Report on the open problems session

Gábor Lukács

Halifax, Nova Scotia, Canada

ARTICLE INFO

Keywords:
Permutation group
Homeomorphism group
Minimal group
Duality theory
Reflexive
Subreflexive
Quasi-convex
$D$-space
Hyperextension

ABSTRACT

This report summarizes the problems presented during the last session of the “Algebra meets Topology: Advances and Applications” conference in honor of Dikran Dikranjan on his 60th birthday, Barcelona, July 20–23, 2010.

© 2012 Elsevier B.V. All rights reserved.

Editors’ preface

At the conference “Algebra Meets Topology” held in Barcelona in July 2010 to honor Dikran Dikranjan’s 60th birthday, the organizers convened a special session for the participants, asking them to present open problems related to the general theme of the conference (but not necessarily linked to presentations given at the meeting). Since the problems proposed at the conference and eventually submitted for publication in the Special Issue of “Topology and its Applications” originated from a great variety of topics, the task of creating a meaningful summary was not easy. The editors of the Special Issue were therefore fortunate to have Gábor Lukács agree to undertake the tasks of collecting and organizing these problems into a text which we are happy to include in the Special Issue. The author of the collection, in turn, benefited from the advice of dedicated referees, as well as from additional input from the proposers of the problems themselves. We are grateful to all those who contributed to the final outcome of this project.

The Editors of the Special Issue of “Topology and its Applications”

1. Permutation and homeomorphism groups, and minimality

The self-isomorphism groups of many mathematical objects have a very rich structure. In this section, open problems related to two instances of such self-isomorphism groups are presented, namely, the permutation group $\mathcal{S}(X)$ of an (infinite) set $X$, and the self-homeomorphism group $\mathcal{H}(X)$ of a Hausdorff topological space $X$.

A Hausdorff topological group $G$ is minimal if it does not admit a coarser Hausdorff group topology. Minimal groups were introduced in the works of Stephenson (cf. [52]) and Doitchinov (cf. [25]), independently. Initially, minimal groups were considered in the abelian case due to the famous conjecture of Prodanov that every minimal abelian group is precompact, which was confirmed and became known as the Prodanov–Stojanov Theorem (cf. [50] and [49]). The permutation group $\mathcal{S}(X)$ of an infinite set $X$ equipped with the topology of pointwise convergence on $X$ is a topological group that is not precompact. In 1975, Doitchinov conjectured that $\mathcal{S}(X)$ is minimal. In 1976, Dierolf and Schwanengel studied various properties of $\mathcal{S}(X)$, and confirmed Doitchinov’s conjecture (cf. [19]).
Neither Doitchinov nor Dierolf and Schwanengel knew at the time that ten years earlier, in 1967, Gaughan proved a much stronger statement, namely, that the topology of pointwise convergence is the coarsest Hausdorff group topology on $\mathcal{F}(X)$ (cf. [35, Theorem 2] and [24, 7.1.9]). In other words, while Dierolf and Schwanengel showed that the pointwise topology is a minimal element in the lattice of Hausdorff group topologies on $\mathcal{F}(X)$, Gaughan proved that it is the smallest one. Gaughan’s work was discovered by Remus only in the mid-1980’s. Gaughan’s result was motivated by a problem Ulam posed in 1935, namely, whether there is a non-discrete locally compact Hausdorff group topology on $\mathcal{F}(\mathbb{N})$ (cf. [53], [43, pp. 177–178], and [24, 7.7]). Gaughan used the aforesaid result to show that the only locally precompact Hausdorff group topology on $\mathcal{F}(X)$ is the discrete one.

Recall that the support of $\sigma \in \mathcal{F}(X)$ is the set $\{x \in X \mid \sigma(x) \neq x\}$. Let $\mathcal{F}_o(X)$ denote the set of permutations of $X$ with a finite support. The following conjecture is motivated by the observation that the proof of Gaughan’s result found in [24] fails to produce a similar result for subgroups $G$ of $\mathcal{F}(X)$ that contains $\mathcal{F}_o(X)$, because [24, 7.1.5] fails for $\mathcal{F}_o(X)$.

