ELSEVIER

Theoretical Computer Science 237 (2000) 1–32

Theoretical Computer Science

www.elsevier.com/locate/tcs

Fundamental Study

The monadic second-order logic of graphs XII: planar graphs and planar maps [1]

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Received April 1998; revised November 1998
Communicated by W. Thomas

Abstract

We prove that we can specify by formulas of monadic second-order logic the unique planar embedding of a 3-connected planar graph. If the planar graph is not 3-connected but given with a linear order of its set of edges, we can also define a planar embedding by monadic second-order formulas. We cannot do so in general without the ordering, even for 2-connected planar graphs. The planar embedding of a graph can be specified by a relational structure called a *map*, which is a graph enriched with a circular ordering of the edges incident with each vertex. This circular ordering, called a *rotation system*, represents a planar embedding of the neighbourhood of each vertex. For each connected map one can define a linear order on its vertices by formulas of monadic second-order logic. Hence, we have for planar graphs, some kind of equivalence between linear orderings and planar embeddings. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Planar graph; Rotation system; Graph drawing; Monadic second-order logic; Ordered structure; Depth-first search tree

Contents

[1] Work supported by the Training and Mobility in Research European network: GETGRATS
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PII: S0304-3975(99)00305-9

0. Introduction

Graphs can be handled as a logical structures. Logical formulas of the appropriate language can thus express graph properties. The classification of graph properties according to the languages able to express them is an essential issue of descriptive complexity. We are interested in the use of monadic second-order logic because it yields efficient algorithms for graphs given with a hierarchical decomposition, and because it behaves well with respect to context-free graph grammars (see [1, 11, 7]).

By means of monadic second-order formulas (*MS formulas*), one can express graph properties; one can also enrich the given graphs by equipping them with some additional information: orientation [4], linear order [5], Tutte decomposition [8]. In this paper we investigate the possibility of specifying a planar embedding of a planar graph, i.e., a *drawing* of this graph without edge crossings. (In [9] we handle drawings of graphs *with edge crossings*.)

We prove that we can specify by MS formulas the unique planar embedding of a 3-connected planar graph (unicity was proved by Whitney, see [12, 17]). If the planar graph is not 3-connected but given with a linear order of its set of edges, we can also define a planar embedding, but we cannot do so in general without the ordering, even for 2-connected graphs.

The planar embedding of a graph can be specified by a relational structure called a *map*, which is a graph enriched with a circular ordering of the edges incident with each vertex. Such a circular ordering is called a *rotation system* in the books [15, 17]. It represents a planar drawing of the neighbourhood of each vertex (where neighbourhood means here the set of incident edges). As a preliminary result, we give a characterization of planar maps in terms of “forbidden submaps” of two types, as in Kuratowski’s theorem. It should be noted that a map can be nonplanar, even if the underlying graph is planar. So Kuratowski’s Theorem does not indicate whether a map is planar.

For each *connected map*, i.e., each map having a connected underlying graph, one can define by MS formulas a linear order on its edges. Hence, for planar graphs, we have some kind of equivalence between linear orderings and planar *embeddings*.

Kuratowski’s planarity criterion yields a monadic second-order expression of *non-planarity*. In the case of a planar graph, knowing that it does not contain subdivisions of K_5 or $K_{3,3}$ does not give any information on the construction of a planar embedding.

Our monadic second-order characterization of planarity is more complicated than the one based on Kuratowski's theorem, but it is "effective" in that, when a graph is planar (and linearly ordered) it specifies a planar embedding of it.

We now present two notions introduced in this paper. The first one is that of a *drawing scheme*, formally defined in Section 2. It is not easy to describe graph drawings in a finite combinatorial way. A planar map of a connected graph specifies actually drawings of this graph on the sphere, whence a *family of drawings in the plane*. One such drawing can be determined (up to homeomorphism of the plane) if we choose a source vertex and an edge incident to this vertex. However, the construction of this drawing needs a traversal of the graph. An actual drawing is immediate if we fix a spanning tree of the graph, because drawing a tree is straightforward from the knowledge of a linear orderings of the set of sons of nodes, and these orderings are specified by the rotation system of the map. A map equipped with a spanning tree is called a *drawing scheme*, and its planarity is easy to formulate in terms of relative positions in the tree of the edges of the graph that are not in the tree. Drawing schemes are useful for our combinatorial expression of planarity of maps, and can be given as input to drawing algorithms.

We now motivate the introduction (in Section 1) of a special extension of first-order logic by transitive closure. Some logical languages lying inbetween first-order and second-order logic have received special attention because they *capture certain complexity classes*, i.e., they can express exactly those properties of finite structures (usually graphs) that are in these classes. (For the links between second-order logic and complexity, we refer the reader to the excellent book by Ebbinghaus and Flum [13].) One of these languages is *existential second-order logic* (consisting of first-order formulas preceded by sequences of existential quantifications on relation variables), which captures the class NP. Its restriction to formulas involving set quantifications is a subset of monadic second-order logic called sometimes *monadic-NP*. It does not capture any known complexity class but it is hoped that some techniques introduced to prove nonexpressibility in monadic-NP can be extended to existential second-order logic and will help to solve open questions in complexity theory by logical tools.

Two extensions of first-order logic, denoted by FO(TC) and FO(DTC) have been introduced to capture NLOGSPACE and LOGSPACE, respectively (for linearly ordered structures). They are defined by introducing as new $2k$-ary relations in first-order formulas, transitive closures of certain binary relations between k-tuples of elements of the domain defined by other formulas. The formulas of these languages built with such relations restricted to $k = 1$ are translatable into MS formulas.

In order to link our constructions, as much as possible, to these extensions of first-order logic, we introduce $\mathrm{FO}^{\mathrm{TC}}$ which is like FO(TC) except that k is limited to 1 and the relations of which one takes the transitive closure are defined by existential first-order formulas. It appears that the closure of these formulas by existential set quantifications is enough to cover a good number of our constructions. These formulas use the two main graph theoretical notions expressible in MS logic: colorings (handled by set quantifications) and existence of paths (handled by transitive closure).

The paper is organized as follows: Section 1 contains definitions concerning graphs and trees on the one hand, and logic on the other. We refer the reader to other papers (in particular [3, 7]) for detailed definitions concerning monadic second-order logic. Section 2 introduces maps, drawing schemes and gives the characterization of planar maps. Section 3 deals with planar 3-connected graphs and Section 4 with general (finite) planar graphs. Section 5 is the conclusion.

1. Preliminaries

1.1. Graphs

Graphs will be finite and loop-free. For a graph G, we will denote by V_G its set of vertices, and by E_G its set of edges; we write $e:x–y$ if e is a directed or undirected edge linking x and y and $e:x\rightarrow y$ if it is directed from x to y (in the latter case, the vertex x is the *source* of e, denoted by $s(e)$, and y is its *target*, denoted by $t(e)$). Two edges e and f form a *pair of multiple edges* if either they are undirected and have the same sets of ends, or they are directed and $s(e)=s(f)$ and $t(e)=t(f)$. Graphs may have multiple edges. A graph without multiple edges is *simple*. We denote by $E_G(x)$ the set of edges incident with x.

We let $H\subseteq G$ denote that H is a *subgraph* of G (i.e., $V_H\subseteq V_G$, $E_H\subseteq E_G$, and the incidence relation of H is the restriction to $V_H\cup E_H$ of that of G). If U is a set of edges of G, we denote by $G[U]$ the subgraph of G having U as set of edges and the set of ends of the edges of U as set of vertices. If X is a set of vertices of G we denote by $G[X]$ the *induced subgraph* of G having X as set of vertices and the set of all edges of G, the two ends of which are in X as set of edges. If H and K are subgraphs of G, we denote by $H\cup K$ the subgraph of G with set of edges $E_H\cup E_K$ and set of vertices $V_H\cup V_K$. We denote by $H\cap K$ the subgraph defined similarly with $\cap$ instead of $\cup$.

We let $inc_G=\{(e,x,y)/e:x–y\}$ if G is undirected and $inc_G=\{(e,x,y)/e:x\rightarrow y\}$ if G is directed. We let $edg_G=\{(x,y)/(e,x,y)\in inc_G$ for some $e\in E_G\}$. A *path* from x to y is a sequence of edges $(e_1,e_2,\dots,e_n)$ such that for some $x_1,\dots,x_n\in V_G$ we have $x_1=x$, $e_i:x_i–x_{i+1}$ for $i=1,\dots,n-1$, $e_n:x_n–y$ and the vertices x_i are pairwise distinct. However x and y may be equal, and the path is called a *cycle*. A *directed path* (in a directed graph) is similar with $e_i:x_i\rightarrow x_{i+1}$ and $e_n:x_n\rightarrow y$. Occasionally we will consider empty paths, denoted by ( ). A *circuit* in a directed graph is a directed path from a vertex to itself.

We will also call *circuit* a pair $C=\langle X,r\rangle$ where X is a set, $r\subseteq X\times X$ is a functional relation defining a cyclic permutation of X. We write $x\rightarrow y$ if $(x,y)\in r$ and we specify a circuit by, for example,

$$a\rightarrow b\rightarrow c\rightarrow d\rightarrow a.$$

If C is a circuit as above with $Card(X)\geqslant 3$, we define a ternary relation on X by letting $x\ll_C y\ll_C z$ if and only if $x\neq y\neq z\neq x$ and $x\rightarrow x_1\rightarrow\cdots\rightarrow x_n\rightarrow y$ where

$n \geq 0$, $z \notin \{x_1, x_2, \dots, x_n\}$. (There is no binary relation $\ll_C$ involved in this definition). An equivalent condition is $x \neq y \neq z \neq x$ and there exist $y_1, \dots, y_p$ such that $x \notin \{y_1, \dots, y_p\}$ and $y \to y_1 \to \cdots \to y_p \to z$.

If $\langle X, r \rangle$ is a circuit C and $Y \subseteq X$ then the circuit $\langle Y, r' \rangle$ induced by C on Y is the one such that $r'(y) = r^n(y)$ where n is the smallest positive integer such that $r^n(y) \in Y$. We write in this case $C' \subseteq C$ and say that C' is a *subcircuit* of C. (Note that it is not a subgraph.)

Example 1.1. If $C = a \to b \to c \to d \to e \to a$ then $a \to d \to e \to a$, $b \to b$, $a \to c \to a$ are subcircuits of C .

Fact 1.2. *If C is a circuit $\langle X, r \rangle$ with $Card(X) \geq 3$, if $x, y, z \in X$, then $x \ll_C y \ll_C z$ if and only if $x \to y \to z \to x$ is a subcircuit of C. If C' is a subcircuit of C and $x, y, z \in C'$ then $x \ll_C y \ll_C z$ if and only if $x \ll_{C'} y \ll_{C'} z$.*

Let $C = \langle X, r \rangle$ and $C' = \langle Y, r' \rangle$ be two circuits with $X \cap Y = \emptyset$. A *merge of* C and C' is a circuit $C'' = \langle X \cup Y, r'' \rangle$ such that $C \subseteq C''$, $C' \subseteq C''$ and where C'' has no subcircuit of the form $x \to y \to x' \to y' \to x$ with $x, x' \in X$, $y, y' \in Y$.

Fact 1.3. *C'' is a merge of C and C' if and only if it can be described as $x_1 \to x_2 \to \cdots \to x_k \to y_1 \to y_2 \to \cdots \to y_l \to x_1$ where $C = x_1 \to x_2 \to \cdots \to x_k \to x_1$ and $C' = y_1 \to y_2 \to \cdots \to y_l \to y_1$.*

Let $\leqslant$ be a linear order on a finite set X. Let $x_1, \dots, x_n$ be the enumeration of X in increasing order with respect to $\leqslant$. We will denote by $Circ(X, \leqslant)$ the circuit $x_1 \to x_2 \to \cdots \to x_n \to x_1$. If $C = Circ(X, \leqslant)$ we have thus $x \ll_C y \ll_C z$ if and only if $x \neq y \neq z \neq x$ and either $x \leqslant y \leqslant z$ or $y \leqslant z \leqslant x$ or $z \leqslant x \leqslant y$. We say that two orders $\leqslant$ and $\leqslant'$ on a finite set X are *equivalent* if $Circ(X, \leqslant) = Circ(X, \leqslant')$. Conversely, if $C = \langle X, r \rangle$ is a circuit and x belongs to X, we denote by $Ord(C, x)$ the linear order on X with least element x and such that $Circ(Ord(C, x)) = C$.

