{I}(A))))\).

**Theorem 3.4.** Let \((X, \tau, I)\) be an ideal topological space and \(\gamma\) be an operation on \(\tau\).

(i) Every \(\tau_\gamma\)-I-open set is \(\tau_\gamma\)-open.
(ii) Every \(\gamma\)-semi-I-open set is \(\gamma\)-semi-open.
(iii) Every \(\gamma\)-\(\beta\)-I-open set is \(\gamma\)-\(\beta\)-open.

**Proof.**

(i) Let \(A\) be an \(\tau_\gamma\)-I-open set in \((X, \tau, I)\). Then, it follows that \(A \subseteq \tau_\gamma - \mathcal{I}(\tau_\gamma - \mathcal{C}(\tau_\gamma - \mathcal{I}(A))) = \tau_\gamma - \mathcal{I}(\tau_\gamma - \mathcal{C}(\tau_\gamma - \mathcal{I}(A)))\).
Theorem 3.5. Let \((X, \tau, I)\) be an ideal topological space and \(\gamma\) be an operation on \(\tau\).

(i) Every \(x, \gamma, I\)-open set is \(\gamma\)-semi-I-open.
(ii) Every \(x, \gamma, I\)-open set is \(\gamma\)-pre-I-open.
(iii) Every \(x, \gamma, I\)-open set is \(\gamma\)-\(\beta\)-I-open.
(iv) Every \(\gamma\)-semi-I-open set is \(\gamma\)-\(\beta\)-I-open.
(v) Every \(\gamma\)-semi-I-open set is \(\gamma\)-\(\beta\)-I-open.

Proof.

(i) Let \(A\) be an \(x, \gamma, I\)-open set in \((X, \tau, I)\). Then, \(A \subseteq (\tau, \gamma, I) - \text{int}(A) \subseteq \tau - \text{cl}(\tau, \gamma, I - \text{int}(A))\). Therefore, \(A\) is \(\gamma\)-open.

(ii) Let \(A\) be a \(\gamma\)-semi-I-open set in \((X, \tau, I)\). Then, \(A \subseteq (\tau, \gamma, I) - \text{int}(A) \subseteq \tau - \text{cl}(\tau, \gamma, I - \text{int}(A))\). Hence, \(A\) is \(\gamma\)-semi-I-open.

(iii) Let \(A\) be a \(\gamma\)-\(\beta\)-I-open set in \((X, \tau, I)\). Then, \(A \subseteq (\tau, \gamma, I) - \text{int}(A) \subseteq \tau - \text{cl}(\tau, \gamma, I - \text{int}(A))\). Hence, \(A\) is \(\gamma\)-\(\beta\)-I-open.

(iv) Let \(B\) be a \(\gamma\)-\(\beta\)-I-open set in \((X, \tau, I)\). Then, \(A \subseteq (\tau, \gamma, I) - \text{int}(A) \subseteq \tau - \text{cl}(\tau, \gamma, I - \text{int}(A))\). Hence, \(A\) is \(\gamma\)-\(\beta\)-I-open.

Theorem 3.6. Every \(\gamma\)-open set of an ideal topological space is \(x, \gamma, I\)-open.

Proof. Let \(A\) be any \(\gamma\)-open set. Then, \(A = (\tau, \gamma, I) - \text{int}(A) \subseteq \tau - \text{cl}(\tau, \gamma, I - \text{int}(A))\). This shows that \(A\) is \(x, \gamma, I\)-open.

Theorem 3.7 [16]. Let \((X, \tau, I)\) be an ideal topological space and \(\gamma\) be an operation on \(\tau\). A subset \(A\) of \(X\) is \(x, \gamma, I\)-open if and only if it is \(\gamma\)-semi-I-open and \(\gamma\)-pre-I-open.

Proof. Necessity. Let \(A\) be an \(x, \gamma, I\)-open set. Then, \(A \subseteq (\tau, \gamma, I) - \text{int}(A) \subseteq (\tau, \gamma, I) - \text{int}(\tau, \gamma, I - \text{int}(A)) = \tau - \text{cl}(\tau, \gamma, I - \text{int}(A))\). This shows that \(A\) is \(x, \gamma, I\)-open.

