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HIGHER APPROXIMATIONS TO THE SOLUTION OF A PROBLEM CONCERNING A HIGH-PRESSURE GAS-DISCHARGE ARC

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Abstract—The method of matched asymptotic expansions is applied to a problem which is of importance in the modelling of high-pressure gas-discharge arcs. In the solution domain, three different regions are distinguished, in each of which an asymptotic solution can be given. These match smoothly in two regions of common validity. The results are used in a companion paper [1].

1. INTRODUCTION

In [2] we studied the solution of the problem defined by

$$\frac{1}{Z}\frac{d}{dZ}Z\frac{dQ}{dZ} = e^{-Q} - (1-\varepsilon)\,e^{-\eta Q} \quad \text{in} \quad Z \geq 0, \tag{1}$$

with

$$Q = 0, \quad \frac{dQ}{dZ} = 0 \quad \text{at} \quad Z = 0, \tag{2}$$

in the limit $\varepsilon \downarrow 0$. The parameter η is assumed to be larger than unity. This problem arose from the study of the temperature distribution in a high-pressure gas-discharge arc. A more detailed description of the background of the model and its motivation can be found in [1]. The analysis of [2] aimed at clarifying the structure of the solution of (1) and (2) for $\varepsilon \downarrow 0$ which, as was shown there, is characterized by three layers. A singular perturbation technique was used to elucidate this structure. However, in that paper, we restricted ourselves to presenting the leading-order terms only. The purpose of this note is to go one step further and show how one can derive more and more accurate solutions. Since small values of ε are not at all uncommon in research into arcs, a more accurate solution may be of practical importance. In fact, the results of this paper are utilized in [1].

2. REGION I

As was shown in [2], the first region, which includes the boundary point $Z = 0$, is characterized by small values of Q. Introducing $Q = \varepsilon P$ and expanding, we find

$$\frac{1}{Z}\frac{d}{dZ}Z\frac{dP}{dZ} - (\eta - 1)\,P = 1 + \varepsilon\left\{\frac{1}{2}(1-\eta^2)P^2 - \eta P\right\} + O(\varepsilon^2), \tag{3}$$

which can be solved by the series expansion $P = P_0 + \varepsilon P_1 + \cdots$. Clearly, P_0, P_1, etc., must satisfy (2). The solution for P_0 was obtained in [2]:

$$P_0 = \frac{I_0\{Z(\eta-1)^{1/2}\} - 1}{\eta - 1}, \tag{4}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where I_0 is a modified Bessel function. We are interested in the behaviour of P for $Z \to \infty$, since that is where the expansion of Region I breaks down, thus defining its outer boundary. We can show that the asymptotic behaviour of P_1 for $Z \to \infty$ is determined by a term on the right-hand side which is proportional to P_0^2. Using Eq. 9.7.1. of [3], we find

$$\frac{Q}{\varepsilon} = P \sim (\eta-1)^{-1} e^{X} (2\pi X)^{-\frac{1}{2}} \left\{ 1 + \frac{1}{8} X^{-1} + \frac{9}{128} X^{-2} + O(X^{-3}) \right\} + O\left(\frac{\varepsilon\, e^{2X}}{X}\right), \tag{5}$$

when

$$X = Z(\eta-1)^{\frac{1}{2}} \to \infty. \tag{6}$$

Inverting (5), we have

$$\begin{aligned} X \sim\ & L + \frac{1}{2}\log(L) + c + \log(Q) + O(Q) + \frac{1}{4}\frac{\log(L)}{L} + \left\{ \frac{1}{2}\log(Q) + \frac{1}{2}c - \frac{1}{8} + O(Q) \right\} \frac{1}{L} \\ & - \frac{1}{16}\frac{\log^2(L)}{L^2} + \left\{ -\frac{1}{4}\log(Q) + \frac{3}{16} - \frac{1}{4}c + O(Q) \right\} \frac{\log(L)}{L^2} \\ & + \left\{ -\frac{1}{4}\log^2(Q) + \left(\frac{3}{8} - \frac{1}{2}c\right)\log(Q) - \frac{1}{4}c_1^2 + \frac{3}{8}c - \frac{1}{8} + O(Q) \right\} \frac{1}{L^2} + \cdots, \end{aligned} \tag{7}$$

where

$$c = \frac{1}{2}\log(2\pi) + \log(\eta-1) \tag{8}$$

and

$$L = \log\left(\frac{1}{\varepsilon}\right) \gg 1.$$

Eq. (7) applies in the region where Q is still much smaller than unity, but with $Q/\varepsilon \gg 1$. The terms $O(Q)$ result from the last term on the right of Eq. (5).