**Conjecture 1.1.** (D. Dikranjan) Let $X$ be an infinite set, and $G$ a subgroup of $\mathcal{F}(X)$ such that $\mathcal{F}_o(X) \subseteq G$. Then the topology of pointwise convergence on $X$ is the coarsest Hausdorff group topology on $G$.

**Remark 1.2.** On December 17, 2011, we learned that Banakh, Guran, and Protasov proved the conjecture by showing that if $\mathcal{F}_o(X) \subseteq G \subseteq \mathcal{F}(X)$, then every $T_1$ topology on $G$ that makes the multiplication $(x, y) \mapsto xy$ and the commutator $(x, y) \mapsto xy^{-1}y^{-1}$ separately continuous is finer than the topology of pointwise convergence (cf. [6]).

Let $X$ be a Hausdorff space, and let $\mathcal{H}(X)$ denote the group of self-homeomorphisms of $X$, equipped with the compact-open topology. In general, the group operations of $\mathcal{H}(X)$ need not be continuous; however, there are some known cases when they turn out to be so:

(a) If $X$ is locally compact, then $\mathcal{H}(X)$ is a paratopological group, that is, composition is continuous.
(b) If $X$ is compact, then $\mathcal{H}(X)$ is a topological group.
(c) If $G$ is a subgroup of $\mathcal{H}(X)$ that is locally compact in compact-open topology inherited from $\mathcal{H}(X)$, then $G$ is a topological group (cf. [40, Theorem 1]).
(d) If every point in $X$ has a compact connected neighborhood, then the compact-open topology is a group topology on $\mathcal{H}(X)$ (cf. [20]).

If $X$ is metric and $\mathcal{H}(X)$ is locally compact, or if $X$ is locally compact and $\mathcal{H}(X)$ is compact, then $\mathcal{H}(X)$ is zero-dimensional (cf. [40, Theorem 3] and [39]).

**Problem 1.3.** (K.H. Hofmann and S.A. Morris) Let $G$ be a compact group. Are the following two statements equivalent?

1. $G$ is zero-dimensional (i.e., profinite).
2. There is a compact connected space $X$ such that $\mathcal{H}(X) \cong G$.

Is there a canonical, if not functorial, algorithm to answer the question?

The following partial answers are available in the literature:

(a) In 1958, de Groot and Wille presented a one-dimensional compact connected metric space whose self-homeomorphism group is trivial (cf. [36]).
(b) Keesling showed that for every cardinal $\alpha$, there is a space $X$ such that $\mathcal{H}(X) \cong (\mathbb{Z}/2\mathbb{Z})^\alpha$ (cf. [40, Theorem 4]).
(c) Gartside and Glyn established that a positive answer applies to all metric compact groups $G$ (cf. [34]).
(d) Hofmann and Morris suggest a variant of the usual algorithm to produce, for every profinite group $G$ with a dense cyclic subgroup, a one-dimensional compact connected metric space $X$ such that $G \cong \mathcal{H}(X)$ (cf. [38]).

**Problem 1.4.** (J.J. Dijkstra and J. Hickmann [21]; communicated by J. van Mill at the conference) Is there a compact space $X$ such that $\mathcal{H}(X)$ is homeomorphic to the complete Erdős space $\{x_n \in \ell^2 \mid \forall n \in \mathbb{N} \ (x_n \in \mathbb{R} \setminus \mathbb{Q})\}$?

**Problem 1.5.** (E. van Douwen and J. van Mill; cf. [44, p. 259]) Is there an infinite homogeneous zero-dimensional separable metrizable space with the fixed-point property for homeomorphisms?

We conclude this section with a brief history of an open problem that relates to all three classes of groups discussed here, which was solved thanks to the conference, and in fact, during the conference. If $K$ is the one-point compactification of discrete set $X$, then $\mathcal{H}(K) \cong \mathcal{F}(X)$, and thus, by Gaughan’s result, $\mathcal{H}(K)$ is minimal. Gamarnik proved that the homeomorphism groups $\mathcal{H}([0, 1])$ and $\mathcal{H}([0, 1]^\omega)$ of the unit interval and the Cantor cube, respectively, are minimal too (cf. [33, Theorems 1 and 2]). During his presentation, Megrelishvili cited a problem due to Stojanov, asking whether the
homeomorphism of every homogeneous compact space is minimal (cf. [3, VI.7] and [16, 3.3.3(a)]). Before the conference was over, van Mill had provided a negative answer to the problem (cf. [45, 1.1]).