Sufficiency. Let \(A\) be a \(\gamma\)-semi-I-open and \(\gamma\)-pre-I-open. Then, \(A \subseteq (\tau, \gamma, I) - \text{int}(A) \subseteq (\tau, \gamma, I) - \text{int}(\tau, \gamma, I - \text{int}(A))\) and hence \(A\) is \(\gamma\)-semi-I-open and \(\gamma\)-pre-I-open.

Lemma 3.8. Let \((X, \tau, I)\) be an ideal topological space and \(\gamma\) be a subset of \(X\). Then, the following properties hold:

(i) If \(O\) is \(\gamma\)-open in \((X, \tau, I)\), then \(O \cap (\tau, \gamma, I) - \text{int}(A) \subseteq (\tau, \gamma, I) - \text{int}(\tau, \gamma, I - \text{int}(A))\).
(ii) If \(A \subseteq \tau X \subseteq \tau X\), then \(A \subseteq (\tau, \gamma, I) - \text{int}(A) \subseteq (\tau, \gamma, I) - \text{int}(\tau, \gamma, I - \text{int}(A))\).

Proof.

(i) If \(O \subseteq \tau \gamma\), then \(O \subseteq (\tau, \gamma, I) - \text{int}(A) \subseteq (\tau, \gamma, I) - \text{int}(\tau, \gamma, I - \text{int}(A))\).

Theorem 3.9. Let \((X, \tau, I)\) be an ideal topological space and \(\gamma\) be an operation on \(\tau\).

(i) If \(V \in \gamma - \text{SIO}(X)\) and \(A \in \tau \gamma\), then \(V \cap A \in \gamma - \text{SIO}(X)\).
(ii) If \(V \in \gamma - \text{PIO}(X)\) and \(A \in \tau \gamma\), then \(V \cap A \in \gamma - \text{PIO}(X)\).

Proof.

(i) Let \(V \in \gamma - \text{SIO}(X)\) and \(A \in \tau \gamma\). By using Lemma 3.8, \(V \cap A \subseteq (\tau, \gamma, I) - \text{int}(V) \cap (\tau, \gamma, I) - \text{int}(A)\). This implies that \(V \cap A \in \gamma - \text{SIO}(X)\).

(ii) Let \(V \in \gamma - \text{PIO}(X)\) and \(A \in \tau \gamma\). By using Theorem 3.9, \(V \cap A \subseteq (\tau, \gamma, I) - \text{int}(V) \cap (\tau, \gamma, I) - \text{int}(A)\). Therefore, \(V \cap A \in \gamma - \text{PIO}(X)\).
Definition 3.11. A subset $A$ of $X$ is said to be $\gamma$-$I$-closed if and only if $X - A$ is $\gamma$-$I$-open, which is equivalently

Let $(X, \tau, I)$ be an ideal topological space and $\gamma$ be an operation on $\tau$ and $A$ be a subset of $X$. Then, $A$ is $\gamma$-$I$-closed and it is denoted by $\tau_{\gamma-I}(A)$.

Definition 3.12. Let $(X, \tau, I)$ be an ideal topological space and $\gamma$ be an operation on $\tau$ and $A$ be a subset of $X$. Then, $\tau_{\gamma-I}$ interior of $A$ is the union of all $\gamma$-$I$-closed sets contained in $A$ and it is denoted by $\tau_{\gamma-I} = \text{int} \left( \tau_{\gamma-I}(A) \right)$.

Definition 3.13. Let $(X, \tau, I)$ be an ideal topological space and $\gamma$ be an operation on $\tau$ and $A$ be a subset of $X$. Then, $\tau_{\gamma-I}$ closure of $A$ is the intersection of all $\gamma$-$I$-closed sets containing $A$ and it is denoted by $\tau_{\gamma-I} = \text{cl} \left( \tau_{\gamma-I}(A) \right)$.