If $(X, \leqslant)$ and $(Y, \leqslant')$ are disjoint linearly ordered finite sets, if $x \in X$, we denote by $X[Y/x]$ the linearly ordered set $(Z, \leqslant'')$ such that $Z = Y \cup X - \{x\}$ and $u \leqslant'' v$ if and only if $u = v$, or $u, v \in X - \{x\}$ and $u \leqslant v$, or $u, v \in Y$ and $u \leqslant' v$, or $u \in Y$, $v \in X - \{x\}$ and $x \leqslant v$, or $u \in X - \{x\}$ and $v \in Y$ and $u \leqslant x$. We call it the result of *the substitution in X of Y for x*.

A *separating vertex* of a connected graph G is a vertex s such that there are two connected induced subgraphs H and K with $G = H \cup K$, $H \cap K$ is the graph reduced to the vertex s, $E_H \neq \emptyset$, $E_K \neq \emptyset$. We denote by S_G the set of separating vertices of G. A graph is 2-*connected* if it is connected and has no separating vertex. A *block* of a graph G is a maximal 2-connected subgraph of G (maximal for subgraph inclusion).

A separating pair in a 2-connected graph G is a set $\{u, v\}$ of two vertices such that there exist connected induced subgraphs H and K such that $G = H \cup K$, $V_H \cap V_K = \{u, v\}$, $E_H \cap E_K = \emptyset$, $Card(V_H) \geq 3$, $Card(V_K) \geq 3$. A graph is 3-*connected* if it is 2-connected, has at least 4 vertices, and has no separating pair.

1.2. Trees

By a *tree* we mean a connected undirected graph without cycles. A *rooted tree* is a pair $T=(G,r)$ of a tree G and one of its vertices r; it will be directed in such a (unique) way that every vertex x is reachable from r by a directed path, and this path will be denoted by $\mu(x)$.

A vertex which is not the root has a unique *father*, one or more *sons* unless it is a *leaf*. We denote by $\leqslant_T$ the partial order such that $x \leqslant_T y$ if and only if x is on the path $\mu(y)$. If $x \leqslant_T y$ we denote by $\mu(y) - \mu(x)$ the path from x to y. (It is empty if $x = y$).

An *ordered tree* is a rooted tree T given with a family $(\leqslant_v)_{v \in V}$ (where $V = V_T$) such that $\leqslant_v$ is a linear order on the set of edges with source v (we recall that T is a directed graph). Of course $\leqslant_v$ is empty if v is a leaf. We let $\leqslant_{\text{lex}}$ denote the *lexicographic order* on the set of directed paths in T originating at r. We let also $x \leqslant_{\text{lex}} y$ if and only if $\mu(x) \leqslant_{\text{lex}} \mu(y)$ for $x, y \in V_T$. We have thus $x \leqslant_{\text{lex}} y$ whenever $x \leqslant_T y$.

Let G be a connected graph, let T be a rooted spanning tree of G. We say that T is a *depth-first spanning* tree of G (a DFS tree of G for short) if for every edge e in $E_G - E_T$ linking x and y, we have $x \leqslant_T y$ or $y \leqslant_T x$, i.e., x and y are on a same branch of T. A DFS tree is always handled as a directed graph. For later reference, we recall the following fact which is an easy consequence of the definitions.

Fact 1.4. *Let T be a DFS tree of a* 3-*connected graph. Its root has a unique son which by itself also has a unique son.*

1.3. Monadic second-order logic

Let R be a finite set of relation symbols where each element r in R has a positive arity $\rho(r)$. An R-(*relational*) *structure* is a tuple $S = \langle \mathbf{D}_S,\ (r_S)_{r \in R} \rangle$ where $\mathbf{D}_S$ is a finite (possibly empty) set, called the *domain* of S, and r_S is a subset of $\mathbf{D}_S^{\rho(r)}$ for each r in R. We will denote by $\mathscr{S}(R)$ the set of R-structures.

We refer the reader to [2–7] for definitions concerning *monadic second-order logic*. Its formulas will be called *MS-formulas* in short. They are intended to describe properties of R-structures. A graph G will be represented in most cases by the logical structure $|G|_2 := \langle V_G \cup E_G,\ inc_G \rangle$ and sometimes by the less informative structure $|G|_1 := \langle V_G,\ edg_G \rangle$. We will say that a property P of the graphs G of a class $\mathscr{C}$ is MS$_i$-*expressible* (where $i = 1$ or 2), if there is an MS-formula φ (written with *edg* or *inc*, respectively) such that, for every G in $\mathscr{C}$ the property $P(G)$ holds if and only if $|G|_i \models \varphi$.

We will denote by $\mathrm{FO}^{\mathsf{TC}}$ the set of first-order formulas constructed with special atomic formulas representing transitive closures of binary relations defined by existential first-order formulas; these atomic formulas will be written $\mathsf{TC}_{x,y}(\varphi)(u,v)$ where φ is an existential first-order formula that can have free variables other than x and y, say $z_1,\ldots,z_k,\ X_1,\ldots,X_n$. We now define the meaning of these new formulas.

For every assignment of values to $z_1,\dots,z_k$, $X_1,\dots,X_n$, for every u,v in the domain $\mathbf{D}_S$ of the considered structure S, $\mathsf{TC}_{x,y}(\varphi)(u,v)$ holds if and only if (u,v) belongs to the transitive closure of the binary relation equal to $\{(x,y)\in\mathbf{D}_S\times\mathbf{D}_S/S\models\varphi(x,y,z_1,\dots,z_k,\ X_1,\dots,X_n)\}$. (This relation depends on fixed values of $z_1,\dots,z_k$, $X_1,\dots,X_n$.)

Note that this transitive closure constructor is used for *binary* relations over domains of structures, which, furthermore, are defined by *existential* first-order formulas. The class $\mathrm{FO}^{\mathsf{TC}}$ does not coincide with the classes FO(TC) and FO(DTC) considered by Immerman; see the introduction. It lies strictly between first-order logic and monadic second-order logic.

The notion of *monadic second-order definable* transduction of structures (*MS-definable* transduction in the sequel) is surveyed in [3, 7]. It consists in defining from a structure S and n subsets of the domain of S (n is fixed) a structure T, the domain of which is a subset of $\mathbf{D}_S\times\{1,\dots,k\}$, where k is fixed. The composition of two MS-definable transductions is an MS-definable transduction. For every MS-transduction, for every MS-formula μ, one can construct from the formulas defining the transduction the *backwards translation* of μ which is an MS formula ψ such that, whenever T is defined from a structure S by the transduction, S satisfies ψ if and only if T satisfies μ. It should be noted that ψ depends on the n parameters used to define T from S.

2. Drawings and maps

2.1. Definitions: Maps

A *map* is a pair $M=\langle G,sigma\rangle$ where G is a loop-free undirected graph (that may have multiple edges) and *sigma* is a mapping: $V_G\to\mathscr{P}(E_G\times E_G)$ associating with every $v\in V_G$ a subset $sigma(v)$ of $E_G(v)\times E_G(v)$ such that $\langle E_G(v),\ sigma(v)\rangle$ is a circuit. (We denote by $E_G(v)$, the set of edges of G incident with v.) We say that M is *a map of* G. (Such an object is called a *rotation system* in [15, 17]; see these books for basic results and references.) To every plane drawing D of an undirected graph G (where edges may cross) corresponds a map $M(D)=\langle G,sigma\rangle$ where $sigma(v)$ represents the order in which the edges incident with v appear in this drawing if we sweep the plane around from v in the trigonometric sense. If we sweep it in the clockwise sense, we get the map $M^{-1}=\langle G,sigma'\rangle$ where $sigma'(v)=sigma(v)^{-1}$ for each v.

An *ordered map* is a pair $\langle G,(\leqslant_v)_{v\in V_G}\rangle$ where each $\leqslant_v$ is a linear order on $E_G(v)$. Its *underlying* map is $\langle G,sigma\rangle$ where $sigma(v)=Circ(E_G(v),\leqslant_v)$. The notion of ordered map contains slightly more information than that of a map because each set $E_G(v)$ is linearly ordered, as opposed to circularly ordered. Two different ordered maps may have the same underlying map.

An example is shown in Fig. 1. This figure shows a drawing D of a graph G; the corresponding map is $M=\langle G,sigma\rangle$ where $sigma(1)=\{a\to d\to e\to a\}$, $sigma(2)=\{a\to c\to b\to a\}$, $sigma(3)=\{b\to f\to e\to b\}$, $sigma(4)=\{c\to d\to f\to c\}$.

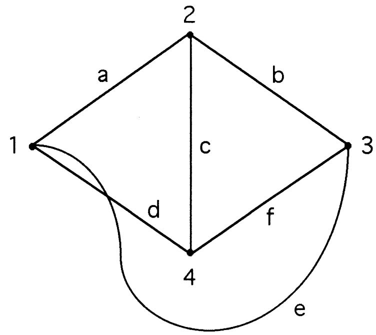

Fig. 1.

A map is *planar* if it is associated with a planar embedding of a graph (see Diestel [12] for precise definition) i.e., an embedding in the plane such that edges are represented by segments that do not cross. We will give below a combinatorial characterization of this topological definition. A map $M = \langle G, sigma \rangle$ where G is a tree is necessarily planar.

Let $M = \langle G, sigma \rangle$ be a (nonnecessarily planar) map. Let G' be a subgraph of G. The *G'-induced submap of M* is the map $M' = \langle G', sigma' \rangle$ where for every $v \in V'_G$, $\langle E'_G(v), sigma'(v) \rangle$ is the subcircuit of $\langle E_G(v), sigma(v) \rangle$ induced by $E'_G(v)$. It is clear that if M represents a planar drawing of G, then M' represents the planar drawing of G' obtained by removing the edges and vertices of G not in G'.

2.2. A linear order on paths originating from a distinguished vertex

Let $M = \langle G, sigma \rangle$ be a map. Let $r \in V_G$; we denote by $Paths_G(r)$ the set of paths in G originating at r. Among these paths, we have the empty path which starts and ends at r. Let us also choose $h \in E_G(r)$. We define as follows a linear order $\leqslant$ on $Paths_G(r)$ (by using the notation $\ll_w$ for $\ll_C$ where C is the circuit $\langle E_G(w), sigma(w) \rangle$):

$p = (e_1, \ldots, e_n) \leqslant p' = (g_1, \ldots, g_m)$ if and only if
either p' extends p, or $e_1 = h$ and $g_1 \neq h$, or $e_1 \neq h$ and
$h \ll_r e_1 \ll_r g_1$, or $e_1 = g_1, \ldots, e_i = g_i$, $i \geqslant 1$, $i < n$, $i < m$, and
$e_i \ll_v e_{i+1} \ll_v g_{i+1}$ (where v is the vertex incident with e_i, e_{i+1} and g_{i+1}).

It is easy to check that $\leqslant$ is a linear order on $Paths_G(r)$. We call $N = (M, r, h)$ a *1-map* ("1-" refers to the pair (r, h) of a distinguished vertex and an edge incident to it; in Section 4, we use 2-maps, defining drawings of graphs with two such pairs of distinguished vertices and incident edges). We denote the order $\leqslant$ by $\leqslant_N$. We call (r, h) the *source* of N.

2.3. Constructing DFS trees from connected maps

Let us now assume that M is *connected*, i.e., by definition, that its underlying graph G is connected. For every vertex $x \in V_G$, there is a unique $\leqslant_N$-minimal path in $Paths_G(r)$ from r to x, that we denote by $\mu_N(x)$. We let $x \leqslant_N y$ if and only if $\mu_N(x) \leqslant_N \mu_N(y)$.

Lemma 2.1. *Let $N=(M,r,h)$ be a connected 1-map of a graph G.*
(1) *The relation $\leqslant_N$ on V_G is a linear order.*
(2) *The subgraph of G defined as the union of the paths $\mu_N(x)$, for all $x \in V_G$ is a depth-first spanning tree of G with root r.*

Proof. (1) Since G is connected, $\mu_N(x)$ is well-defined for all $x \in V_G$. That $\leqslant_N$ is a linear order follows then from the corresponding fact for $\leqslant_N$ on $Paths_G(r)$.