2. Duality theory of abelian groups

Let $G$ be an abelian topological group, and let $\hat{G}$ denote the Pontryagin dual of $G$, that is, the group of continuous homomorphisms $\chi : G \to \mathbb{T}$ equipped with the compact-open topology, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The evaluation $\alpha_G : G \to \hat{G}$ defined by $(\alpha_G(g))(\chi) = \chi(g)$ is a group homomorphism, but it need not be continuous, open, surjective, or injective in general. If $\alpha_G$ is a topological isomorphism, we say that $G$ is reflexive.

Let $G$ be an abelian topological group. For $E \subseteq G$ and $A \subseteq \widehat{G}$, the polars of $E$ and $A$ are defined as

$$E^c = \{ x \in \hat{G} \mid \chi(E) \subseteq T_+ \}$$

and

$$A^\circ = \{ x \in G \mid \forall \chi \in A \ (\chi(x) \in T_+) \},$$

where $T_+ := [-\frac{1}{2}, \frac{1}{2}] + \mathbb{Z}$. The set $E$ is said to be quasi-convex if $E = E^{c^c}$. The group $G$ is locally quasi-convex (LQC) if it admits a base of quasi-convex neighborhoods at zero. This property can be related to the evaluation homomorphism $\alpha_G$ as follows (cf. [4, 6.10] and [41, I.12]):

(a) If $G$ is LQC, then $\alpha_G$ is open onto its image, and if in addition $G$ is Hausdorff, then $\alpha_G$ is injective.

(b) If $\alpha_G$ is an embedding, then $G$ is LQC. In particular, every reflexive group is LQC.

Recall that a topological group $G$ is precompact if for every neighborhood $U$ of the identity, there is a finite subset $F \subseteq G$ such that $G = F + U$. It is well known that (Hausdorff) precompact groups are subgroups of compact ones, and thus they are LQC, which is necessary (but not sufficient) for reflexivity. Nevertheless, the existence of non-compact reflexive precompact groups used to be an open problem until recently (cf. [13, p. 641]). First examples of such groups were found by Galindo and Macario (cf. [32, 6.1]), and shortly thereafter by Bruguera and Tkachenko (cf. [10]). The following problem is a special case of a problem due to Bruguera and Tkachenko, who asked whether there exists an infinite reflexive precompact group that is countable (cf. [10, 5.3]). Since it is known that $\mathbb{Z}$ (and in fact, every abelian group of an infinite exponent) admits a non-discrete reflexive group topology (cf. [31, Theorem 1]), the assumption that the topology be precompact is essential.

**Problem 2.1.** (M. Tkachenko) Is there a precompact Hausdorff group topology $\tau$ on $\mathbb{Z}$ such that $(\mathbb{Z}, \tau)$ is reflexive?

A map $f : X \to Y$ between Hausdorff spaces is $k$-continuous if for every compact subset $K \subseteq X$, the restriction $f|_K$ is continuous, and $X$ is a $k$-space if every $k$-continuous map with domain $X$ into a Hausdorff space is continuous.

One says that $G$ is subreflexive if $\alpha_G$ is an embedding (cf. [15]). Although in general, the evaluation homomorphism $\alpha_G$ need not be continuous, it is always $k$-continuous, and thus if $G$ is a Hausdorff $k$-space and LQC, then $\alpha_G$ is an embedding, and so $G$ is subreflexive. In particular, every metrizable LQC group is subreflexive. For every group $G$, its Pontryagin dual $\hat{G}$ is LQC (cf. [7, p. 2], [11, I] and [41, I.11]). If in addition $G$ is metrizable, then $\hat{G}$ is a $k$-space and $\hat{G}$ is complete and metrizable (cf. [4, 4.7], [12, Theorem 1]); in particular, $\alpha_G$ and $\alpha_G^{-1}$ are continuous (cf. [4, 5.21] and [41, I.31]), and thus $\hat{G}$ is subreflexive.