Proof. Since $\phi$, $X \in \tau_{\gamma-I}$, this follows from the Theorem 3.10. \hfill \Box

Theorem 3.15. Let $(X, \tau, I)$ be an ideal topological space. Then,

(i) If $I = \{ \phi \}$, then $\tau_{\gamma-I} = \tau_{\gamma}$

(ii) If $I = P(X)$, then $\tau_{\gamma-I} = \tau$

(iii) If $I = N$, then $\tau_{\gamma-I} = \tau_{\gamma}$

Proof.

(i) Let $I = \{ \phi \}$. Then, $\tau_{\gamma-I}(A) = A$ and hence $\tau_{\gamma-I}(A) = A \cup \tau_{\gamma-I}(A)$ for every subset $A$ of $X$. Therefore, $\tau_{\gamma-I}(A) = \tau_{\gamma-I}(A)$ and hence $\tau_{\gamma-I} = \tau_{\gamma-I}$.

(ii) Let $I = P(X)$. Then $\tau_{\gamma-I}(A) = A$ for every subset $A$ of $X$. Therefore, $\tau_{\gamma-I}(A) = \tau_{\gamma-I}(A)$ and hence $\tau_{\gamma-I} = \tau_{\gamma-I}$.

(iii) Let $I = N$. Then $\tau_{\gamma-I}(A) = \tau_{\gamma-I}(A)$ for every subset $A$ of $X$. Therefore, $\tau_{\gamma-I}(A) = \tau_{\gamma-I}(A)$ and hence $\tau_{\gamma-I} = \tau_{\gamma-I}$.

Theorem 3.16. Let $(X, \tau, I)$ be an ideal topological space and $A$ a subset of $X$. Then, the following hold:

(a) If $I = \{ \phi \}$, then

(i) $\gamma$-$I$-open, $\gamma$-pre-$I$-open and $\gamma$-preopen are all equivalent.

(ii) $A$ is $\gamma$-semi-$I$-open if and only if $A$ is $\gamma$-semi-open.

(iii) $A$ is $\gamma$-$B$-$I$-open if and only if $A$ is $\gamma$-open.

(b) If $I = P(X)$, then $A$ is $\gamma$-$B$-$I$-open if and only if $A$ is $\gamma$-semi-open.

(c) If $I = N$, then $A$ is $\gamma$-$B$-$I$-open if and only if $A$ is $\gamma$-semi-open, where $N$ is the ideal of nowhere dense sets.

Theorem 3.17. Let $(X, \tau, I)$ be an ideal topological space. If $A \in \tau_{\gamma-I}$ and $A \subseteq X_0 \in \tau_{\gamma-I}$, then $A \in \tau_{\gamma-I}(X_0)$.

Proof. By using Lemma 3.8(ii), we obtain $A \subseteq \tau_{\gamma-I}(A \cap X_0), \tau_{\gamma-I}(A \cap X_0) \subseteq \tau_{\gamma-I}(A \cap X_0)$.

Theorem 3.18. Let $(X, \tau, I)$ be an ideal topological space. If $X_0 \in \tau_{\gamma-I}$ and $A \in \tau_{\gamma-I}(X_0)$, then $A \in \tau_{\gamma-I}$.

Proof. Since $A \subseteq \tau_{\gamma-I}(X_0), A \subseteq \tau_{\gamma-I}(X_0)$ for some $U \in \tau$. Since $X_0 \in \tau_{\gamma-I}$, by Lemma 3.8, we obtain $A \subseteq X_0 \cap U \subseteq A \cap \tau_{\gamma-I}(X_0) \subseteq \tau_{\gamma-I}(X_0)$.

4. $\gamma$-$I$-continuity

Definition 4.1. A function $f(X, \tau, I) \rightarrow (Y, \sigma)$ is said to be:

(i) $\gamma$-$I$-continuous if for every $V \in \sigma, f^{-1}(V)$ is an $\gamma$-$I$-open set in $(X, \tau, I)$.