(2) Let H be the union of the paths $\mu_N(x)$ for $x \in V_G$. Consider $y \in V_G$ on $\mu_N(x)$. Let us express $\mu_N(x)$ as the *concatenation* of p and q (written simply as pq) where p is a path from r to y and q is one from y to x. If $p \neq \mu_N(y)$ this means that $\mu_N(y)$ is strictly smaller that p. Hence $\mu_N(y)q$ is strictly smaller than $\mu_N(x)$ and links r to x; this contradicts the definition of $\mu_N(x)$. Hence every initial part of $\mu_N(x)$ is also of the form $\mu_N(y)$. It follows that H is a tree and, even, a spanning tree.

If H is not depth-first (with respect to r taken as root), this means that there is an edge e of $E_G - E_H$ that links two vertices, say x, y, not on a same branch. Hence none of $\mu_N(x)$ and $\mu_N(y)$ is a prefix of the other. Let $\mu_N(x)$ be smaller than $\mu_N(y)$ by one of the last 3 cases of the definition of $\leqslant_N$. Then $\mu_N(x)e$ is a path from r to y that is smaller than $\mu_N(y)$ (by the same case showing that $\mu_N(x) \leqslant_N \mu_N(y)$). This contradicts the definition of $\mu_N(y)$. Hence H is a depth-first spanning tree. □

We will denote H by $T(N)$. It is actually an ordered tree: for every internal vertex v, for outgoing edges $e: v \to x, f: v \to y$ we let $e \leqslant_v f$ if and only if $x \leqslant_N y$. The corresponding order $\leqslant_{\text{lex}}$ on the paths in $T(N)$ starting at r is the restriction of $\leqslant_N$ to this set; the corresponding order $\leqslant_{\text{lex}}$ on $V_{T(N)}$ is identical to $\leqslant_N$ on V_G $(= V_{T(N)})$ and the operator μ_N is equal to the operator μ associated with $T(N)$ considered as a rooted tree (see Section 1).

If G is a tree and N is a 1-map of G, then $T(N)$ is an ordered tree, with underlying tree G. Conversely, if T is an ordered tree with underlying tree G, there is a unique 1-map N of G such that $T(N) = T$.

In the example of the map M of Fig. 1 let us take $r=2$ and $h=a$. The tree $T(N)$ consists of edges a,d,f. Path (a,d,f) is strictly smaller than Path (a,e). If we take $r=3$ and $h=e$ then we get the tree consisting of edges e,a and c.

2.4. Drawing schemes

Let G be a connected undirected graph (finite and without loops by the initial convention). Let T be a rooted spanning tree of G with root r_T. For every $v \in V_G$, we let $E_{G/T}(v)$ be the set $E_G(v)$ minus the edge linking v and its father. Hence $E_{G/T}(r_T) = E_G(r_T)$.

A *drawing scheme of* G is a pair $D=(T,(\leqslant_v)_{v\in V_G})$ where T is a rooted spanning tree of G and each $\leqslant_v$ is a linear order on $E_{G/T}(v)$. (The motivation for introducing drawing schemes is presented in the introduction.)

We consider T as an ordered tree (the edges of T outgoing of v are ordered by the restriction of $\leqslant_v$) with linear ordering $\leqslant_{\text{lex}}$ on the set of paths of T starting from the root. This lexicographical ordering will be extended to the paths starting from r, having all their edges in T except possibly the last one, and will still be denoted by $\leqslant_{\text{lex}}$. We denote by $\mu(x)$ the unique path in T from r to x. We let $B=\{(x,e)/x\in V_G,\ e\in E_G(x)-E_T\}$ and we order this set lexicographically by

$$(x,e)\leqslant_{\text{lex}}(y,f) \quad \text{if and only if} \quad \mu(x)e\leqslant_{\text{lex}}\mu(y)f$$

(i.e., if and only if $\{x=y$ and $e\leqslant_x f\}$ or $\{x<_T y$ and $e<_x g$ where g is the first edge of $\mu(y)-\mu(x)\}$. (See Section 1.2 for the notation $\mu(y)-\mu(x)$). We say that D as given above is a *planar drawing scheme* if there are no two edges $e,f\in E_G-E_T$ and vertices x,x',y,y' (with $e:x-x'$ and $f:y-y'$) such that

$$(x,e)\leqslant_{\text{lex}}(y,f)\leqslant_{\text{lex}}(x',e)\leqslant_{\text{lex}}(y',f). \tag{1}$$

From a planar drawing scheme D of G, we get a planar drawing of G as follows. We make G into a tree H by cutting every edge $e\in E_G-E_T$ into two. More precisely, if e links x and y and, without loss of generality, we have $\mu(x)e<_{\text{lex}}\mu(y)e$ (otherwise we interchange x and y; we have $x\neq y$ since G has no loop), we introduce in G two new vertices $\bar{e}_1$ and $\bar{e}_2$ and we replace e by two edges: $e_1:x-\bar{e}_1$ and $e_2:y-\bar{e}_2$. We get a rooted tree H with root r, $V_H=V_G\cup\{\bar{e}_1,\bar{e}_2/e\in E_G-E_T\}$ and $E_H=E_T\cup\{e_1,e_2/e\in E_G-E_T\}$.

We make H into an ordered tree (ordered by $(\leqslant'_v)_{v\in V_G}$) by replacing in each $\leqslant_v$, each $e\in E_G-E_T$ by the relevant e_i; formally, for edges f,f' of H outgoing of v we let

$$f\leqslant'_v f' \quad \text{if and only if} \quad g\leqslant_v g'$$

where $g=e$ if $f=e_i$, $e\in E_G-E_T$, $i\in\{1,2\}$ and $g=f$ otherwise (and similarly for f' and g'). The notations μ' and $\leqslant'_{\text{lex}}$ will refer to H. In particular

$\bar{e}_i\leqslant'_{\text{lex}}\bar{f}_j$
if and only if $\mu(x)e_i\leqslant'_{\text{lex}}\mu(y)f_j$ (since $\mu'(\bar{e}_i)=\mu(x)e_i$
where e_i links x and $\bar{e}_i$ in H and, similarly, $\mu'(\bar{f}_j)=\mu(y)f_j$)
if and only if $\mu(x)e<_{\text{lex}}\mu(y)f$ (by the definition of the orderings
$\leqslant'_v$ from $\leqslant_v$)
if and only if $(x,e)<_{\text{lex}}(y,f)$.

Let P be a planar drawing of H respecting the linear orders $\leqslant'_v$. Since D is a planar drawing scheme, by this observation, we have in H no 4-tuple of leaves $\bar{e}_1,\bar{e}_2,\bar{f}_1,\bar{f}_2$ with $e,f\in E_G-E_T$ and

$$\bar{e}_1\leqslant'_{\text{lex}}\bar{f}_1\leqslant'_{\text{lex}}\bar{e}_2\leqslant'_{\text{lex}}\bar{f}_2. \tag{2}$$

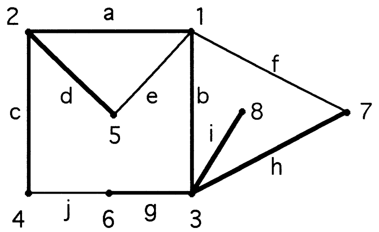

Fig. 2(a).

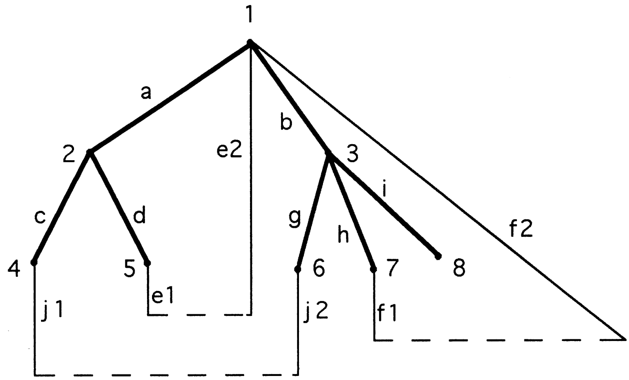

Fig. 2(b).

Note that $\leqslant'_{\text{lex}}$ is the linear order of the leaves of H in the drawing P. Hence we can add to P new segments (not necessarily straight) representing new edges e' between $\bar{e}_1$ and $\bar{e}_2$ (for all $e \in E_G - E_T$), that do not cross one another and do not cross those of P. We obtain a planar drawing of a supergraph H' of H. Since each path of H' of the form $(\bar{e}_1, e', \bar{e}_2)$ is a subdivision of the edge e of G, we get actually a planar drawing of G. This drawing is a planar drawing of the map $M(D) := (G, sigma)$ of G where for every $v \in V_G, sigma(v)$ is the set of pairs $(e, e') \in E_G(v) \times E_G(v)$ such that either e' is the sucessor of e in $\leqslant_v$, or e links v to its father in T and e' is the $\leqslant_v$-smallest element of $E_{G/T}(v)$, or e' links v to its father in T and e is the $\leqslant_v$-largest element of $E_{G/T}(v)$.

As an example, consider the graph G of Fig. 2a; let T be its spanning tree with root 1 and edges a, c, d, b, g, h, i (in bold). It defines a drawing scheme if we associate with its nodes 1, 2, 3 the orders $a \to e \to b \to f, c \to d$, and $g \to h \to i$, respectively (the arrow represents the successor function, like for circuits).

Fig. 2b shows the graph H' of the previous construction, with the new edges e', j', f' in broken lines. The corresponding drawing is homeomorphic to the one of Fig. 2a.

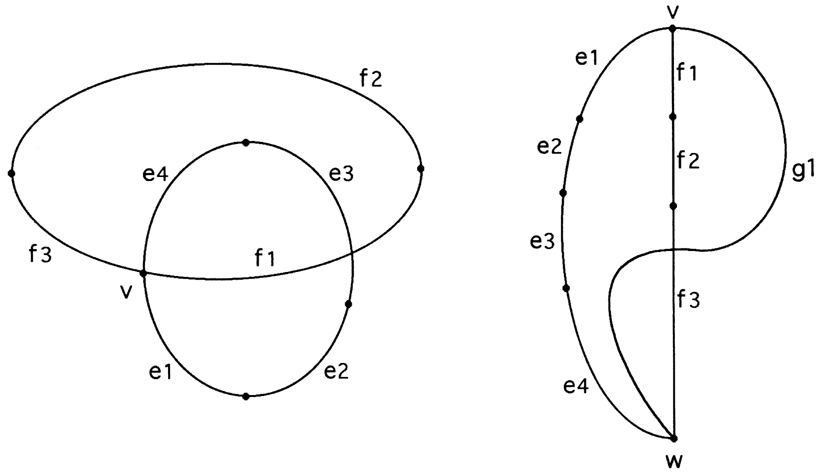

Fig. 3.

2.5. Planar maps: a combinatorial characterization

Let $M=\langle G, sigma\rangle$. A *pair of crossing cycles* in M is a pair of paths (p,q) both from some vertex v to itself, having no other common vertex than v such that $p=(e_1,e_2,\ldots,e_n)$, $q=(f_1,f_2,\ldots,f_m)$, $n,m\geqslant 2$, $e_1 \ll_v f_1 \ll_v e_n$ and $e_n \ll_v f_m \ll_v e_1$, where $\ll_v$ is relative to $sigma(v)$ (as in the beginning of this section). A *triple of crossing paths* in M is a triple of paths (p,q,r) from v to w (where $w\neq v$) having no other vertex in common than v and w and such that $p=(e_1,\ldots,e_n)$, $q=(f_1,f_2,\ldots,f_m)$, $r=(g_1,\ldots,g_t)$, $n,m,t\geqslant 1, e_1 \ll_v f_1 \ll_v g_1$ and $e_n \ll_w f_m \ll_w g_t$

Fig. 3 illustrates these two notions with $n=4, m=3, t=1$.

Proposition 2.2. *A map is planar if and only if it has no pair of crossing cycles and no triple of crossing paths.*

Proof. The condition is clearly necessary. Let us now assume that $M=(G, sigma)$ satisfies it. It suffices to prove the planarity of each connected component of M since the condition holds for each connected component. Thus, we assume that M is connected, and of course, not reduced to a single vertex.

Let us choose $r\in V_G$ and $h\in E_G(r)$. Let $N=(M,r,h)$ and $T=T(N)$ (defined after Lemma 2.1). Then $D:=(T,(\leqslant_v)_{v\in V_G})$ is a drawing scheme of G where for $e,e'\in E_{G/T}(v)$:

$$e \leqslant_v e' \quad \text{if and only if} \quad \mu(v)e \leqslant_{\text{lex}} \mu(v)e'.$$

Two claims will prove that D is a planar drawing scheme. Note immediately that $N=M(D)$.

