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Parameter Augmentation for Basic Hypergeometric Series, II

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In a previous paper, we explored the idea of parameter augmentation for basic hypergeometric series, which provides a method of proving q-summation and integral formula based special cases obtained by reducing some parameters to zero. In the present paper, we shall mainly deal with parameter augmentation for q-integrals such as the Askey–Wilson integral, the Nassrallah–Rahman integral, the q-integral form of Sears transformation, and Gasper's formula of the extension of the Askey–Roy integral. The parameter augmentation is realized by another operator, which leads to considerable simplications of some well known q-summation and transformation formulas. A brief treatment of the Rogers–Szegö polynomials is also given. © 1997 Academic Press

1. INTRODUCTION

Since the umbral calculus was developed by G.-C. Rota and his collaborators [30, 31], there has been extensive interest in an operator approach to basic hypergeometric series, as in the work of Goldman and Rota [16, 17], Andrews [2], and Roman [29]. Instead of aiming at a general approach to polynomials of q-binomial type, we focus our attention on some specific operators which can be simply defined as certain exponential operators. It turns out that such simple operators play a fundamental

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role in the theory of basic hypergeometric series. In the present paper we continue the study of parameter augmentation based on an exponential operator which is dual to the operator introduced in a previous paper [12]. By using an augmentation operator an identity on basic hypergeometric series with multiple parameters may be recovered from its special case obtained by setting some parameters to zero. Many important results on q-summations and q-integrals naturally fall into the framework of this augmentation operator such as the Askey–Wilson integral, the Nassrallah–Rahman integral, the q-integral form of Sears transformation, Gasper's formula of the extension of the Askey–Roy integral, the q-Gauss ${}_2\phi_1$ summation formula, the q-Pfaff–Saalschütz formula, Jackson's q-analogue of the Euler transform, some identities on Rogers–Szegö polynomials, or their equivalent forms for the q-Hermite polynomials, and an identity of Andrews generalizing the Lebesgue identity.

We shall follow the notation and terminology in [15]. Let $|q|<1$ and the q-shifted factorial be defined by

$$(a;q)_0=1, \qquad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k). \tag{1.1}$$

Clearly,

$$(a;q)_n=(a;q)_\infty/(aq^n;q)_\infty. \tag{1.2}$$

Throughout we shall adopt the following notation of multiple q-shifted factorials:

$$(a_1, a_2, ..., a_m; q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$(a_1, a_2, ..., a_m; q)_\infty=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}=\frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}.$$

The basic hypergeometric series ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, ..., a_{r+1} \\ b_1, ..., b_r \end{matrix}; q, x\right)=\sum_{n=0}^{\infty}\frac{(a_1, ..., a_{r+1}; q)_n\, x^n}{(q, b_1, ..., b_r; q)_n}.$$

In this paper, we will frequently use the Cauchy identity and its special cases [15]:

$$\frac{(ax;q)_\infty}{(x;q)_\infty}=\sum_{n=0}^{\infty}\frac{(a;q)_n\,x^n}{(q;q)_n}, \tag{1.3}$$

$$\frac{1}{(x;q)_\infty}=\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}, \tag{1.4}$$

$$(-x;q)_\infty=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}\,x^n}{(q;q)^n}. \tag{1.5}$$

2. THE EXPONENTIAL OPERATOR $T(bD_q)$

The usual q-differential operator, or q-derivative, is defined by

$$D_q\,f(a)=\frac{f(a)-f(aq)}{a}. \tag{2.1}$$

By convention, D_q^0 is understood as the identity.

The Leibniz rule for D_q is the following identity, which is a variation of the q-binomial theorem [29]:

$$D_q^n\{f(a)\,g(a)\}=\sum_{k=0}^{n}q^{k(k-n)}\begin{bmatrix}n\\k\end{bmatrix}D_q^k\{f(a)\}\,D_q^{n-k}\{g(q^k a)\}. \tag{2.2}$$

The following property of D_q is straightforward, but important:

THEOREM 2.1.

$$D_q\left\{\frac{1}{(at;q)_\infty}\right\}=\frac{t}{(at;q)_\infty}, \tag{2.3}$$

$$D_q^k\left\{\frac{1}{(at;q)_\infty}\right\}=\frac{t^k}{(at;q)_\infty}. \tag{2.4}$$

The operator to be used in this paper, denoted T, is constructed based on D_q:

$$T(bD_q)=\sum_{n=0}^{\infty}\frac{(bD_q)^n}{(q;q)_n}. \tag{2.5}$$

From the above theorem, we may obtain the following.