(ii) $\gamma$-semi-$I$-continuous if for every $V \in \sigma, f^{-1}(V)$ is an $\gamma$-semi-$I$-open set in $(X, \tau, I)$.

(iii) $\gamma$-pre-$I$-continuous if for every $V \in \sigma, f^{-1}(V)$ is an $\gamma$-pre-$I$-open set in $(X, \tau, I)$.

Example 4.2. Let $X = \{ a, b, c \}$, $\tau = \{ \phi, X, \{ a \}, \{ c \}, \{ a, b \}, \{ a, c \} \}$ and $I = \{ \phi, \{ b \} \}$. We define an operation $\gamma: \tau \rightarrow P(X)$ as follows: for every $A \in \tau$,
\[ A'^* = \begin{cases} A \cup \{c\} & \text{if } A \neq \{a\} \\ A & \text{if } A = \{a\} \end{cases} \]

Then, \( \tau_{\gamma-} = \{\phi, X, \{a\}, \{c\}, \{a, c\}\} \) and \( \tau_{\gamma-L} = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \).

Let \( Y = \{a, b, c\} \), \( \sigma = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\} \). We define \( f(x, \tau, I) \to (Y, \sigma) \) as \( f(a) = c, f(b) = b, f(c) = a \). Then, for every \( V \in \sigma \), \( f^{-1}(V) \) is \( \gamma-L \)-open set in \((X, \tau, I)\). Hence, \( f \) is \( \gamma-L \)-continuous.

**Theorem 4.3.** If a function \( f:(X, \tau, I) \to (Y, \sigma) \) is \( \gamma-L \)-continuous (resp. \( \gamma-semi-L \)-continuous, \( \gamma-pre-L \)-continuous), then \( f \) is \( \gamma-L \)-continuous (resp. \( \gamma-semi-L \)-continuous, \( \gamma-pre-L \)-continuous).

**Proof.** Proof follows from the Theorem 3.4. \( \Box \)

**Theorem 4.4.** Let \( f:(X, \tau, I) \to (Y, \sigma) \) be a function, then the following statements are equivalent:

(i) \( f \) is \( \gamma-L \)-continuous.
(ii) For each \( x \in X \) and each open set \( V \subseteq Y \) containing \( f(x) \), there exists \( W \in \tau_{\gamma-L} \) such that \( x \in W \) and \( f(W) \subseteq V \).
(iii) The inverse image of each closed set in \( Y \) is \( \gamma-L \)-closed.
(iv) \( f^{-1}(\tau_{\gamma-L}(f(A)))) \subseteq \tau_{\gamma-L}(f(A))) \) for each \( A \subseteq X \).

**Proof.**

(i) \( \Rightarrow \) (ii): Let \( x \in X \) and \( V \) be any open set of \( Y \) containing \( f(x) \). Set \( W = f^{-1}(V) \), then by Definition 4.1, \( W \) is an \( \gamma-L \)-open set containing \( f(W) \subseteq V \).

(ii) \( \Rightarrow \) (iii): Let \( F \) be a closed set of \( Y \). Set \( V = Y - F \), then \( f(V) \) is open in \( Y \). Let \( x \in f^{-1}(V) \), by (ii), there exists an \( \gamma-L \)-open set \( W \) of \( X \) containing \( x \) such that \( f(W) \subseteq V \). Thus, we obtain \( x \in W \subseteq \tau_{\gamma-L}(f^{-1}(f(W))) \subseteq \tau_{\gamma-L}(f^{-1}(f(W))) \) and hence \( f(V) \subseteq \tau_{\gamma-L}(f^{-1}(f(W))) \). This shows that \( f^{-1}(V) \subseteq \tau_{\gamma-L}(V) \) in \( X \). Hence, \( f^{-1}(F) = X - f^{-1}(Y - F) = X - f^{-1}(V) \) is \( \gamma-L \)-closed in \( X \).