Claim 2.2.1. *If e in $E_G - E_T$ links x and y, then, either $y <_T x$ or $x <_T y$. In the first case, we have $\mu(x)e <_{\text{lex}} \mu(y)e$.*

Proof. Since T is a DFS tree of G (by Lemma 2.1) and G has no loop, either $y <_T x$ or $x <_T y$. Let us assume the first. We let $p = (f_1, \dots, f_m) = \mu(x) - \mu(y)$ (i.e. $\mu(x) = \mu(y)p$ and p is a path in T from y to x). We have $m \geqslant 1$ since $x \neq y$. Let us consider the path $q = \mu(y)e$ from r to x. We have $\mu(x) <_N q$, (by the definition of $\mu(x)$) and we have three cases (we let $N = (M, r, h)$) : either $y = r$ and $f_1 = h \neq e$, or $y = r$ and $h \ll_y f_1 \ll_y e$, or $y \neq r$ and $g \ll_y f_1 \ll_y e$ where g is the last edge of $\mu(y)$. In each case we have $\mu(x)e = \mu(y)pe <_N \mu(y)e$. $\square$

Claim 2.2.2. *There are no two edges $e, f \in E_G - E_T$ and no four vertices x, z, y, t such that*

$$(x, e) <_{\text{lex}} (z, f) <_{\text{lex}} (y, e) <_{\text{lex}} (t, f). \tag{3}$$

Proof. Assume there are two such edges $e : x - y$ and $f : z - t$. By Claim 2.2.1 we have $y <_T x$ and $t <_T z$ since $\mu(x)e <_{\text{lex}} \mu(y)e$ and $\mu(z)f <_{\text{lex}} \mu(t)f$.

There are several cases to consider, each of which leads to a contradiction. We let $\mu(y) = (g_1, \dots, g_n)$, $(n \geqslant 0)$, and $\mu(x) = \mu(y)(h_1, \dots, h_m)$, $(m \geqslant 1)$, where $e, f \notin \{g_1, \dots, g_n, h_1, \dots, h_m\}$. We recall that $N = (M, r, h)$.

Case 1: $x = z$ and $y = t$

We first assume that $y \neq r$. By (3) we have $h_m \ll_x e \ll_x f$ and $g_n \ll_y e \ll_y f$. Let us compare h_1 with g_n, e and f. We have $g_n \ll_y h_1 \ll_y e$ because otherwise $g_n \ll_y e \ll_y h_1$ and $\mu(y)e <_N \mu(x)$, and $\mu(x)$ would not be $\leqslant_N$-minimal as a path from r to x. Hence, we have $h_1 \ll_y e \ll_y f$ and $((h_1, \dots, h_m), e, f)$ is a triple of crossing paths in M from y to x contradicting the assumption on M.

It remains to consider the case where $y = r$ (i.e., $n = 0$). Note the $e \neq h$ since h is an edge of T. We have by (3) $h_m \ll_x e \ll_x f$ and $h \ll_y e \ll_y f$. We also have $h_1 \ll_y e \ll_y f$. This is clear if $h = h_1$. Otherwise, we would have $e \ll_y h_1 \ll_y f$, whence $h \ll_y e \ll_y h_1$ and $\mu(x)$ would not be $\leqslant_N$-minimal (because e would be $\leqslant_N$-smaller than $\mu(x)$). We also have a contradiction.

Case 2: $x = z$ *and* $t <_T y$.

We have $\mu(y) - \mu(t) = (g_{n-i}, \dots, g_n)$ for some $1 \leqslant i < n$. As in Case 1 we have $h_m \ll_x e \ll_x f$ and $g_n \ll_y h_1 \ll_y e$, hence $h_1 \ll_y e \ll_y g_n$. We have now the triple of crossing paths $((h_1, \dots, h_m), e, (g_n, g_{n-1}, \dots, g_{n-i})f)$ from y to x. Contradiction.

Case 3: $x = z$ *and* $y <_T t$.

The condition $y <_T t$ implies that $\mu(t) = \mu(y)(h_1, \dots, h_\ell)$ for some $\ell \geqslant 1$ hence $\mu(t)f <_{\text{lex}} \mu(y)e$ (since $\mu(y)h_1 <_{\text{lex}} \mu(y)e$)), which contradicts (3).

Case 4: $z <_T x$ *and* $y = z$.

We have $t <_T y$ (because if $t = y$ then f is a loop) and we let $\mu(y) := \mu(t)(g_{n-i}, \dots, g_n)$. Thus $((h_1, \dots, h_m)e, f(g_{n-i}, \dots, g_n))$ is a pair of crossing cycles with common vertex $y = z$, because $g_n \ll_y h_1 \ll_y e$, and $h_1 \ll_y f \ll_y e$. This contradicts the hypothesis on M.

Case 5: $z <_T x$ *and* $z <_T y$.

From (3) we have $\mu(z)f <_{\mathrm{lex}} \mu(y)e$ hence $\mu(z)f <_{\mathrm{lex}} \mu(z)g$ where g is the first edge in $\mu(y) - \mu(z)$ (because $z <_T y$, $\mu(z)$ is a prefix of $\mu(y)$). We have also $\mu(z)f <_{\mathrm{lex}} \mu(x)e$ since g is also the first edge of $\mu(x) - \mu(z)$. Hence $(z, f) <_{\mathrm{lex}} (x, e)$ but this contradicts (3) (which yields $(x, e) <_{\mathrm{lex}} (z, f)$. This case cannot happen.

Case 6: $z <_T x$ *and* $y <_T z$.

Thus we have $t \leqslant_T y$ (otherwise if $y <_T t$ we have $(t, f) <_{\mathrm{lex}} (y, e)$). We let $\mu(z) - \mu(y) = (h_1, \dots, h_p)$, $p < m$ (we recall that $\mu(x) - \mu(y) = (h_1, \dots, h_m)$). We obtain a triple of crossing paths $((\mu(x) - \mu(z))e,\ f(\mu(y) - \mu(t)),\ (h_p,, \dots, h_1))$ from z to y. Contradiction.

Case 7: $x <_T z$.

The argument is similar to the previous cases.

Case 8: x and z are incomparable with respect to $\leqslant_T$.

We let u be the $\leqslant_T$-largest common ancestor of x and z. We have $y \leqslant_T u$ (otherwise $(y, e) <_{\mathrm{lex}} (z, f)$), and $t \leqslant_T y$ (otherwise $(t, f) <_{\mathrm{lex}} (y, e)$). If $u = y$, we get a pair of crossing cycles with common vertex u and if $y <_T u$, we get a triple of crossing paths from u to y. Contradiction.

In all cases we get a contradiction. Hence (3) cannot happen and this proves the claim. □

We can now complete the proof of Proposition 2.2. Since D is a planar drawing scheme, there is a planar drawing of $M(D)$. But $N = M(D)$. □

2.6. *Logical representation of maps*

Let $M = \langle G,\ sigma \rangle$ be a map. We let $|M|_2$ be the relational structure $\langle V_G \cup E_G, inc_G, sig_M \rangle$ where $inc_G = \{(e, x, y)/e \in E_G,\ e$ links x and $y\}$ and $sig_M = \{(x, e, f)/x \in V_G,\ e, f \in E_G(x),\ (e, f) \in sigma(x)\}$. (Thus $(e, x, y) \in inc_G$ implies $(e, y, x) \in inc_G$ since we deal with undirected graphs).

The structure $|M|_2$ contains the structure $|G|_2 = \langle V_G \cup E_G, inc_G \rangle$ which represents G. The component inc_G is actually redundant in $|M|_2$ because an element y of its domain is an edge iff there exists in sig_M a triple of the form (x, y, z); furthermore, the ends of y are the elements x in such triples. Hence, inc_G is definable from sig_M by a first-order formula. However, it is convenient to keep it in order to handle $|M|_2$ as an enrichment of $|G|_2$.

An ordered map $M = \langle G,\ (\leqslant_v)_{v \in V_G} \rangle$ will be represented by the structure $|M|_2 = \langle V_G \cup E_G,\ inc_G,\ \leqslant\text{-}sig_M \rangle$ where $\leqslant\text{-}sig_M[v] = \leqslant_v$ for each vertex v. (If R is a ternary relation, we denote by $R[x]$ the set of pairs (y, z) such that (x, y, z) belongs to R.)

Proposition 2.3. *Let $M = \langle G,\ sigma \rangle$ be a map represented by the structure $|M|_2$.*

(1) *The planarity of M is expressible by the negation of an existential MS-formula.*

(2) *Let $r \in V_G$ and $h \in E_G(r)$. The linear order on V_G associated with (M, r, h) is definable by an MS-formula taking r, h as parameters.*

Proof. (1) That the nonplanarity of M is expressible by an MS-formula is a straightforward consequence of Proposition 2.2. For such a formula of the form $\exists X_1,\ldots,X_k.\varphi$, where φ is first-order, to be constructed a little care is required. The notion of "path" is not first-order definable but we will manage with the weaker notion of "quasi-path" which is first-order.

Let H be an undirected graph; let $x,y\in V_H$ and $X\subseteq E_H$. We say that X is a *quasi-path from* x to y if and only if $x\neq y$, each of x and y is incident to a unique edge in X, and any vertex v incident to an edge of X and not in the set $\{x,y\}$ is incident to exactly two edges of X. These conditions are first-order; since, we deal with finite graphs they express that X is the set of edges of a path linking x and y, augmented possibly by those of pairwise disjoint cycles that are also disjoint from the path. We denote by $Q(X,x,y)$ the corresponding first-order formula. The existence of three disjoint paths in M between x and $y\neq x$ can be expressed as follows:

$$\exists X_1,X_2,X_3,x,y[\theta_1(X_1,X_2,X_3,x,y)\wedge x\neq y],$$

where θ_1 is

$Q(X_1,x,y)\wedge Q(X_2,x,y)\wedge Q(X_3,x,y)\wedge$ "X_1,X_2,X_3 are pairwise disjoint"
$\wedge$ "no vertex except x or y belongs to an edge of X_i and one of X_j for $i\neq j$".

A first-order formula $\theta_2(X_1,X_2,X_3,x,y,e_1,e_2,e_3,e'_1,e'_2,e'_3)$ can express that e_i is the unique edge in X_i incident to x, and e'_i is the unique edge in X_i incident to y, for all $i=1,2,3$. Assuming this, it remains to express that

$$e_1\ll_x e_2\ll_x e_3 \quad\text{and}\quad e'_1\ll_y e'_2\ll_y e'_3$$

in order to get a triple of crossing paths from x to y. This can be done by an adaptation of the quasipath trick. By a similar construction, an existential MS-formula can express the existence of a pair of crossing cycles. Hence, by Proposition 2.2, nonplanarity can be expressed by an existential MS-formula.

(2) The paths can be specified by sets of edges. In order to express that a path from r to x defined by a set of edges X is smaller than a path from r to y defined by a set of edges Y, we write that, either X is a subset of Y, or $X=U\cup V, Y=U\cup W$ where U defines a path from r to some u, V and W are paths from u to x and y, respectively, and the first edge of V is strictly "smaller" than the first edge of W. From these hints, the MS definition of the desired linear order can be obtained by straightforward translation of the definition. □

Remark 2.4. It is proved in [6] and by using Kuratowski's theorem that the nonplanarity of a graph is expressible by an existential MS-formula. The proof also uses the notion of a quasi-path.

3. Planar 3-connected graphs

Our objective is to MS-define in a structure $|G|_2$ representing a planar graph, a ternary relation sig_M such that $|M|_2 = \langle V_G \cup E_G, inc_G, sig_M \rangle$ defines a planar map of G. We consider 3-connected simple planar graphs in this section. The general case is considered in Section 4.

By a theorem of Whitney, a simple planar 3-connected graph G has a unique embedding in the plane. Unicity holds up to homeomorphism (see for instance Diestel [Die]). It follows that there are exactly two planar maps M of G. (For every planar map M we have also the planar map M^{-1}; it corresponds to the same class of homeomorphic drawings but is different as a relational structure.)

Theorem 3.1. *For every simple planar* 3*-connected graph* G, *the two structures* $|M|_2$ *representing its planar maps are MS-definable in* $|G|_2$.

This result is no longer true for simple planar 2-connected graphs as proved at the end of this section.