3. REGION II

Interchanging the roles of Q and X as dependent and independent variables, respectively, we find after a single integration and an application of one of the boundary conditions:

$$\frac{1}{2(X')^2} = \frac{f^2(Q)}{\eta-1} - \int_0^Q \frac{dq}{X(q)X'(q)}, \tag{9}$$

where a prime stands for differentiation with respect to Q. The term of $O(\varepsilon)$, which appears on the right-hand side of (1), can be disregarded in Region II, since its contribution is asymptotically equal to zero in comparison with the terms included in the expansion. Furthermore,

$$f(Q) = \left\{ 1 - e^{-Q} - \frac{1-e^{-\eta Q}}{\eta} \right\}^{\frac{1}{2}}. \tag{10}$$

Eq. (9) will be used in Region II which adjoins Region I. In Region II we have $X = O(\log(1/\varepsilon))$. Therefore, the integral appearing in (9) is small in comparison with the first term on the right of (9). Thus we may expand this equation as follows:

$$\begin{aligned} X' = & \left(\frac{\eta-1}{2}\right)^{\frac{1}{2}} f(Q)^{-1} \left\{ 1 + \frac{(\eta-1)}{2} f(Q)^{-2} \int_0^Q \frac{dq}{X(q)X'(q)} \right. \\ & \left. + \frac{3}{8}(\eta-1)^2 f(Q)^{-4} \left(\int_0^Q \frac{dq}{X(q)X'(q)} \right)^2 + \cdots \right\}. \end{aligned} \tag{11}$$

In view of (7) we shall try to solve this equation with

$$X = L + \frac{1}{2}\log(L) + \zeta_0(Q) + \frac{1}{4}\frac{\log(L)}{L} + \zeta_1(Q)\frac{1}{L} - \frac{1}{16}\frac{\log^2(L)}{L^2} + \zeta_2(Q)\frac{\log(L)}{L^2} + \zeta_3\frac{1}{L^2} + \cdots. \quad (12)$$

The solution for ζ_0 has already been given in [2]:

$$\zeta_0 = \left(\frac{\eta}{2}\right)^{\frac{1}{2}} Q + \alpha + \int_Q^\infty \left\{ \left(\frac{\eta}{2}\right)^{\frac{1}{2}} - \left(\frac{\eta-1}{2}\right)^{\frac{1}{2}} f(q)^{-1} \right\} dq, \quad (13)$$

where

$$\alpha = c - (\eta/2)^{\frac{1}{2}} + \int_1^\infty \left\{ \left(\frac{\eta-1}{2}\right)^{\frac{1}{2}} f(q)^{-1} - \left(\frac{\eta}{2}\right)^{\frac{1}{2}} \right\} dq + \int_0^1 \left\{ \left(\frac{\eta-1}{2}\right)^{\frac{1}{2}} f(q)^{-1} - q^{-1} \right\} dq. \quad (14)$$

The equation for ζ_1 reads

$$\zeta_1' = \frac{\eta-1}{2} f(Q)^{-3} \int_0^Q f(q)\, dq\,, \quad (15)$$

where, according to (7), ζ_1 should satisfy the condition

$$\zeta_1 \to \frac{1}{2}\log(Q) + \frac{c}{2} - \frac{1}{8} + O(Q) \quad \text{when } Q \downarrow 0. \quad (16)$$

This leads to the following solution

$$\zeta_1 = \frac{1}{2}\log(Q) + \frac{c}{2} - \frac{1}{8} + \int_0^Q \left\{ \frac{\eta-1}{2} f(q)^{-3} \int_0^q f(\tilde q)\, d\tilde q - \frac{1}{2} q^{-1} \right\} dq. \quad (17)$$