THEOREM 2.2.

$$T(bD_q)\left\{\frac{1}{(at;q)_\infty}\right\}=\frac{1}{(at,bt;q)_\infty}. \tag{2.6}$$

Employing the Leibniz formula, we may derive the following theorem.

THEOREM 2.3.

$$T(bD_q)\left\{\frac{1}{(as,at;q)_\infty}\right\}=\frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}. \tag{2.7}$$

Proof of Theorem 2.3. Applying the Leibniz formula for D_q, the left-hand side of (2.7) equals

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{b^n}{(q;q)_n}D_q^n\left\{\frac{1}{(as,at;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{b^n}{(q;q)_n}\sum_{k=0}^{n}q^{k(k-n)}\begin{bmatrix}n\\k\end{bmatrix}D_q^k\left\{\frac{1}{(as;q)_\infty}\right\}D_q^{n-k}\left\{\frac{1}{(atq^k;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{b^n}{(q;q)_n}\sum_{k=0}^{n}q^{k(k-n)}\begin{bmatrix}n\\k\end{bmatrix}\frac{s^k}{(as;q)_\infty}\frac{(tq^k)^{n-k}}{(atq^k;q)_\infty}\\
&=\frac{1}{(as,at;q)_\infty}\sum_{k=0}^{\infty}\frac{(at;q)_k\,(bs)^k}{(q;q)_k}\sum_{n=k}^{\infty}\frac{(bt)^{n-k}}{(q;q)_{n-k}}\\
&=\frac{1}{(as,at;q)_\infty}\frac{(abst;q)_\infty}{(bs;q)_\infty}\frac{1}{(bt;q)_\infty}\\
&=\frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty},
\end{aligned}$$

as desired. ∎

Theorem 2.3 reduces to Theorem 2.2 when $s=0$. These two theorems may lead to many important results in the theory of hypergeometric series.

3. THE q-PFAFF–SAALSCHÜTZ FORMULA AND THE JACKSON TRANSFORM

To give the reader a taste of the applications of the operator T, we will present probably the simplest proofs of the well-known q-Pfaff–Saalschütz formula, the Gauss ${}_2\phi_1$ summation formula, and the Jackson q-analogue of the Euler transform. Using the idea of parameter augmentation, the q-Pfaff–Saalschütz formula can easily be derived from the q-Chu–Vandermonde convolution formula, the Gauss ${}_2\phi_1$ summation formula can easily be derived from the Cauchy binomial theorem, and the Jackson q-analogue of the Euler transform can also be recovered from the Cauchy binomial theorem written in a slightly different form.

Recall that the usual q-Chu–Vandermonde convolution can be rewritten as the following basic hypergeometric series form (see, for example, [13]):

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q\right) = \frac{a^n(c/a; q)_n}{(c; q)_n}. \tag{3.1}$$

Using the following relation

$$a^n(c/a; q)_n = (-c)^n q^{\binom{n}{2}} \frac{(aq^{1-n}/c; q)_\infty}{(aq/c; q)_\infty}, \tag{3.2}$$

(3.1) can be written as

$$\sum_{k=0}^{n} \frac{(q^{-n}; q)_k\, q^k}{(q; q)_k\, (c; q)_k\, (aq^k, aq^{1-n}/c; q)_\infty} = (-c)^n \frac{q^{\binom{n}{2}}}{(c; q)_n\, (aq/c, a; q)_\infty}. \tag{3.3}$$

Applying the operator $T(bD_q)$ on both sides of (3.3), it follows that

$$\begin{aligned} &\sum_{k=0}^{n} \frac{(q^{-n}; q)_k\, q^k}{(q, c; q)_k} T(bD_q)\left\{\frac{1}{(aq^k, aq^{1-n}/c; q)_\infty}\right\} \\ &\quad = \frac{(-c)^n q^{\binom{n}{2}}}{(c; q)_n} T(bD_q)\left\{\frac{1}{(aq/c, a; q)_\infty}\right\}. \end{aligned}$$