Let G be a simple planar 3-connected graph with planar map $M = \langle G, sigma \rangle$. Let $r \in V_G$, $h \in E_G(r)$, $N = (M, r, h)$ and $T = T(N)$. We let L be the set of edges of T and $R = E_G - L$. We let D be the drawing scheme $(T, (\leqslant_v)_{v \in V_G})$, as defined in Section 2.4. We recall from Lemma 2.1 that T is a DFS tree.

We prove that we can reconstruct the orders $\leqslant_v$ from G, r, h, and L, and thus the mapping *sigma* (whence M). We prove later that this reconstruction can be done by MS formulas taking r, h and L as parameters.

In order to describe the construction, we let G' be the orientation of G defined as follows: the edges of L are directed as in T (which is a rooted ordered tree); an edge in R is directed from u to v such that $v <_T u$ (i.e., v is on the branch of T from r to u; this is possible since T is DFS and G has no loop).

Hence G' has circuits (unless G is a tree). With respect to this orientation, we let $out(v)$ denote the set of edges of G with source v, and $in(v)$ denote the set of edges of G with target v. In the next lemmas, we denote by r' the unique son of the root r of T, (see Fact 1.4).

Lemma 3.2. *Let* G *be a simple planar* 3*-connected graph with planar map* $M = \langle G, sigma \rangle$. *Let* $r \in V_G$, $h \in E_G(r)$, $N = (M, r, h)$, $T = T(N)$, r' *be the son of* r *in* T, L *be the set of edges of* T *and* $R = E_G - L$. *Let* $v \in V_G$ *be such that* $v \notin \{r, r'\}$ *and* L' *be such that* $\emptyset \neq L' \subseteq out(v) \cap L$. *There exist paths* p, p' *in* $L'L^*R$ *such that* p *links* v *to* u, p' *links* v *to* u', $u, u' <_T v$ *and* $u \neq u'$.

(Since we consider a path as a sequence of edges, we can use regular expressions over sets of edges to specify paths.)

Proof. Let H be the subgraph of G' consisting of all directed paths in $L'L^*R$. Let K be the subgraph of G' spanned by $E_{G'} - E_H$. It is nonempty since it contains at least the path in T from r to v. Hence $V_H \cap V_K$ consists of v and vertices $u_1, u_2, \ldots, u_n$ on this path (by the fact that T is a depth-first spanning tree). If $n \geqslant 2$ we are done, with $u = u_1$, and $u' = u_2$.

If $n \leqslant 1$ we will get a contradiction as follows. We first note that $V_G - V_H$ is not empty: it contains at least r or r' (otherwise $n \geqslant 2$); we note also that $V_G - V_K$ is nonempty: it contains the target of some $e \in L'$. Then the set $\{v\}$ if $n = 0$, or the set $\{v, u_1\}$ if $n = 1$ separates G. This contradicts the 3-connectivity assumption on G. □

The following lemma shows that the restriction of $<_v$ to $out(v)$ can be defined in terms of G, r, h, and L.

Lemma 3.3. *Let G, T, L, R be as in Lemma 3.2. Let $v \in V_G$ and $e, e' \in out(v)$ with $e \neq e'$. There exist two paths, p in $\{e\} \cup eL^*R$ and p' in $\{e'\} \cup e'L^*R$ such that p links v to u, p' links v to u', $u <_T v$, $u' <_T v$ and $u \neq u'$. Then $e <_v e'$ if and only if $u <_T u'$.*

Proof. We first note from Fact 1.4 that $v \notin \{r, r'\}$. We now consider $e : v \to w$ and $e' : v \to w'$. We have $w \neq w'$ since G is simple.

Case 1: $e, e' \in R$.

We let $p = e$, $u = w$ and $p' = e'$, $u' = w'$ and we are done.

Case 2: $e \in L$, $e' \in R$.

We let $u' = w'$ and $p' = e'$. If $out(w) \cap L \neq \emptyset$, by Lemma 3.2 applied to w and $L' = out(w) \cap L$, there are two paths p_1, p_2 in L^+R from w to u_1, u_2, respectively, with $u_2 <_T u_1 <_T w$. At least one of u_1, u_2 is not equal to u', say u_i. We let then $p = e\, p_i$ and $u = u_i$ and we are done.

Case 3: $e, e' \in L$ (see Fig. 4).

As in Case 2 we have p_1, p_2 and similarly p'_1, p'_2, respectively, from w' to u'_1, u'_2. We have $u_2 <_T u_1 <_T v$, $u'_2 <_T u'_1 <_T v$. We can choose $u \in \{u_1, u_2\}$ and $u' \in \{u'_1, u'_2\}$ such that $u \neq u'$, say $u = u_i$ and $u' = u'_j$. We let then $p = ep_i$, $p' = e'p'_j$.

In all the three cases, we have $e <_v e'$ if and only if $u <_T u'$. Because if $e <_v e'$ and $u' <_T u$ then we have a triple of crossing paths from v to u, $(p, p'(\mu(u) - \mu(u')), q)$ where q is the reverse path of $\mu(v) - \mu(u)$ linking u to v. This completes the proof. □

The next lemma is similar and shows that the restriction of $<_v$ to $in(v) \cap R$ can be defined from G', r, h, and L.

Lemma 3.4. *Let G, T, L, R be as in Lemma 3.2. Let $v \in V_G$, let $e, e' \in in(v) \cap R$, $e \neq e'$. Let u and u' be the sources of e and e', respectively. Then $e <_v e'$ if and only if either $u <_T u'$ or there is a vertex $w \geqslant_T v$ and paths p, p' linking w to v, such that $p \in gL^*e$, $p' \in g'L^*e'$. Furthermore, $e <_v e'$ iff either $w \neq v$ and $g' <_w g$ or $w = v$ and $g <_v g'$.*

Proof. We cannot have $u = u'$ since G is simple. If u and u' are on a same branch of T then either $u <_T u'$ and $e <_v e'$ or $u' <_T u$ and $e' <_T e$ since otherwise, we get

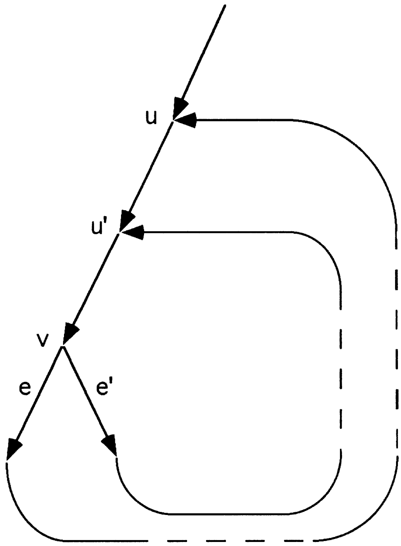

Fig. 4.

a triple of crossing paths with common vertices v, and either u or u'. Assume now they are not. Then $v <_T u$, $v <_T u'$ and there is a $<_T$-largest w such that $w <_T u$, $w <_T u'$. We have thus unique paths $p \in gL^*e$ and $p' \in g'L^*e'$ from w to v for some g, $g' \in out(w) \cap L$. Here we have two cases.

Case 1: $v \neq w$. See Fig. 5.

If we have $g <_w g'$ and $e <_v e'$ then we have a triple of crossing paths from w to v namely (p, p', q) where q is the reversal of $\mu(w) - \mu(v)$ linking v to w. Hence, we have $g' <_w g$ if and only if $e <_v e'$.

Case 2: $v = w$.

We have thus two cycles p, p' from v to v such that, $p \in gL^*e$ and $p' \in g'L^*e'$. We have $g <_v e$ and $g' <_v e'$ by the definition of T as $T(M, r, h)$.

If $g <_v g'$ and $e' <_v e$ then we have $g <_v g' <_v e' <_v e$. We now apply Lemma 3.2 with $L' = \{g'\}$ (since $g \neq g', v \notin \{r, r'\}$). We obtain the existence of a path $q \in g'L^*R$ from v to s for some $s <_T v$. But $(p, q(\mu(v) - \mu(s)))$ is a pair of crossing cycles with common vertex v, contradicting the hypothesis on M. Hence $g <_v g'$ if and only if $e <_v e'$. □

Finally, in the next lemma, we compare with respect to $<_v$ an edge of $out(v)$ with the one of $in(v) \cap R$. If $e \in R$ and $e : u \to v$, we denote by $org(e)$ the first edge of the path in T from v to u. Hence $org(e) \in out(v) \cap L$.

Lemma 3.5. *Let $v \in V_G$, $e \in in(v) \cap R$, $g \in out(v)$. Then $e <_v g$ if and only if $org(e)$ $<_v g$.*

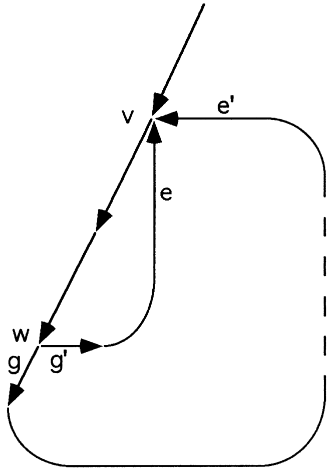

Fig. 5.

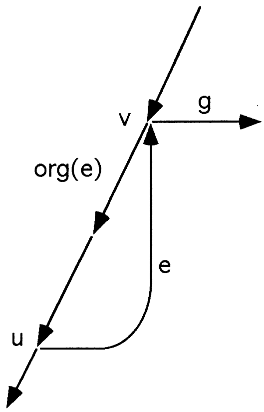

Fig. 6.

Proof. We compare org(e) with g and this gives three cases.

Case 1: $g = \text{org}(e)$.

We have thus $g <_v e$ by the definition of T as a depth-first spanning tree. Hence neither $e <_v g$ nor $\text{org}(e) <_v g$ holds.

Case 2: $g <_v \text{org}(e)$.

By Case 1, since we have $\text{org}(e) <_v e$ and by transitivity, we also have $g <_v e$. Hence again neither $e <_v g$ nor $\text{org}(e) <_v g$ holds.

Case 3: $\text{org}(e) <_v g$ (see Fig. 6).

Since $g \neq \text{org}(e)$, this implies that $v \notin \{r, r'\})$ by Fact 1.4. Assume that we have $g <_v e$. We have at least one path $p \in gL^*R$ from v to w with $w <_T v$ (by Lemma 3.2).

Since we have also a path, say p' in $\mathrm{org}(e)L^*e$ from v to itself, we would have a pair of crossing cycles $(p', p(\mu(v) - \mu(w)))$. This contradicts the initial hypothesis that M is a planar map. Hence $e <_v g$. □

Proof of Theorem 3.1. The conditions of Lemmas 3.3–3.5 can be combined in order to give a necessary and sufficient condition on (v, e, e') with $e, e' \in E(v)$ to satisfy $e <_v e'$. This condition can be expressed by means of an MS-formula $\theta(L, r, h, v, e, e')$. However, this formula expresses correctly that $e <_v e'$ under the assumption that L is indeed the set of edges of $T(M, r, h)$ where (M, r, h) is a 1-map of G. It remains to show that an auxiliary formula $\alpha(L, r, h)$ can express that a given triple (L, r, h) is indeed "correct" in the above sense.

We take as $\alpha(L, r, h)$ an MS-formula expressing the following conditions:

(1) $r \in V_G$, $h \in E_G(r)$, L is the set of edges of a DFS tree T of G with root r and $h \in L$,

(2) for each $v \in V_G$, the binary relation R_v on E_G defined by "$(e, e') \in R_v$ if and only if $\theta(L, r, h, v, e, e')$" is a linear order $\leqslant_v$ on the set $E_{G/T}(v)$,

(3) there are no two edges $e, f \in E_G - L$ where e links x and y, f links z and t such that:

$$(x, e) <_{lex} (z, f) <_{lex} (y, e) <_{lex} (t, f)$$

where $<_{lex}$ is defined as in the beginning of Section 2 from T (which is a DFS tree by (1)) and $\leqslant_v$ (which is a linear order on each set $E_{G/T}(v)$ by (2)). These conditions express that $(T, (R_v)_{v \in V_G})$ is a planar drawing scheme D of G.

It is now easy to build from θ an MS-formula $\theta'(L, r, h, v, e, e')$ which defines $sig_{M(D)}$. □

This construction depends on parameters L, r, h. See Remark 3.8 below for a discussion. We now review its steps in order to establish the following more precise version of Theorem 3.1 announced in [6, Theorem 5.1]. The language $\mathrm{FO}^{\mathrm{TC}}$ is defined in Section 1 and is motivated in the Introduction.