(iii) \( \Rightarrow \) (iv): Let \( B \) be any subset of \( Y \). Since \( \tau_{\gamma-L}(B) \) is closed in \( Y \), by (iii), \( f^{-1}(\tau_{\gamma-L}(B)) \) is \( \gamma-L \)-closed and \( X - f^{-1}(\tau_{\gamma-L}(B)) \) is \( \gamma-L \)-open. Thus \( X - f^{-1}(\tau_{\gamma-L}(B)) \subseteq \tau_{\gamma-L}(\tau_{\gamma-L}(f^{-1}(X - f^{-1}(\tau_{\gamma-L}(B)))) = X - \tau_{\gamma-L}(\tau_{\gamma-L}(f^{-1}(f(B)))) \subseteq \tau_{\gamma-L}(f^{-1}(\tau_{\gamma-L}(B))) \). Hence, \( \tau_{\gamma-L}(\tau_{\gamma-L}(f^{-1}(\tau_{\gamma-L}(B)))) \subseteq \tau_{\gamma-L}(f^{-1}(\tau_{\gamma-L}(B))) \).

(iv) \( \Rightarrow \) (v): Let \( A \) be any subset of \( X \). By (iv), we have \( \tau_{\gamma-L}(\tau_{\gamma-L}(f^{-1}(f(A)))) \subseteq \tau_{\gamma-L}(\tau_{\gamma-L}(f^{-1}(f(A)))) \subseteq f^{-1}(\tau_{\gamma-L}(f(A))) \) and hence \( \tau_{\gamma-L}(\tau_{\gamma-L}(f^{-1}(\tau_{\gamma-L}(A)))) \subseteq \tau_{\gamma-L}(f^{-1}(A)) \).

(v) \( \Rightarrow \) (i): Let \( V \) be any open set of \( Y \). Then, by (v), \( \tau_{\gamma-L}(\tau_{\gamma-L}(f^{-1}(f(V)))) \subseteq \tau_{\gamma-L}(f^{-1}(f(V))) \subseteq \tau_{\gamma-L}(f^{-1}(f(V))) \subseteq \tau_{\gamma-L}(f^{-1}(f(V))) \). Therefore, \( \tau_{\gamma-L}(\tau_{\gamma-L}(f^{-1}(f(V)))) \subseteq f^{-1}(\tau_{\gamma-L}(f^{-1}(f(V)))) \). We obtain that, \( f^{-1}(V) \subseteq \tau_{\gamma-L}(f^{-1}(f^{-1}(V))) \). This implies that \( f^{-1}(V) \) is \( \gamma-L \)-open set. Hence, \( f \) is \( \gamma-L \)-continuous. \( \Box \)

**Corollary 4.5.** Let \( f:(X, \tau, I) \to (Y, \sigma) \) be \( \gamma-L \)-continuous, then \( f^{-1}(V) \subseteq \tau_{\gamma-L}(f^{-1}(f(U))) \) for each \( U \in \gamma-L \)-open set in \( Y \).

**Proof.**

(i) Let \( U \subseteq \gamma-L \)-open set in \( Y \). Therefore, by Theorem 4.3 we have \( f^{-1}(\gamma-L \)-open set \( U \subseteq \tau_{\gamma-L}(\gamma-L \)-open set \( f(U) \subseteq \tau_{\gamma-L}(f(U)) \).

(ii) Let \( V \subseteq \gamma-L \)-open set in \( Y \). By Theorem 4.3, we have \( \tau_{\gamma-L}(f^{-1}(f(V))) \subseteq \tau_{\gamma-L}(f^{-1}(f(V))) \subseteq \tau_{\gamma-L}(f^{-1}(f^{-1}(f(V)))) \subseteq f^{-1}(\tau_{\gamma-L}(f^{-1}(f(V)))) \).

**Theorem 4.6** [16]. A function \( f:(X, \tau, I) \to (Y, \sigma) \) is \( \gamma-L \)-continuous if and only if it is \( \gamma-semi-L \)-continuous and \( \gamma-pre-L \)-continuous.