Using the relations

$$T(bD_q)\left\{\frac{1}{(aq^k, aq^{1-n}/c; q)_\infty}\right\} = \frac{(abq^{1-n+k}/c; q)_\infty}{(aq^k, aq^{1-n}/c, bq^k, bq^{1-n}/c; q)_\infty},$$

$$T(bD_q)\left\{\frac{1}{(aq/c, a; q)_\infty}\right\} = \frac{(abq/c; q)_\infty}{(aq/c, a, bq/c, b; q)_\infty},$$

we obtain the q-Pfaff–Saalschütz formula [5, 15, 20, 35]:

THEOREM 3.1. *We have*

$$ {}_3\phi_2\left(\begin{matrix} a, & b, & q^{-n} \\ c, & abc^{-1}q^{1-n} \end{matrix}; q, q\right) = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}. \tag{3.4}$$

Next we proceed to give an augmentation argument for the q-Gauss theorem. By (1.2), the Cauchy binomial theorem can be written as

$$\frac{1}{(x;q)_\infty (a;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n (aq^n, ax;q)_\infty}.$$

Applying $T(bD_q)$ on both sides of the above identity, we get

$$\frac{1}{(x;q)_\infty} T(bD_q) \left\{ \frac{1}{(a;q)_\infty} \right\} = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} T(bD_q) \left\{ \frac{1}{(aq^n, ax;q)_\infty} \right\}. \tag{3.5}$$

Since

$$T(bD_q) \left\{ \frac{1}{(a;q)_\infty} \right\} = \frac{1}{(a,b;q)_\infty}$$

and

$$T(bD_q) \left\{ \frac{1}{(aq^n, ax;q)_\infty} \right\} = \frac{(abxq^n;q)_\infty}{(aq^n, ax, bq^n, bx;q)_\infty},$$

it follows from (3.5) that

$$\frac{1}{(x,a,b;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} \frac{(abxq^n;q)_\infty}{(aq^n, ax, bq^n, bx;q)_\infty}.$$

Using (1.2), the above identity can be written as

$${}_2\phi_1\left(\begin{matrix} a, b \\ abx \end{matrix}; q, x \right) = \frac{(ax, bx;q)_\infty}{(x, abx;q)_\infty},$$

which is equivalent to the following form of the q-Gauss theorem [3, 5, 15, 20, 29]:

THEOREM 3.2. *We have*

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab} \right) = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}. \tag{3.6}$$

By the Cauchy binomial theorem, we have

$$\frac{(abx;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(ab;q)_n}{(q;q)_n} x^n,$$

$$\frac{(abx;q)_\infty}{(b;q)_\infty} = \sum_{n=0}^{\infty} \frac{(ax;q)_n}{(q;q)_n} b^n.$$

It follows that

$$\sum_{n=0}^{\infty} \frac{(ab;q)_n}{(q;q)_n} x^n = \frac{(b;q)_\infty}{(x;q)_\infty} \sum_{n=0}^{\infty} \frac{(ax;q)_n}{(q;q)_n} b^n,$$

which can be rewritten as

$$\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n (abq^n, ax;q)_\infty} = \frac{(b;q)_\infty}{(x;q)_\infty} \sum_{n=0}^{\infty} \frac{b^n}{(q;q)_n (axq^n, ab;q)_\infty}.$$

Applying $T(cD_q)$, we have

$$\sum_{n=0}^{\infty} \frac{x^n (abcxq^n;q)_\infty}{(q;q)_n (abq^n, ax, bcq^n, cx;q)_\infty} = \frac{(b;q)_\infty}{(x;q)_\infty} \sum_{n=0}^{\infty} \frac{b^n (abcxq^n;q)_\infty}{(q;q)_n (axq^n, ab, cxq^n, bc;q)_\infty},$$

which can be rewritten as

$${}_2\phi_1\left(\begin{matrix} ab, bc \\ abcx \end{matrix}; q, x\right) = \frac{(b;q)_\infty}{(x;q)_\infty} {}_2\phi_1\left(\begin{matrix} ax, cx \\ abcx \end{matrix}; q, b\right). \tag{3.7}$$

Note that (3.7) is equivalent to the Jackson q-analogue of the Euler transform [15]:

THEOREM 3.3. *We have*

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, x\right) = \frac{(abx/c;q)_\infty}{(x;q)_\infty} {}_2\phi_1\left(\begin{matrix} c/a, c/b \\ c \end{matrix}; q, \frac{abx}{c}\right). \tag{3.8}$$