**Problem 2.2.** (L. Außenhofer) Let $G$ be an LQC metrizable abelian group. Is $\hat{G}$ reflexive?

**Remarks 2.3.** (L. Außenhofer)

(a) In the setting of Problem 2.2, if $\hat{G}$ is reflexive, then all higher character groups are so, and if $\hat{G}$ is not reflexive, then all higher character groups fail to be reflexive (cf. [4, 5.22]).

(b) For $1 < p < \infty$, let $L_p^0[0, 1]$ denote the subgroup of $L^p[0, 1]$ consisting of the almost-everywhere integer valued functions. Although $L_p^0[0, 1]$ is not reflexive (cf. [4, 11.9, 11.15]), its dual $\overline{L_p^0[0, 1]}$ is topologically isomorphic to $L_p^0[0, 1]$ (cf. [4, 11.14]), and thus it is reflexive. This example demonstrates that $\hat{G}$ may be reflexive even when $G$ itself is only subreflexive, and fails to be reflexive.

(c) In the setting of Problem 2.2, if $G$ admits a structure of a topological vector space over $\mathbb{R}$, then $\hat{G}$ is reflexive (cf. [7, 15.2]).

Recall that the precompactness index (also known as boundedness number) of a group $G$ is the smallest infinite cardinal $\kappa$ such that $G$ can be covered by at most $\kappa$ many translates of every neighborhood of the neutral element. Although subreflexive groups may fail to be reflexive even in the presence of completeness (and in particular, they need not be dense in their biduals), nevertheless, in several ways they are as “big” as their biduals: Subreflexivity implies that the character [15, 5.12] and the precompactness index [15, 7.6] of $G$ and $\hat{G}$ coincide, and consequently, so does their weight [15, 7.7]. Thus, it is natural to inquire about the extent of this subreflexivity phenomenon, and ask how far $\hat{G}$ and $G$ are apart for a subreflexive group, both quantitatively and qualitatively.
Problem 2.4. (B. Tsaban) Suppose that $G$ is subreflexive.  
(a) Which other cardinal invariants of $G$ and $\hat{G}$ coincide?  
(b) Are there topological algebraic properties $\mathcal{P}$ such that $G$ satisfies $\mathcal{P}$ if and only if $\hat{G}$ satisfies $\mathcal{P}$?  
(c) Suppose that, in addition, $G$ is complete. How would this affect the answers to (a) and (b)?

Problem 2.5. (L. Außenhofer; see also [4, 5.23]) Let $G$ be a subreflexive abelian group. Is $\alpha_G(G)$ $qc$-dense in $\hat{G}$; in other words, is $\alpha_G(G)^{\ast\ast} = \hat{G}$?  

A sequence $\{u_n\}$ in an abelian group $A$ is a $TB$-sequence if there is a precompact Hausdorff group topology $\tau$ on $A$ such that $u_n \xrightarrow{\tau} 0$ (cf. [8]). If $X$ is a compact metrizable abelian group and $u = \{u_n\} \subseteq X$ is a $TB$-sequence, then the group $\mathcal{S}_u(X) := \{x \in X \mid u_n(x) \to 0\}$ admits a Polish group topology (cf. [30]); let $G_u$ denote $\mathcal{S}_u(X)$ equipped with this topology.

Problem 2.6. (S.S. Gabriyelyan) When is $G_u$ reflexive?

Let $CCAb$ denote the class of complete countable Hausdorff abelian groups.

Problem 2.7. (S.S. Gabriyelyan) Describe the class $\{G \mid G \in CCAb\}$.

Problem 2.8. (S.S. Gabriyelyan) Let $H$ be a Polish abelian group. Is there $G \in CCAb$ such that:

(a) $\hat{H} \cong \hat{G}$?  
(b) $\hat{H} \cong G$?