**Proof.** This is an immediate consequence of Theorem 3.7. \( \Box \)

**Theorem 4.7.** A function \( f:(X, \tau, I) \to (Y, \sigma) \) is \( \gamma-L \)-continuous if and only if \( f:(X, \tau_{\gamma-L}) \to (Y, \sigma) \) is continuous.

**Proof.** This is an immediate consequence of Corollary 3.14. \( \Box \)

**Theorem 4.8.** If a function \( f:(X, \tau, I) \to (Y, \sigma) \) is \( \gamma-L \)-continuous and \( X_{0} \in \tau_{\gamma-L} \), then the restriction \( f|_{X_{0}:(X_{0}, \tau_{\gamma-L})} \to (Y, \sigma) \) is \( \gamma-L \)-continuous.

**Proof.** Let \( V \) be any open set of \( Y \). Since \( f \) is \( \gamma-L \)-continuous, \( f^{-1}(V) \) is \( \gamma-L \)-open in \( (X, \tau, I) \) and by Theorem 3.10 \( f^{-1}(V) \cap X_{0} = (f|_{X_{0}})^{-1}(V) \subseteq \tau_{\gamma-L} \). By Theorem 3.17 \( (f|_{X_{0}})^{-1}(V) \subseteq \tau_{\gamma-L} \). This implies that \( f|_{X_{0}} \) is \( \gamma-L \)-continuous. \( \Box \)

**Theorem 4.9.** Let \( (X, \tau, I) \) be an ideal topological space and \( \{V_{\zeta}\}_{\zeta \in \sigma} \) be a cover of \( X \) by \( \gamma-L \)-open sets of \( (X, \tau, I) \). A function \( f:(X, \tau, I) \to (Y, \sigma) \) is \( \gamma-L \)-continuous if and only if the restriction \( f|_{V_{\zeta}}:(V_{\zeta}, \tau_{\gamma-L}) \to (Y, \sigma) \) is \( \gamma-L \)-continuous for each \( \zeta \in \sigma \).

**Proof.** Necessity. Let \( f \) be \( \gamma-L \)-continuous, then by Theorem 4.7 \( f|_{V_{\zeta}} \) is \( \gamma-L \)-continuous for each \( \zeta \in \sigma \).

Sufficiency. Let \( f|_{V_{\zeta}} \) be \( \gamma-L \)-continuous for each \( \zeta \in \sigma \). For any open set \( V \subseteq Y \), \( f|_{V_{\zeta}}^{-1}(V) \subseteq \tau_{\gamma-L} \) for each \( \zeta \in \sigma \) and hence \( f^{-1}(V) \subseteq \bigcup\{f|_{V_{\zeta}}^{-1}(V)\|_{\zeta \in \sigma}\} \subseteq \tau_{\gamma-L} \) by Theorem 3.10 and Theorem 3.18. This implies that \( f \) is \( \gamma-L \)-continuous. \( \Box \)

**5. \( \gamma-L \)-open functions**

**Definition 5.1.**

(i) A function \( f:(X, \tau, I) \to (Y, \sigma, I) \) is said to be \( \gamma-L \)-open if the image of each open set in \( X \) is \( \gamma-L \)-open set of \( Y \).
(ii) A function \( f(X, \tau) \to (Y, \sigma, I) \) is said to be \( \gamma \)-semi-I-open (resp. \( \gamma \)-pre-I-open, \( \gamma \)-beta-I-open) if the image of each open set in \( X \) is \( \gamma \)-semi-I-open (resp. \( \gamma \)-pre-I-open, \( \gamma \)-beta-I-open) set of \( Y \).