Theorem 3.6. *The planarity of a simple* 3*-connected graph* G *can be expressed by a formula* ψ *of the form* $\exists L, r, h.\alpha(L, r, h)$ *where* α *is* $\mathrm{FO}^{\mathrm{TC}}$. *There exists a formula* θ *in* $\mathrm{FO}^{\mathrm{TC}}$ *such that whenever* L, r, h *satisfy* α, *the ternary relation* $\{(v, e, f)/|G|_2 \models \theta(L, r, h, v, e, f)\}$ *is* sig_M *where* M *is a planar map of* G.

Proof. The proof is straightforward by inspection of the proof of Theorem 3.1 as soon as one has noted that the following properties are $\mathrm{FO}^{\mathrm{TC}}$:

(P1) L is the set of edges of a DFS tree of G with root r,

(P2) $u <_T v$ where T is the tree defined by L and r assumed to satisfy (P1). □

This construction yields another class of graphs on which a linear order is MS-definable. See [5] for a general study of this question, and Grohe [14] for a different construction for planar 3-connected graphs using another logical language. We recall from Section 1 that $|G|_1$ is a logical representation of a graph where the edges are not members of the domain.

Corollary 3.7. *One can MS-define a linear order on V_G for every simple 3-connected planar graph G represented by $|G|_1$.*

Proof. We compose the following MS-definable transductions: $\tau_1 : |G|_1 \to |G|_2$; it exists since G is simple and planar by [2], $\tau_2 : |G|_2 \to |M|_2$ where M is a planar map of g: it exists by Theorem 3.1, $\tau_3 : |M|_2 \to \langle V_G, \leqslant_N \rangle$ where $\leqslant_N$ is the linear order associated with $N = (M, r, h)$ (for arbitrary $r \in V_G$ and $h \in E_G(r)$) by Proposition 2.3. Hence $\tau_3 \circ \tau_2 \circ \tau_1$ produces from $|G|_1$ a linear order of V_G, definable by an MS-formula using parameters r, h and L. □

Remark 3.8. There are only two planar maps M_1 and M_2 of G if G is simple, planar and 3-connected. However, we have many choices of r, h, L satisfying $\alpha(L, r, h)$ that yield the same map, either M_1 or M_2. In order to distinguish $M_1 = (G, sigma_1)$ from the reverse map $M_2 = M_1^{-1}$, we need only choose another edge $h_1 \in E_G(r)$ such that $(h, h_1) \in sigma_1(r)$.

In this case M_1 is the unique map $(G, sigma_1)$ such that $(h, h_1) \in sigma_1(r)$, and M_2 is the unique one $(G, sigma_2)$ such that $(h_1, h) \in sigma_2(r)$. Hence the map M_1 can be obtained from (r, h, h_1) and the map M_2 from (r, h_1, h) by the same MS-transduction.

Assume now that G is a simple graph given with a linear order $\leqslant$ on V_G and a fixed vertex r. We can MS-define from $\leqslant$ a linear order on edges (by ordering lexicographically their pairs of ends), denoted by $\leqslant$, and we can choose for h the $\leqslant'$-smallest edge in $E_G(r)$ and h_1 is thus determined in a unique way. It follows that the MS-transduction of Theorem 3.1 can be transformed into one using r and $\leqslant$ as the only inputs in addition to $|G|_2$. We thus have the following corollary of Theorem 3.1 (to be used in Section 4):

Corollary 3.9. *There exists an MS-transduction without parameters that associates with the structure $(|G|_1, \leqslant, r)$ (where G is planar, simple and 3-connected, r is a vertex and $\leqslant$ is a linear order on V_G) a structure $|M|_2$ representing a planar map of G.*

Proposition 3.10. *There is no MS-transduction associating with every structure $|G|_2$ representing a simple planar 2-connected graph a structure $|M|_2$ such that M is a planar map of G.*

Proof. Let V be a set with cardinality at least 4; let x, y, z be pairwise distinct elements of V. Let $G(V, x, y, z)$ be the graph G with $V_G = V$ and the two edges $x - z$, $y - z$, and the edges $u - x$, $u - y$ for every $u \in V_G - \{x, y, z\}$; the graph G is simple, planar, 2-connected. The mapping associating $|G(V, x, y, z)|_1$ with V is an MS-transduction T

with parameters x, y, z (it is easy to define formally). Assume the existence of an MS-transduction ω associating with $|G|_2$ a structure $|M|_2$ such that M is a planar map of G for any simple planar 2-connected graph G; then, we would get an MS-transduction λ associating with V a linear order of this set, by taking the composition of τ, the MS-transduction β defined in [2] associating with $|G|_1$ a structure isomorphic to $|G|_2$ for every simple planar graph G, the MS-transduction ω and then the transduction of Proposition 2.3 associating with every connected map a linear order of the set of vertices. But we know from [5] that no MS-transduction like λ can exist (even allowing parameters). This contradiction concludes the proof. □

4. Planar ordered graphs

The aim of this section is to establish the existence of an MS-transduction associating with $(|G|_2, \leqslant)$ (where G is a connected planar graph and $\leqslant$ is a linear ordering of E_G) a structure $|M|_2$ where M is a planar map of G.

In Section 3 we have given a construction for the special case of 3-connected graphs, without needing to use any ordering of edges or vertices. The linear ordering $\leqslant$ will be useful in the general case to define by MS-formulas a hierarchical decomposition of a connected planar graph G in terms of planar 3-connected subgraphs (called 3-*blocks* [18, 8]) and graph operations like substitution for an edge, series-composition and parallel-composition of 2-graphs. A map M of G is then obtained by appropriate combinations of those of the 3-connected pieces of the decomposition. Since the decomposition is MS-definable and since maps of the 3-blocks are MS-definable, the construction of M can be done by MS-formulas.

We first handle the decomposition of a connected graph in 2-connected components (also called *blocks* [18, 8]). Thus we reduce the general case to the special case where G is 2-connected. Then we handle the case of 2-connected graphs and we get the main theorem that we now state.

Theorem 4.1. *One can construct an MS-definable transduction that associates with* $(|G|_2, \leqslant)$ *a structure* $|M|_2$ *representing a planar map* M *of* G *where* G *is a connected planar graph and* $\leqslant$ *is a linear order of* E_G.

Lemma 4.2. *Let* G, H, K *be graphs, such that* $G = H \cup K$, *and* $H \cap K$ *consists of a single vertex* s. *Let* $N = \langle H, sigma\rangle$, $P = \langle K, sigma'\rangle$ *be planar maps. The pair* $M = \langle G, sigma''\rangle$ *where* $sigma''(v) = sigma(v)$ *if* $v \in V_H - \{s\}$, $sigma''(v) = sigma'(v)$ *if* $v \in V_K - \{s\}$, $sigma''(s)$ *is a merge of* $sigma(s)$ *and* $sigma'(s)$ *is a planar map. If* H *and* K *are 2-connected, then every planar map of* G *is of this form*

Proof. Let M be defined as in the statement. If it is not planar then it has a triple of crossing paths from v to v' or a pair of crossing cycles with common vertex v. There are several cases, all of which lead to a contradiction.

Case 1: $v = s$, $v' \in H$ (or K).

The three paths are all in H (or K) and thus N (or P) is not planar.

Case 2: $v \in H$, $v' \in K$, $s \notin \{v, v'\}$.

Impossible because the three paths should go through s.

Case 3: $v, v' \in V_H - \{s\}$ (or $V_K - \{s\}$).

The three paths are in H (or K) and thus N (or P) is not planar.

Hence, we cannot have 3 crossing paths. Assume now we have two crossing cycles.

Case 4: $v \in V_H - \{s\}$ (or $V_K - \{s\}$).

The two circuits are in H (or in K) and N (or P) is not planar.

Case 5: $v = s$.

The two circuits cannot be both in H or in K (see Case 4). Hence one is in H and the other is in K. But the condition that $sigma''(s)$ is a merge of $sigma(s)$ and $sigma'(s)$ excludes that they are actually crossing. (See Fig. 3: we can take H consisting of e_1, e_2, e_3, e_4, K of f_1, f_2, f_3, but the circuit $e_1 \to f_1 \to e_4 \to f_3 \to e_1$ is not a merge of $e_1 \to e_4 \to e_1$ and $f_1 \to f_3 \to f_1$.)

For proving the second assertion, consider a planar map M of G, let N be the H-induced submap of M and P its K-induced submap. If H and K are 2-connected and $sigma_M(s)$ is not a merge of $sigma_N(s)$ and $sigma_P(s)$ then this means that we have in $sigma_M(s)$ a subcircuit $e \to f \to e' \to f' \to e$ where $e, e' \in E_H(s)$ and $f, f' \in E_K(s)$. By the 2-connectivity assumptions, we have a path in H from s to itself of the form $(e, e_1, \ldots, e_k, e')$ and one in K from s to itself of the form $(f, f_1, \ldots, f_l, f')$. They form a pair of crossing circuits in M, contradicting its planarity. □

If one of H and K is not 2-connected then the second assertion of the lemma does not hold: take H consisting of a circuit (a, b, c) from s to s and K consisting of two edges d and e incident to s, together with $sigma_M(s) = a \to d \to c \to e \to a$; M is not a merge of N and P, but is planar. We recall that S_G is the set of separating vertices of G.

Lemma 4.3. *Let G be a graph; let $R \subseteq V_G \times E_G \times E_G$ be a relation such that, for each block B of G, the restriction $R[B]$ of R to $V_B \times E_B \times E_B$ defines a planar map $\langle B, R[B] \rangle$. Let us assume that $\leqslant$ is a linear order of E_G. One can MS-define from $|G|_2$, $\leqslant$ and R a relation sigma such that $\langle G, sigma \rangle$ is a planar map.*

Proof. We define $sigma = S_1 \cup S_2 \subseteq V_G \times E_G \times E_G$ as follows.

We define $S_1 = \{(v, e, f) \in R / v \in V_G - S_G\}$. Each vertex v of $V_G - S_G$ belongs to one and only one block B. Hence $\{(e, f)/(v, e, f) \in S_1\} = \{(e, f)/(v, e, f) \in R[B]\}$ and is a circuit on $E_G(v)$.

We will now define the circular order of edges around a separating vertex $s \in S_G$. We let $E = E_G(s)$; for $e, f \in E$ we let $e \approx f$ if and only if e and f belong to the same block, denoted $B(e)$; we let $T(e)$ be the circuit on $E_{B(e)}(s)$ defined as $\{(h, k)/R(s, h, k), h, k \in E_{B(e)}\}$.

For every $e \in E$, we let $\bar{e}$ be the $\leqslant$-smallest element $f \in E$ such that $e \approx f$; we let $e \leqslant_{\mathrm{E}} f$ if and only if either $e = f$ or $\bar{e} < \bar{f}$ (which means that e and f are not in the same block) or $\bar{e} = \bar{f}$ and $e = \bar{e}$ or $\bar{e} = \bar{f}$ and $\bar{e} \ll_{T(e)} e \ll_{T(e)} f$.

We now define the circuit $C = Circ(E_G(s), \leqslant_{\mathrm{E}})$.

Let $B(e_1), \ldots, B(e_m)$ be the list of blocks containing s ordered in such a way that $\bar{e}_1 < \bar{e}_2 < \cdots < \bar{e}_n$. It follows from the definition of $\leqslant_{\mathrm{E}}$ that C is obtained by merging successively $T(e_2)$ with $T(e_1)$, the resulting circuit with $T(e_3)$, the resulting circuit with $T(e_4)$, etc. Hence, on account of the restrictions of R to these blocks, which define planar maps, their merge defined in this way defines also a planar map of the union of these blocks, by Lemma 4.2.

We let $r(s)$ be the set of triples (s, e, f) defining the circuit C . We let S_2 be the union of the sets $r(s)$ for all vertices $s \in S_G$. We let $sigma = S_1 \cup S_2$. It follows from Lemma 4.2 and an induction on the tree of blocks of G that $\langle G, sigma \rangle$ is a planar map. It is clear that S_1 and S_2, whence $sigma$, are MS-definable in $|G|_2$ from R and $\leqslant$. □

Definition 4.1 (2-*dags*). Our objective is now to deal with 2-connected graphs. We need graphs with pairs of distinguished vertices. A 2-*graph* is a graph G given two distinguished vertices, $s_1(G)$ and $s_2(G)$. We let $G//e$ denote the graph G augmented with a new directed edge e from $s_1(G)$ to $s_2(G)$ ("new" means that $e \notin E_G$). A 2-*dag* is a directed acyclic graph such that every vertex is on a path from a unique vertex s of indegree 0 to a unique vertex t of outdegree 0. Hence, it is a 2-graph with source $s_1(G) = s$ and $s_2(G) = t$. We recall from [8] that every 2-connected graph has an orientation making into a 2-dag.