For a topological group $(G, \tau)$, let $(G, \tau)^{\ast}$ denote the group of continuous characters (without a topology). One says that $(G, \tau)$ is a Mackey group in a class $\mathcal{C}$ of topological groups if $\tau$ is the finest group topology on $G$ in the class $\mathcal{C}$ with the given group of continuous characters. In other words, $(G, \tau)$ is Mackey in the class $\mathcal{C}$ if for every group topology $\tau'$ on $G$ such that $(G, \tau') \in \mathcal{C}$ and $(G, \tau)^{\ast} = (G, \tau')^{\ast}$ as subgroups of $\text{hom}(G, \mathbb{T})$, one has $\tau' \subseteq \tau$ (cf. [14, 23, 5]). Every locally compact abelian group, and in particular every compact abelian group, is Mackey in the class LQC of locally quasi-convex groups (cf. [14]). Part (a) of the next question is asking whether compact abelian groups are also Mackey in the class of all (Hausdorff) topological groups.

Problem 2.9. (V.I. Tarieladze and G. Lukács) Let $K$ be an (infinite) abelian group with a compact Hausdorff group topology $\tau$. Is there another Hausdorff group topology $\eta$ on $K$ such that:

(a) $(K, \tau)^{\ast} = (K, \eta)^{\ast}$ as subgroups of $\text{hom}(G, \mathbb{T})$?  
(b) $(K, \tau) \cong (K, \eta)$ as topological groups and $(K, \eta)$ is maximally almost periodic?  
(c) $(K, \eta)$ is discrete and $(K, \tau)^{\ast} = (K, \eta)^{\ast}$ as subgroups of $\text{hom}(G, \mathbb{T})$?

Remarks 2.10. (G. Lukács; simplified by L. Außenhofer)

(a) In parts (a) and (c) of Problem 2.9, the topology $\eta$ is necessarily finer than $\tau$, because $\tau$ is the coarsest group topology on $K$ that makes each character $\tau$-continuous. In particular, $(G, \eta)$ is maximally almost periodic. As noted earlier, since $(G, \tau)$ is a compact abelian group, it is Mackey in LQC. Therefore, the topology $\eta$, if it exists, cannot be LQC.

(b) If one replaces the condition that $(K, \tau)^{\ast} = (K, \eta)^{\ast}$ as subgroups of $\text{hom}(G, \mathbb{T})$ in Problem 2.9(a) with the requirement that $(K, \tau) \cong (K, \eta)$ as topological groups, then $\eta$ no longer has to be maximally almost periodic, as the counterexample below shows. Consequently, in order to ensure that Problem 2.9(b) is meaningful, we also require that $(K, \eta)$ be maximally almost periodic.

Let $K := \mathbb{T}$ with its standard topology. Algebraically, one has $K \cong \mathbb{Q}/\mathbb{Z} \times \mathbb{R}$. If the factor $\mathbb{Q}/\mathbb{Z}$ is equipped with the subgroup topology induced by $\mathbb{T}$, then $\mathbb{Q}/\mathbb{Z} = \mathbb{T} = \mathbb{Z}$ not only algebraically, but also topologically, because $\mathbb{Q}/\mathbb{Z}$ is a dense subgroup of the metrizable group $\mathbb{T}$ (cf. [4, 4.5] and cf. [12, Theorem 2]). Nielsen has shown that there is a Hausdorff group topology $\eta_0$ on $\mathbb{R}$ such that $(\mathbb{R}, \eta_0) = \{0\}$ (cf. [46]). Let $\eta$ denote the product topology on $K \cong \mathbb{Q}/\mathbb{Z} \times \mathbb{R}$, with the standard topology in the first component, and $\eta_0$ in the second. Then $(K, \eta) \cong \mathbb{Q}/\mathbb{Z} \times (\mathbb{R}, \eta_0) = \mathbb{Z} \times \{0\} \cong \mathbb{Z} = \mathbb{R}$,

but $(K, \eta)$ is not maximally almost periodic.

Remark 2.11. (S.S. Gabriyelyan) If $(P, \tau)$ is a countably infinite metrizable precompact group, then $P$ admits a strictly finer group topology $\eta$ such that $(P, \eta)$ is discrete and $(P, \tau)^{\ast} = (P, \eta)^{\ast}$ as subgroups of $\text{hom}(P, \mathbb{T})$ (cf. [22]). Thus, if one replaces the requirement that $K$ be compact in Problem 2.9(c) with “countably infinite precompact metrizable”, then the answer to the problem is positive.
3. Subdirect products and coding theory