Again, we are interested in the behaviour of ζ_1 for $Q \to \infty$. After some manipulations, we find

$$\zeta_1 \sim \frac{\eta Q^2}{4} - \beta Q + \gamma + \text{exponentially small terms}\,, \qquad (Q \to \infty), \quad (18)$$

where

$$\beta = \frac{\eta^{3/2}}{2} (\eta - 1)^{-1/2} \int_0^\infty \{f_\infty - f(q)\}\, dq \quad (19)$$

and

$$\gamma = \frac{c}{2} - \frac{1}{8} - \frac{\eta}{4} + \beta + \int_0^1 \left\{ \frac{\eta-1}{2} f(q)^{-3} \int_0^q f(\tilde q)\, d\tilde q - \frac{1}{2} q^{-1} \right\} dq + \int_1^\infty \left\{ \frac{\eta-1}{2} f(q)^{-3} \int_0^q f(\tilde q)\, d\tilde q - \frac{1}{2}\eta q + \beta \right\} dq. \quad (20)$$

Although the analysis becomes increasingly cumbersome, we shall proceed a few steps further and present partial solutions of the next few perturbations. The reason is that we wish to use our results elsewhere in a practical context. In view of this, it is desirable to have a final result which is as accurate as one is able to make it. We shall restrict our calculations to the first derivatives of the functions ζ_2 and ζ_3. The equation for the first of these is simple enough

$$\zeta_2' = -\frac{1}{2}\zeta_1'. \quad (21)$$

The next equation is

$$\zeta_3' = \frac{3}{2}\left(\frac{2}{\eta-1}\right)^{1/2} (\zeta_1')^2 f(Q) - \left(\frac{\eta-1}{2}\right)^{3/2} f(Q)^{-3} \int_0^Q \left(\zeta_0 + \frac{\zeta_1'}{\zeta_0'}\right) \frac{dq}{\zeta_0'}. \tag{22}$$

What we need is the behaviour of ζ_3 for $Q \to \infty$. After a lengthy calculation, we obtain

$$\zeta_3' \sim \frac{1}{2}\left(\frac{\eta}{2}\right)^{3/2} Q^2 - \left\{(2\eta)^{1/2}\beta + \frac{1}{2}\alpha\eta\right\} Q + \delta + \text{exponentially small terms}, \quad (Q \to \infty), \tag{23}$$

where

$$\begin{aligned}\delta = &\frac{3}{2}\left(\frac{2}{\eta}\right)^{1/2} \beta^2 \\ &- \left(\frac{\eta}{2}\right)^{3/2} \left\{ \int_0^\infty \{f_\infty f(q)^{-1} - 1\} q\, dq + \int_0^\infty \left(f(q)^{-1} \int_0^q \{f(\tilde{q}) - f_\infty\}\, d\tilde{q} + \frac{2\beta}{\eta}\right) dq \right. \\ &+ \left. \int_0^\infty f(q) \int_q^\infty \{f_\infty^{-1} - f(\tilde{q})^{-1}\}\, d\tilde{q}\, dq + \int_0^\infty \left\{q + \left(\frac{2}{\eta}\right)^{1/2} \alpha\right\}\{f_\infty^{-1} f(q) - 1\}\, dq \right\}\end{aligned} \tag{24}$$

4. REGION III

In Region III the differential equation is (see [2]):

$$\frac{d^2Q}{dX^2} + \frac{1}{X}\frac{dQ}{dX} = 0, \tag{25}$$

which has the solution

$$Q = A \log\left(\frac{X}{B}\right). \tag{26}$$

It was shown in [2] that, to a leading order, the constants A and B are both $O(L)$. Expanding (26) for $O(Q) \ll O(L)$, we have

$$X \sim B\left\{1 + \frac{Q}{A} + \frac{1}{2}\frac{Q^2}{A^2} + \cdots\right\}. \tag{27}$$

According to (12), (13), (18), (21) and (23) this must be equal to

$$\begin{aligned}X \sim\ & L + \frac{1}{2}\log(L) + \left\{\left(\frac{\eta}{2}\right)^{1/2} Q + \alpha\right\} + \frac{1}{4}\frac{\log(L)}{L} + \left(\frac{1}{4}\eta Q^2 - \beta Q + \gamma\right)\frac{1}{L} \\ &- \frac{1}{16}\frac{\log^2(L)}{L^2} + \left\{-\frac{1}{8}\eta Q^2 + \frac{1}{2}\beta Q + O(1)\right\}\frac{\log(L)}{L^2} \\ &+ \left[\frac{1}{6}\left(\frac{\eta}{2}\right)^{3/2} Q^3 - \left\{\beta\left(\frac{\eta}{2}\right)^{1/2} + \frac{1}{4}\eta\alpha\right\} Q^2 + \delta Q + O(1)\right]\frac{1}{L^2} + \cdots,\end{aligned} \tag{28}$$