The substitution of a 2-dag G for a directed edge e in a graph is the result of the deletion of e and its replacement by a disjoint copy of G such that $s_1(G)$ is identified with $s(e)$ and $s_2(G)$ with $t(e)$. We denote by $K[G_1/e_1, \ldots, G_k/e_k]$ the result of the substitution of the 2-dags $G_1, \ldots, G_k$ in a directed graph K for $e_1, \ldots, e_k$. Two important special cases are those where K consists of two parallel edges (yielding the operation of *parallel-composition*, denoted by //), and of two edges in series (yielding the operation of *series-composition* denoted by $*$). We refer the reader to [8] for formal details.

A *standard decomposition* of a 2-dag G is a pair (T, g) where T is a rooted ordered tree with root r_T and g is a mapping from N_T (the set of nodes of T) to subgraphs of G satisfying the following conditions.

S1: Each $g(x)$ is a *factor* i.e., is a subgraph of G which is a 2-dag and has no vertex apart from the two ends incident with an edge of G not in $g(x)$; furthermore, $g(r_T) = G$,

S2: If x is a leaf of T then $g(x)$ is a 2-dag reduced to a single edge,

S3: If x is an internal node of T with sons $x_1, \ldots, x_k$, we have one of the following cases:

S3.1 $g(x) = g(x_1)//g(x_2)$, $k = 2$

S3.2 $g(x) = g(x_1) * g(x_2)$, $k = 2$

S3.3 $g(x) = K[g(x_1)/e_1, \ldots, g(x_k)/e_k]$

where K is a *substitution atom*, i.e., a 2-dag such that $K//e$ is 3-connected.

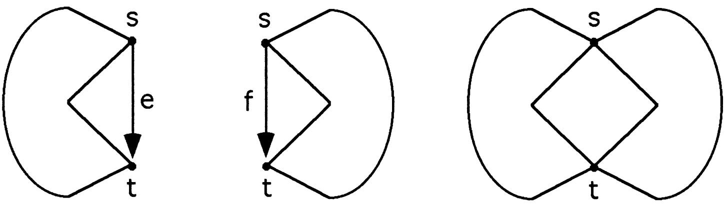

Fig. 7.

For logical constructions, we will handle a decomposition (T,g) of a 2-dag G as the following logical structure: $|(G,T,g)|_2 = \langle V_G \cup E_G \cup N_T,\ inc_G,\ son,\ bth,\ comp\rangle$ where N_T is the set of nodes of T (disjoint from $V_G \cup E_G$), $son(x,y)$ holds if and only if y is a son of x in T, $bth(y,z)$ holds if and only if y and z have the same father and z follows y in the sequence of sons (T is an ordered tree), $comp \subseteq N_T \times (V_G \cup E_G)$ is such that for every $x \in N_T$, the set $\{y \in V_G \cup E_G / comp(x,y) \text{ holds}\}$ is the set of vertices and edges of the factor $g(x)$ of G (called the *component* of the decomposition *defined* by x). We use the structure $|(G,\leqslant,T,g)|_2$ consisting of $|(G,T,g)|_2$ augmented with a linear order $\leqslant$ on the edges of G.

Definition 4.2 (*Map gluing*). Let $N=(H,sigma)$ and $P=(K,sigma')$ be maps where H and K are directed graphs. We assume that $E_H \cap E_K = \emptyset$, $V_H \cap V_K = \{s,t\}$, where $s=s(e)=s(f)$, $t=t(e)=t(f)$, for edges $e \in E_H$, and $f \in E_K$. We let $M=(G,sigma'')$ be the map, also denoted by $(N,e)\square(P,f)$, and defined as follows (see Fig. 7): $G = H \cup K - \{e,f\}$:

$sigma''(v) = sigma(v)$ if $v \in V_H - \{s,t\}$,
$sigma''(v) = sigma'(v)$ if $v \in V_K - \{s,t\}$,
$sigma''(s) = e_1 \to e_2 \to \cdots \to e_n \to f_1 \to \cdots \to f_m \to e_1$
  if $sigma(s) = e \to e_1 \to \cdots \to e_n \to e$ and
  $sigma'(s) = f \to f_1 \to \ldots \to f_m \to f$
$sigma''(t) = e'_1 \to e'_2 \to \cdots \to e'_p \to f'_1 \to \cdots \to f'_q \to e'_1$
  if $sigma(t) = e \to e'_1 \to \cdots \to e'_p \to e$
  $sigma'(t) = f \to f'_1 \to \cdots \to f'_p \to f$

Lemma 4.4. *If N and P are planar, then M is planar.*

Proof. We apply Proposition 2.2. The verification is lengthy because there are several cases but straightforward. □

Definition 4.3 (*2-Maps*). Let G be a 2-graph. A 2-map of G is a map M of G // e. We will also denote it by (M,e) in order to specify the special edge e. A planar 2-map of a 2-graph G is thus a planar map of G such that the two distinguished vertices belong to the same face.

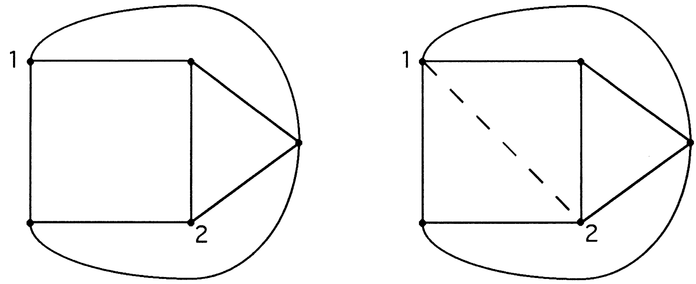

Fig. 8.

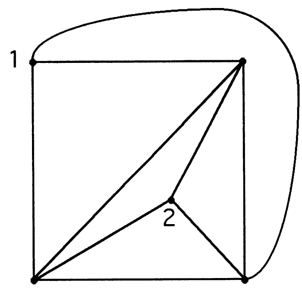

Fig. 9.

Fig. 8 shows a planar 2-graph G and a 2-map of G (represented as a planar map of $G//e$ where e is in broken line). We now consider an example.

The 2-graph G' of Fig. 9 (with $s_1(G')=1$, $s_2(G')=2$) has a planar map but no planar 2-map (because $G'//e$ is K_5 and hence has no planar map).

Definition 4.4 (*Map substitutions*). Let N be a 2-map of H with distinguished edge $h: s_1(H)\rightarrow s_2(H)$; let $e\in E_H$ (thus $e\neq h$); let P be a 2-map of a 2-graph K with distinguished edge f, such that $M=(N,e)\square(P,f)$ is well-defined. The pair (M,h) is a 2-map of $G=H[K/e]$, the 2-graph obtained as the result of the substitution in the 2-graph H of the edge e by the 2-graph K. We denote (M,h) by $N[P/e]$.

We write $M=N[P_1/e_1,\ldots,P_n/e_n]$ for $N[P_1/e_1][P_2/e_2]\ldots[P_n/e_n]$ where $e_1,e_2,\ldots,e_n\in E_H$ and N is a 2-map of H. It is easy to check that M is a 2-map of $H[K_1/e_1,\ldots,K_n/e_n]$ (if P_i is a 2-map of K_i).

Letting Q and S be the maps shown in Fig. 10 we obtain two special operations on 2-maps $P_1//P_2=Q[P_1/e_1,\ P_2/e_2]$ and $P_1*P_2=S[P_1/e_1,P_2/e_2]$.

Lemma 4.5. *If N, $P_1,\ldots,P_n$ are planar 2-maps, then $N[P_1/e_1,\ldots,P_n/e_n]$, $P_1//P_2$ and P_1*P_2 are planar 2-maps whenever they are defined.*

Proof. Immediate consequence of Lemma 4.4. □

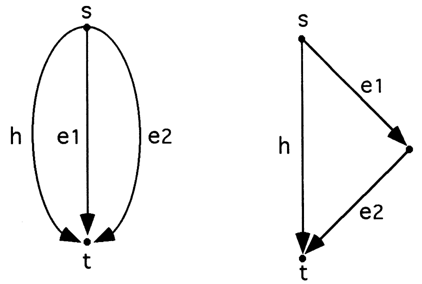

Fig. 10.

For 2-graphs, we have $G_1//G_2$ equal to $G_2//G_1$. For 2-maps, if $P_1//P_2$ is defined, then so is $P_2//P_1$ but they are not equal (except in degenerated cases) although they are 2-maps of the same 2-graph.

We will now formalize these notions in MS logic, by using ordered maps (see Section 1).

Definition 4.5 (*Ordered 2-maps*). An *ordered 2-map* of a 2-graph G is a structure $S = \langle V_G \cup E_G, inc_G, \leqslant\text{-}sig_S \rangle$ as defined in Section 2. We recall that $\leqslant\text{-}sig_S \subseteq V_G \times E_G \times E_G$ and for each $x \in V_G$, $\leqslant\text{-}sig_S[x]$ is a linear order on the set $E_G(x)$. We say that S *represents* a 2-map $M = \langle G//e, sigma_M \rangle$ if and only if

(1) $x \in \{s_1(G), s_2(G)\}$ and $y, z \in E_G(x)$ then (x, y, z) belongs to $\leqslant\text{-}sig_S$ if and only if $y = z$ or $y = e$ or $e \ll_C y \ll_C z$ where C is the circuit on $E_G(x) \cup \{e\}$ defined by M.

(2) $x \notin \{s_1(G), s_2(G)\}$ then the circuit $sigma_M(x)$ is $Circ(E_G(x), \leqslant\text{-}sig_S[x])$.

Several nonisomorphic ordered 2-maps can represent the same 2-map.

Fact 4.6. *If $G \subseteq G'$ with $s_1(G) = s_1(G'), s_2(G) = s_2(G')$ and S' represents an ordered 2-map M' of G', then the restriction of S' to $V_G \cup E_G$ is the ordered 2-map of G which is induced by M'.*

Fact 4.7. *Let $G = H[K/e]$ where G, H, K are 2-graphs. Let S be an ordered 2-map of H, let U be an ordered 2-map of K. One obtains an ordered 2-map T of G by letting $\leqslant\text{-}sig_T \subseteq V_G \times E_G \times E_G$ be such that*

(1) if $x \in V_K - \{s_1(K), s_2(K)\}$ then $\leqslant\text{-}sig_T[x] = \leqslant\text{-}sig_U[x]$.

(2) if $x \in V_H$ then $\leqslant\text{-}sig_T[x] = \leqslant\text{-}sig_S[x][\leqslant\text{-}sig_U[x]/e]$

(if $e \notin E_H(x)$ then $\leqslant\text{-}sig_T[x] = \leqslant\text{-}sig_S[x]$).

We use the substitution of linear orders defined in Section 1.

Lemma 4.8. *One can construct an MS-definable transduction that associates with $|(G, \leqslant, T, g)|_2$ representing a standard decomposition of a planar 2-dag G a structure $|M|_2$ representing a planar map M of G.*

Proof. Let (T, g) be a standard decomposition of a planar 2-graph G. Each 2-dag K associated with a node x of T satisfying S3.3 is planar (because $K//e$ is a minor of the planar graph $G//e$), and has an ordered set of edges (enumerated as a sequence $e_1, \dots, e_\kappa$ that is well-defined since T is an ordered tree). For each such K we have an ordered 2-map K^M of it (definable from $|K|_1$ and this ordering by an MS-transduction θ; see Corollary 3.9: we apply it with r defined as the unique vertex of K of indegree 0; it follows that θ has no parameter). One can associate with (T, g) an ordered 2-map M defined as follows:

$M = M(r_T)$ (r_T is the root of T) where for each node x of T we let $M(x)$ be defined inductively bottom-up in the tree as follows:

- if x is a leaf then $g(x)$ is an edge 2-dag and $M(x)$ is the unique (evident) ordered 2-map of $g(x)$,
- if x is a $*$-node, then we let $M(x) = M(x_1) * M(x_2)$
- if x is a $//$-node, then we let $M(x) = M(x_1)//M(x_2)$
- if x is a K-node with sons $x_1, \dots, x_\kappa$ corresponding to edges $e_1, \dots, e_\kappa$ of K then we let

$$M(x) = K^M[M(x_1)/e_1, \dots, M(x_k)/e_\kappa].$$

By induction, each $M(x)$ is a planar ordered 2-map. It remains to prove that $|M(r_T)|_2$ is MS-definable from $|(G, \leqslant, T, g)|_2$. The result follows then because it is easy by an MS-transduction to obtain from $|M|_2$, an ordered 2-map, the structure $|M'|_2$ where M' is the map represented by M.