**Example 5.2.** Let \( X = \{a, b, c\} \), \( \tau = \{\phi, X, \{a, c\}\} \), \( Y = \{a, b, c\} \), \( \sigma = P(Y) \) and \( I = \{\phi, \{c\}\} \). We define an operation \( \gamma \sigma \to P(Y) \) as follows: for every \( A \in \tau \),

\[
A' = \begin{cases}
  A \cup \{c\} & \text{if } A = \{a\} \text{or } \{b\} \\
  A \cup \{a\} & \text{if } A = \{c\} \\
  A & \text{if } A \neq \{a\}, \{b\} \text{ and } \{c\}
\end{cases}
\]

Then, \( \sigma_{\gamma \sigma} = \{\phi, X, \{a, b\}, \{b, c\}, \{a, c\}\} \).

We define \( f(X, \tau) \to (Y, \sigma, I) \) as \( f(a) = c, f(b) = b, f(c) = a \). Then, the image of each open set in \( X \) is \( \gamma \)-I-open set in \( (Y, \sigma, I) \). Hence, \( f \) is an \( \gamma \)-I-open function.

**Theorem 5.3** [16]. A function \( f(X, \tau) \to (Y, \sigma, I) \) is said to be \( \gamma \)-I-open if and only if \( f \) is \( \gamma \)-semi-I-open and \( \gamma \)-pre-I-open.

**Proof.** The proof follows from the Theorem 3.7.

**Theorem 5.4.** A function \( f(X, \tau) \to (Y, \sigma, I) \) is \( \gamma \)-I-open if and only if for each subset \( W \subseteq Y \) and each closed set \( F \) of \( X \) containing \( f^{-1}(W) \), there exists an \( \gamma \)-closed set \( H \subseteq Y \) containing \( W \) such that \( f^{-1}(H) \subseteq F \).

**Proof.** Necessity. Let \( H = Y - f(X - F) \). Since \( f^{-1}(W) \subseteq F \), we have \( f(X - F) \subseteq Y - W \). Since \( f \) is \( \gamma \)-I-open, then \( H \) is \( \gamma \)-I-closed and \( f^{-1}(H) = X - f^{-1}(f(X - F)) \subseteq X - (X - F) = F \).

Sufficiency. Let \( U \) be any open set of \( X \) and \( W = Y - f(U) \). Then, \( f^{-1}(W) = X - f^{-1}(f(U)) \subseteq X - U \) and \( X - U \) is closed. By the hypothesis, there exists an \( \gamma \)-closed set \( H \) of \( Y \) containing \( W \) such that \( f^{-1}(H) \subseteq X - U \). Then, we have \( f^{-1}(H) \cap U = \phi \) and \( H \cap f(U) = \phi \). Therefore, we obtain \( Y - f(U) \subseteq H \subseteq Y - f = H \subseteq Y \). This implies that \( f \) is \( \gamma \)-I-open.

**Corollary 5.5.** If \( f(X, \tau) \to (Y, \sigma, I) \) is \( \gamma \)-I-open, then the following properties hold:

(i) \( f^{-1}(\tau, \text{cl}(\gamma, \text{int}^\ast(\tau, \text{cl}(B)))) \subseteq \tau, \text{cl}(f^{-1}(B)) \) for each set \( B \subseteq Y \).

(ii) \( f^{-1}(\tau, \text{cl}(V)) \subseteq \tau, \text{cl}(f^{-1}(V)) \) for each \( \gamma \)-preopen set \( V \) of \( Y \).

**Proof.**

(i) Let \( B \) be any subset of \( Y \), then \( \tau, \text{cl}(f^{-1}(B)) \) is closed in \( X \). By Theorem 5.4, there exists an \( \gamma \)-closed set \( H \subseteq Y \) containing \( \tau, \text{cl}(f^{-1}(B)) \).

(ii) Let \( V \) be any \( \gamma \)-preopen set of \( Y \). By (i), we obtain \( f^{-1}(\tau, \text{cl}(f^{-1}(V))) \subseteq f^{-1}(\tau, \text{cl}(V)) \subseteq f^{-1}(\tau, \text{cl}(\gamma, \text{int}^\ast(\tau, \text{cl}(V)))) \subseteq f^{-1}(\tau, \text{cl}(f^{-1}(V))) \subseteq f^{-1}(\tau) \).