or, rearranging

$$\begin{aligned}X \sim & \left\{L + \frac{1}{2}\log(L) + \alpha + \frac{1}{4}\frac{\log(L)}{L} + \frac{\gamma}{L} + \cdots\right\} + \left\{\left(\frac{\eta}{2}\right)^{1/2} - \frac{\beta}{L} + \frac{1}{2}\beta\frac{\log(L)}{L^2} + \frac{\delta}{L^2} + \cdots\right\} Q \\ &+ \left\{\frac{1}{4}\frac{\eta}{L} - \frac{1}{8}\eta\frac{\log(L)}{L^2} - \frac{(\eta/2)^2\beta + \frac{1}{4}\eta\alpha}{L^2} + \cdots\right\} Q^2 + \left\{\frac{1}{6}\frac{(\eta/2)^{3/2}}{L^2} + \cdots\right\} Q^3 + \cdots.\end{aligned} \tag{29}$$

Comparing (27) and (29), and matching the terms which are $O(1)$ and $O(Q)$, we find

$$B = L + \frac{1}{2}\log(L) + \alpha + \frac{1}{4}\frac{\log(L)}{L} + \frac{\gamma}{L} + \cdots, \tag{30}$$

$$\begin{aligned}A =\ & B\left\{\left(\frac{\eta}{2}\right)^{1/2} - \frac{\beta}{L} + \frac{\beta}{2}\frac{\log(L)}{L^2} + \frac{\delta}{L^2} + \cdots\right\}^{-1} \\ &\sim \left(\frac{2}{\eta}\right)^{1/2}\left\{L + \frac{1}{2}\log(L) + \kappa + \frac{1}{4}\frac{\log(L)}{L} + \frac{\lambda}{L} + \cdots\right\}\end{aligned} \tag{31}$$

where

$$\kappa = \alpha + \left(\frac{2}{\eta}\right)^{1/2} \beta \tag{32}$$

and

$$\lambda = \gamma + 2\frac{\beta^2}{\eta} + (\alpha\beta - \delta)\left(\frac{2}{\eta}\right)^{1/2}. \tag{33}$$

It can easily be shown that the coefficients of Q^2 and Q^3 match automatically, which may serve as a useful check.

5. RESULTS

Referring to the analysis of [1], we know that the asymptotic behaviour of Q for $Z \to \infty$ is important. Indeed, to describe this behaviour as accurately as possible is the main purpose of this note. Using the notation [1], we have

$$Q \sim g_1 \log(Z) - g_2, \quad \text{for } Z \to \infty. \tag{34}$$

Comparing this with (22), using (6), we have

$$g_1 = A, \tag{35}$$

$$g_2 = A\left\{\log(B) - \frac{1}{2}\log(\eta - 1)\right\}. \tag{36}$$

Both A and B are available as expansions in terms of known functions of ε. The dependence on η appears through the parameter functions α, β, γ and δ, some parameter functions derived from these (κ and λ), and some explicit terms. For further usage in [1], we list values of these parameter functions in Table 1. For reasons explained in [1] we restrict ourselves to the interval $1 < \eta \leq 2$. For $\eta = 2$ the function $f(Q)$ reduces to $(1 - e^{-Q})/2^{1/2}$. The integrals can then be evaluated analytically, which provides another useful check, in this case for the correctness of the numerical computations. The results are:

$$\begin{aligned} \alpha(2) &= \frac{1}{2}\log(2\pi), & \beta(2) &= 1, \\ \gamma(2) &= \frac{1}{4}\log(2\pi) + \frac{13}{8} + \frac{\pi^2}{6}, & \delta(2) &= \frac{3}{2} + \frac{1}{2}\log(2\pi). \end{aligned} \tag{37}$$