Claim 4.8.1. *Let x be an ancestor of y in T. Then $\leqslant$-$sig_{M(y)}$ is the restriction of $\leqslant$-$sig_{M(x)}$* to $V_{g(y)} \times E_{g(y)} \times E_{g(y)}$.

Proof. Immediate from the definition of substitutions in ordered 2-maps. □

Claim 4.8.2. *There exists an* MS-*formula $\rho(u, h, h')$ such that $|(G, \leqslant, T, g)|_2 \models \rho(u, h, h')$ if and only if $u \in V_G$, $h, h' \in E_G(u)$ and for every node x of T such that u, h, h' belong to the subgraph $g(x)$ of G, the triple (u, h, h') belongs to $\leqslant$-$sig_{M(g(x))}$.*

Proof. An MS-formula can define from u, h, h' in $E_G(u)$ the node x of T which is deepest in T and is such that u, h, h' belong to $g(x)$. It follows from condition S3 that h belongs to $g(x_i), h'$ belongs to $g(x_j)$ with $i \neq j$, $1 \leqslant i, j \leqslant k$. There are several cases.

Case 1: $g(x) = g(x_1)//g(x_2)$.

Then u is a source of $g(x)$. We let $\rho(u, h, h')$ hold if $i = 1$, $j = 2$ and $\rho(u, h', h)$ hold if $j = 1$, $i = 2$.

Case 2: $g(x) = g(x_1) * g(x_2)$.

Here we must have $u = s_2(g(x_1)) = s_1(g(x_2))$ and u is not a source of $g(x)$. We let $\rho(u, h, h')$ or $\rho(u, h', h)$ hold exactly as above.

Case 3: $g(x) = K[g(x_1)/e_1, \dots, g(x_k)/e_k]$.

In this case u is a vertex of K incident with e_i and e_j. We let $\rho(u,h,h')$ holds if $\leqslant$-$sig_{K^M}(u,e_i,e_j)$ holds and $\rho(u,h',h)$ holds if $\leqslant$-$sig_{K^M}(u,e_j,e_i)$ holds. Since K^M is MS-definable the information whether $\leqslant$-$sig_{K^M}(u,e_j,e_j)$ holds or not is expressible by an MS-formula.

It follows that ρ can actually be constructed as an MS-formula. From the definition of operations on ordered 2-maps, it defines actually the ternary relation requested. □

Proposition 4.9. *One can construct an* MS-*transduction that associates with* $(|G|_2, \leqslant)$ *a structure* $|M|_2$ *representing a planar map* M *of* G *where* G *is a planar* 2-*connected graph and* $\leqslant$ *is a linear ordering of* E_G.

Proof. By Theorem 3.12 of [8] one can construct an MS-transduction which associates with $|G|_2$ a structure $|(G,T,g)|_2$ representing a decomposition of G (made into a 2-dag by the choice of an orientation, this can be done by an MS-transduction as proved in [4]), where T is unordered, and which satisfies conditions S1, S2, S3.3 and

S3.1′ $g(x)=g(x_1)//\cdots//g(x_\kappa), \quad \kappa\geqslant 2,$

S3.2′ $g(x)=g(x_1)*\cdots*g(x_\kappa), \quad \kappa\geqslant 2.$

Conditions S3.1 and S3.2 are stronger than S3.1′ and S3.2′ in that they require $\kappa=2$, and T must be ordered.

In the case of a node x of T such that $g(x)=g(x_1)*g(x_2)*\cdots*g(x_\kappa)$, the linear order $x_1,\ldots,x_\kappa$ on $Sons(x)$ (the set of sons of a node x of T) is definable by the condition $s_1(g(x_{i+1}))=s_2(g(x_i))$. We need ensure that the condition $\kappa=2$. To do so, when $\kappa>2$, we transform T by adding $\kappa-2$ intermediate nodes $u_1,u_2,\ldots,u_{\kappa-2}$ between x and x_κ, and we extend g to these new nodes in order to have

$$g(x)=g(x_1)*g(u_1),$$
$$g(u_1)=g(x_2)*g(u_2),$$
$$g(u_{\kappa-2})=g(x_{\kappa-1})*g(x_\kappa).$$

In order to do a similar transformation for // nodes x with $\kappa>2$, we define on each set $Sons(x)$ a linear order by means of an MS-formula using $\leqslant$. We let for $y,z\in Sons(x)$: $y \lll z$ if $y=z$ or the $\leqslant$-smallest edge of $g(y)$ is smaller with respect to $\leqslant$ than the $\leqslant$-smallest edge of $g(z)$.

This is a linear order because the graphs $g(y), y\in Sons(x)$ are edge-disjoint and $\leqslant$ is a linear order on E_G. Furthermore, it is MS-definable in $|(G,T,g)|_2$ in terms of $\leqslant$. It follows that whenever in T, we have $g(x)=g(x_1)//\cdots//g(x_\kappa)$, $\kappa>2$ we introduce intermediate nodes as above for $*$-nodes between x and x_κ, by using $\lll$ and MS-formulas.

The order $\lll$ is also useful to order linearly the sons of nodes satisfying S3.3. Thus, we obtain from $|G|_2$ and $\leqslant$, and by an MS-transduction, a standard decomposition of G, with an ordered tree. The result follows then from Lemma 4.8. □

Proof of Theorem 4.1. Let G be a planar graph given by the structure $|G|_2$ and a linear order $\leqslant$ on E_G. One can construct by Proposition 4.9 an MS-formula $\varphi(X,x,y,z)$ expressing that X is the set of edges of a block H of G, that $x \in V_H, y,z \in E_H$ and that the ternary relation $B_X = \{(u,v,w)/(|G|_2, \leqslant) \models \varphi(X,u,v,w)\}$ such that $\langle H, B_X \rangle$ is a planar map of H.

We now let $B = \bigcup\{B_X/X$ is a block$\}$. Since the blocks are edge-disjoint, for each of them, say H, the restriction of B to $V_H \times E_H \times E_H$ is of the form B_X where $X = V_H$. We note that B is defined by the MS-formula:

$$\psi(x,y,z) : \Leftrightarrow \exists X \text{ [“}X \text{ is the set of edges of a block and } \varphi(X,x,y,z) \text{ holds”].}$$

We can thus apply Lemma 4.3 and we obtain a map of G. In this lemma, the MS-transduction takes B as input. Since B is MS-definable, we obtain the existence of an MS-transduction defining a map of G from $|G|_2$ and the linear order $\leqslant$ on E_G. □

There exists an MS-transduction that transforms an arbitrary map of a planar graph into a planar map of the same graph. This is an immediate consequence of Proposition 2.3, from which a linear order on the edges of each connected component is MS-definable, and Theorem 4.1 for which linear orders are necessary only on each connected component and not globally.

5. Open questions

The logical structure representing a graph contains no “drawing information”. On the other hand, the one representing a map contains redundant “drawing information”. For the purpose of *concise representation of maps*, it is useful to store the minimum number of tuples, while being able to compute in a unique way the remaining necessary information.

The redundancy of the “map information” is quite clear for a simple planar 3-connected graph, since its two planar maps are definable from the graph by MS-formulas taking as parameters a pair of adjacent edges. (For some applications, one may also want to fix the “infinite face”, and this is possible by giving one vertex r and one incident edge h to this vertex. Assuming fixed the orientation on the plane and known the map M, the left-most branch of the DFS tree $T(M,r,h)$ defines the boundary of the infinite region of the plane defined by the drawing based on the drawing scheme constructed in the proof of Proposition 2.2.)

What about other cases? Let us define a *drawing constraint* for a graph G as a 4-tuple (x,e,f,h) in $V_G \times E_G \times E_G \times E_G$ such that e,f,h are three edges incident with x. A map $\langle G, sigma \rangle$ satisfies this constraint if $e \to f \to h \to e$ is a subcircuit of $sigma(x)$. A set of drawing constraints for G *is realizable* if there exists a planar map of G which satisfies all the constraints. It can be given as a 4-ary relation C on the domain of the structure $|G|_2$.

Problem 5.1. *Is the realizability of a set C of drawing constraints for a graph G given by a structure* $(|G|_2, C)$ *expressible by an MS-formula*?

This is even not immediate for a "star", i.e., a tree consisting of one root and several leaves: one has to express in MS-logic that a set of subcircuits of length 3 can be merged into a single circuit.

In Section 3 we have defined a planar map for any 3-connected simple graph with formulas of the language $\mathrm{FO}^{\mathrm{TC}}$. We do not know whether an alternative proof of Theorem 4.1 can be done with such formulas.

A future paper will establish that these results can be extended to graph embeddings on the torus. The corresponding maps can be characterized in terms of finitely many forbidden configurations as planar maps can be by Proposition 2.2, whence by universal monadic second-order formulas. Furthermore, such maps can be defined by MS-formulas when the given graphs are ordered, which extends Theorem 4.1.

References

[1] S. Arnborg, J. Lagergren, D. Seese, Easy problems for tree-decomposable graphs, J. Algorithms 12 (1991) 308–340.

[2] B. Courcelle, The monadic second-order logic of graphs VI: on several representations of graphs by relational structures, Discrete Appl. Math. 54 (1994) 117–149.

[3] B. Courcelle, Monadic second-order definable graph transductions: a survey, Theoret. Comput. Sci. 126 (1994) 53–75.

[4] B. Courcelle, The monadic second-order logic of graphs VIII: orientations, Annals Pure Appl. Logic 72 (1995) 103–143.

[5] B. Courcelle, The monadic second-order logic of graphs X: Linear orders, Theoret. Comput. Sci. 160 (1996) 87–143.

[6] B. Courcelle, The expression of graph properties in some fragments of monadic second-order logic, in: P. Kolaitis, N. Immerman (Eds.), DIMACS Series in Discrete Mathematics, Vol. 31, AMS, New York, 1997, pp. 33–62.

[7] B. Courcelle, The expression of graph properties and graph transformations in monadic second-order logic, in: G. Rozenberg (Ed.), Handbook of Graph Grammars and Computing by Graph Transformations, Vol. 1: Foundations, Ch. 5, World Scientific, New Jersey, London, 1997, pp. 313–400.

[8] B. Courcelle, The monadic second-order logic of graphs XI: Hierarchical decompositions of connected graphs, Theoret. Comput. Sci. 224 (1999) 35–58.

[9] B. Courcelle, The monadic second-order logic of graphs XIII: Graph drawings with edge crossings, Preliminary version http://dept-info.labri.u-bordeaux.fr/ ~courcell/ActSci.html, submitted for publication, 1998.

[10] B. Courcelle, M. Mosbah, Monadic second-order evaluation of tree-decomposable graphs, Theoret. Comput. Sci. 109 (1993) 49–82.

[11] B. Courcelle, J. Makowsky, U. Rotics, Linear time solvable optimization problems on certain graph families, Proc. WG'98, Lecture Notes Comput. Sci., Springer, Berlin, Vol. 1517, pp. 1–16, 1999.

[12] R. Diestel, Graph Theory, Springer, New York, 1997.

[13] H.-D. Ebbinghaus, J. Flum, Finite Model Theory, Springer, Berlin, 1995.

[14] M. Grohe, Fixed-point logics on planar graphs, in: Proc. 13th IEEE Symp. on Logic in Computer Science, Indianapolis, USA, 1998 (J. Mitchell, ed.).

[15] J. Gross, T. Tucker, Topological Graph Theory, Wiley, New York, 1987.

[16] S. Hedetniemi, Efficient algorithms and partial k-trees, References. Discrete Appl. Math. 54 (1994) 281–290.
[17] B. Mohar, C. Thomassen, Graphs on Surfaces, John Hopkins University Press, to appear.
[18] W. Tutte, Graph Theory, Addison-Wesley, Reading, MA, 1984.