For a family \( \{G_i\}_{i \in \mathbb{N}} \) of topological groups, let \( \bigoplus_{i \in \mathbb{N}} G_i \) denote the subgroup of elements \( (g_i) \) in the product \( \prod_{i \in \mathbb{N}} G_i \) such that \( g_i = e \) for all but finitely many indices \( i \in \mathbb{N} \). A subgroup \( G \subseteq \prod_{i \in \mathbb{N}} G_i \) is called controllable (or, in earlier papers, weakly controllable) if \( G \cap (\bigoplus_{i \in \mathbb{N}} G_i) \) is dense in \( G \), that is, if \( G \) is topologically generated by its “finite sequences”, and \( G \) is observable if \( G \cap (\bigoplus_{i \in \mathbb{N}} G_i) = G \cap (\bigoplus_{i \in \mathbb{N}} G_i) \), where \( G \) stands for the closure of \( G \) in \( \prod_{i \in \mathbb{N}} G_i \) in the product topology. Although the notion of (weak) controllability was coined by Fagnani earlier in a broader context (cf. [26]), both notions were introduced in the area of coding theory by Forney and Trott (cf. [27]). They observed that if the groups \( G_i \) are locally compact abelian, then controllability and observability are dual properties with respect to the Pontryagin duality: If \( G \) is a closed subgroup of \( \prod_{i \in \mathbb{N}} G_i \), then it is controllable if and only if its annihilator \( G^\perp := \{ \chi \in \prod_{i \in \mathbb{N}} G_i \mid \chi(G) = \{0\} \} \) is an observable subgroup of \( \prod_{i \in \mathbb{N}} G_i \) (cf. [27, 4.8]).

Examples 3.1. A subgroup \( G \) of the product \( \prod_{i \in \mathbb{N}} G_i \) is rectangular if there are subgroups \( H_i \) of \( G_i \) such that \( G = \prod_{i \in \mathbb{N}} H_i \). We say that \( G \) is weakly rectangular if there are finite subsets \( F_i \subseteq \mathbb{N} \) and subgroups \( H_i \) of \( \bigoplus_{j \in F_i} G_j \) that satisfy \( G = \prod_{i \in \mathbb{N}} H_i \).

(a) Weakly rectangular subgroups and rectangular subgroups of \( \prod_{i \in \mathbb{N}} G_i \) are controllable.
(b) If each \( G_i \) is a pro-\( p_i \)-group for some prime \( p_i \), and all \( p_i \) are distinct, then every closed subgroup of the product \( \prod_{i \in \mathbb{N}} G_i \) is rectangular, and thus controllable.
(c) If each \( G_i \) is a finite simple non-abelian group, then every closed normal subgroup of the product \( \prod_{i \in \mathbb{N}} G_i \) is rectangular, and thus controllable.

The following problem asks to what extent the converse of these observations hold.

Problem 3.2. (S. Hernández) Let \( \{G_i\}_{i \in \mathbb{N}} \) be a family of compact metrizable groups, and \( G \) a closed subgroup of the product \( \prod_{i \in \mathbb{N}} G_i \). If \( G \) is controllable, that is, \( G \cap \bigoplus_{i \in \mathbb{N}} G_i \) is dense in \( G \), what can be said about the structure of \( G \)? In particular, under what additional conditions is the group \( G \) weakly rectangular, that is, of the form \( \prod_{i \in \mathbb{N}} H_i \), where each \( H_i \) is a subgroup of \( \bigoplus_{j \in F_i} G_j \) for some finite \( F_i \subseteq \mathbb{N} \)?

4. \( D \)-spaces

Let \( X \) be a topological space with topology \( T \). An open neighborhood assignment for \( X \) is a map \( \eta : X \to T \) such that \( x \in \eta(x) \) for every \( x \in X \), and \( X \) is a \( D \)-space if for every open neighborhood assignment \( \eta \) for \( X \), there is a closed discrete subset \( D \) of \( X \) such that \( \bigcup \{ \eta(x) \mid x \in D \} = X \). The notion of a \( D \)-space has proven to be a useful tool in the context of addition theorems in topology (cf. [1,2,37,48]). While every compact \( T_1 \)-space is a \( D \)-space, the following problem, posed by van Douwen, is a longstanding open question.