Another quantity needed in [1] is

$$g_3 = \int_0^\infty Z\, e^{-Q(Z)}\, dZ = \frac{1}{2}\int_0^\infty Z^2(Q)\, e^{-Q}\, dQ, \tag{38}$$

where we derived the second integral through partial integration, using the monotinicity of Q as a function of Z and the boundary conditions $Q(0) = 0$ and $Q \to \infty$ when $Z \to \infty$. It was argued in [2] that, in an asymptotic sense, the value of the integral is fully determined by the representation of the function $Z(Q)$ in the transition Region II. Therefore, substituting (12) in (38), using (6) we have

$$g_3 = \frac{1}{2}(\eta - 1)^{-1}\{L^2 + L\log(L) + 2\mu L + \frac{1}{4}\log^2(L) + \left(\mu + \frac{1}{2}\right)\log(L) + \nu + \cdots\}, \tag{39}$$

where

$$\mu = \int_0^\infty \zeta_0(Q)\, e^{-Q}\, dQ, \tag{40}$$

$$\nu = \int_0^\infty \{\zeta_0^2(Q) + 2\zeta_1(Q)\}\, e^{-Q}\, dQ. \tag{41}$$

Table 1. Some characteristic parameters for various values of η.

η	α	β	γ	δ	κ	λ	μ	ν
1.05	−1.8394	0.6888	1.8789	−0.2848	−0.8888	1.4270	−2.2689	6.0971
1.10	−1.1683	0.7055	2.2207	0.1825	−0.2170	1.7682	−1.5654	4.1687
1.15	−0.7833	0.7222	2.4231	0.4660	0.1691	1.9696	−1.1497	3.5217
1.20	−0.5145	0.7388	2.5705	0.6769	0.4393	2.1156	−0.8517	3.2899
1.25	−0.3088	0.7554	2.6890	0.8493	0.6466	2.2325	−0.6183	3.2473
1.30	−0.1428	0.7719	2.7901	0.9983	0.8146	2.3318	−0.4257	3.3064
1.35	−0.0038	0.7884	2.8799	1.1318	0.9558	2.4195	−0.2614	3.4258
1.40	0.1156	0.8048	2.9618	1.2544	1.0776	2.4991	−0.1177	3.5836
1.45	0.2202	0.8213	3.0381	1.3690	1.1847	2.5730	0.0103	3.7667
1.50	0.3133	0.8376	3.1102	1.4777	1.2805	2.6424	0.1258	3.9673
1.55	0.3971	0.8540	3.1790	1.5817	1.3671	2.7085	0.2312	4.1801
1.60	0.4733	0.8703	3.2454	1.6822	1.4463	2.7719	0.3283	4.4017
1.65	0.5433	0.8866	3.3098	1.7798	1.5194	2.8333	0.4184	4.6296
1.70	0.6079	0.9029	3.3726	1.8752	1.5872	2.8930	0.5026	4.8622
1.75	0.6680	0.9191	3.4341	1.9687	1.6506	2.9514	0.5816	5.0981
1.80	0.7243	0.9353	3.4946	2.0607	1.7102	3.0086	0.6561	5.3366
1.85	0.7771	0.9515	3.5543	2.1515	1.7664	3.0649	0.7267	5.5767
1.90	0.8269	0.9677	3.6132	2.2414	1.8198	3.1203	0.7938	5.8182
1.95	0.8741	0.9839	3.6716	2.3305	1.8705	3.1752	0.8578	6.0604
2.00	0.9189	1.0000	3.7294	2.4189	1.9189	3.2294	0.9189	6.3033

Values of μ and ν are listed in Table 1. Again, the integrals can be evaluated analytically for $\eta = 2$:

$$\mu(2) = \frac{1}{2}\log(2\pi), \qquad \nu(2) = 1 + \frac{1}{4}\{1 + \log(2\pi)\}^2 + \frac{\pi^2}{3}. \tag{42}$$

Finally, we need an asymptotic expression for the important efficiency parameter W defined in [2] as

$$W = \left(\int_0^\infty Z\, e^{-\eta Q}\, dZ\right)\left(\int_0^\infty Z\, e^{-Q}\, dZ\right)^{-1}. \tag{43}$$

It can be shown that W can be expanded as follows:

$$W = 1 - 2(\eta - 1)\left(\frac{2}{\eta}\right)^{1/2}\frac{1}{L}\left(1 - \frac{1}{2}\frac{\log(L)}{L} + \{\alpha + \beta\left(\frac{2}{\eta}\right)^{1/2} - 2\mu\}\frac{1}{L} + \cdots\right). \tag{44}$$

Alternatively, it can be shown that

$$W = 1 - \frac{g_1}{g_3}$$

Using (31), (35) and (39), we may derive an even more accurate expansion for W.

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