Problem 4.1. (E. van Douwen; communicated by J.C. Martínez at the conference) Is every regular Lindelöf space a \( D \)-space?

Recently, Soukup and Szeptycki proved that under the set-theoretic axiom \( \diamondsuit \), there is a Hausdorff hereditarily Lindelöf space that is not a \( D \)-space (cf. [51]). However, the space that their consistency result provides is not regular, and thus Problem 4.1 remains open.

For \( U \) an open cover of \( X \) and \( x \in X \), let \( \text{ord}(x, U) := \{|U \subseteq U \mid x \in U\} \). The space \( X \) is submetacompact if for every open cover \( U \) of \( X \), there is a countable family \( |U_0| \) of open covers of \( X \) such that \( \bigcup U_0 \) is a refinement of \( U \), and for every \( x \in X \), there is \( n \) with \( \text{ord}(x, U_n) < \omega \). Recall that \( X \) is meta-Lindelöf if every open cover of \( X \) has a point-countable open refinement. A space \( X \) is \( C \)-scattered if every non-empty closed subspace \( Y \) of \( X \) has a point with a compact neighborhood in \( Y \). (Clearly, locally compact spaces and scattered spaces are \( C \)-scattered.)

It is known that every regular submetacompact \( C \)-scattered space is a \( D \)-space (cf. [47] and [42]). The next problem aims at replacing submetacompactness with meta-Lindelöfness.

Problem 4.2. (J.C. Martínez) Is every regular meta-Lindelöf \( C \)-scattered space a \( D \)-space?

5. Topological hyperextensions

Let \( X \) be a discrete space. A \( T_1 \)-space \( *X \) is a topological extension of \( X \) if the following conditions are simultaneously satisfied (cf. [9]):

(i) \( X \) is a discrete dense proper subset of \( *X \);
(ii) every map \( f : X \to X \) has a distinguished continuous extension \( *f : *X \to *X \) that satisfies the next two conditions;
(iii) \( *f(g) = *g \cdot *f \) for every \( f, g : X \to X \);
(iv) if \( A \subseteq X \) and \( f|A = \text{id}_A \), then \( *f|_A = \text{id}_A \), where \( \overline{A} = \text{cl}_X A \).
One says that \( *X \) is a topological hyperextension of \( X \) if it is a topological extension, and:

- \( f, g : X \to X \) are such that \( f(x) \neq g(x) \) for every \( x \in X \), then \( *f(*\xi) \neq *g(*\xi) \) for every \( \xi \in *X \);
- there exist \( p, q : X \to X \) with the property that for every pair \( \xi, \eta \in *X \), there exists \( \zeta \in *X \) such that \( \xi = *p(\zeta) \) and \( \eta = *q(\zeta) \).

**Problem 5.1.** (M. Forti) Is the existence of a Hausdorff hyperextension provable in ZFC? Is it derivable from set-theoretic hypotheses weaker than those providing selective ultrafilters?

The existence of hyperextensions as in Problem 5.1 is equivalent to the existence of Hausdorff ultrafilters over \( \mathbb{N} \) (cf. [29] and [18]).

**Problem 5.2.** (M. Forti) Are there large cardinal hypotheses that provide Hausdorff topological hyperextensions \( *\mathbb{R} \) of the real line \( \mathbb{R} \) with the property that there exists \( \xi \in *\mathbb{R} \) such that \( \xi \) is not in the closure of any \( A \subseteq \mathbb{R} \) with \( |A| < \aleph_1 \)?

If \( \xi \) is as in Problem 5.2, then the family \( U_\xi := \{ A \subseteq \mathbb{R} | A \notin \mathcal{A} \} \) is an irregular ultrafilter over \( \mathbb{R} \) (cf. [17] and [18]).

**Problem 5.3.** (M. Forti) Given a set \( X \), is there a non-Hausdorff topological hyperextension of \( X \) such that every function \( f : X \to X \) has a unique continuous extension \( *f : *X \to *X \)?

A topological hyperextension cannot be compact (cf. [18] and [28]).

**Problem 5.4.** (M. Forti) Is the existence of a countably compact hyperextension consistent with ZFC?

**Acknowledgement**

I am grateful to Karen Kipper for her kind help in proofreading this note for grammar and punctuation.

**References**