4. AN IDENTITY OF ANDREWS

In [1], Andrews gives an identity which contains as a special case the Lebesgue identity. This identity is also called the q-analogue of the Gauss second theorem. Here we point out that the Andrews identity can be recovered from the Lebesgue identity by parameter augmentation. To this end, we present a proof of the Lebesgue identity for completeness. It is stated as follows [1, 3]:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{\binom{n+1}{2}}}{(q;q)_n} = (-q;q)_\infty (aq;q^2)_\infty. \tag{4.1}$$

Proof. The left hand side of (4.1) equals

$$
\begin{aligned}
(a;q)_\infty \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q)_n} \frac{1}{(aq^n;q)_\infty} &= (a;q)_\infty \sum_{j=0}^{\infty} \frac{a^j}{(q;q)_j} \sum_{n=0}^{\infty} \frac{q^{nj} q^{\binom{n+1}{2}}}{(q;q)_n} \\
&= (a;q)_\infty \sum_{j=0}^{\infty} \frac{a^j}{(q;q)_j} (-q^{j+1};q)_\infty \\
&= (-q,a;q)_\infty \sum_{j=0}^{\infty} \frac{a^j}{(q^2;q^2)_j} \\
&= \frac{(-q,a;q)_\infty}{(a;q^2)_\infty} \\
&= (-q;q)_\infty\,(aq;q^2)_\infty,
\end{aligned}
$$

as desired. ∎

Let us rewrite (4.1) in the following form:

$$
\sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q)_n\,(aq^n, aq^{n+1};q^2)_\infty} = \frac{(-q;q)_\infty}{(a;q^2)_\infty}.
$$

Setting q to $q^{1/2}$, the above identity becomes

$$
\sum_{n=0}^{\infty} \frac{q^{1/2\binom{n+1}{2}}}{(q^{1/2};q^{1/2})_n\,(aq^{n/2}; aq^{(n+1)/2};q)_\infty} = \frac{(-q^{1/2};q^{1/2})_\infty}{(a;q)_\infty}. \tag{4.2}
$$

Applying the operator $T(bD_q)$ on both sides of (4.2) leads to the following identity:

$$
\sum_{n=0}^{\infty} \frac{q^{1/2\binom{n+1}{2}}(abq^{n+(1/2)};q)_\infty}{(q^{1/2};q^{1/2})_n\,(aq^{n/2}, aq^{(n+1)/2}, bq^{n/2}, bq^{(n+1)/2};q)_\infty} = \frac{(-q^{1/2};q^{1/2})_\infty}{(a,b;q)_\infty}. \tag{4.3}
$$

Finally, setting $q^{1/2}$ back to q, we get the Andrews identity [3]:

THEOREM 4.1. *We have*

$$
\sum_{n=0}^{\infty} \frac{(a,b;q)_n\, q^{\binom{n+1}{2}}}{(q;q)_n\,(abq;q^2)_n} = \frac{(-q;q)_\infty\,(aq,bq;q^2)_\infty}{(abq;q^2)_\infty}. \tag{4.4}
$$

5. THE ASKEY–WILSON INTEGRAL AND THE NASSRALLAH–RAHMAN INTEGRAL

In this section, we shall present a treatment of the Askey–Wilson integrals via parameter augmentation. The point of departure from the usual Askey–Wilson integral is the orthogonality relation obtained from the Cauchy binomial theorem and the Jacobi triple product identity. Once we have the identity with one parameter, while the Askey–Wilson integral involves four parameters, the augmentation turns out to be an easy task. Moreover, one more augmenation argument on the Askey–Wilson integral leads to a generalization due to Ismail–Stanton–Viennot. We also point out that the Nassrallah–Rahman integral can be easily derived from the result of Ismail–Stanton–Viennot.

5.1. *The Askey–Wilson Integral*

In the Cauchy binomial theorem, setting $a=q^{-2N}$, $x=q^N$, where N is an nonnegative integer, one obtains the following orthogonality relation:

$$\sum_{n=-\infty}^{+\infty} (-1)^n q^{\binom{n}{2}} \begin{bmatrix} 2N \\ N+n \end{bmatrix} = \delta_{0,N}, \tag{5.1}$$

where $\delta_{n,m}$ is the Kronecker delta.

By the Cauchy binomial identity we have

$$\begin{aligned} \frac{1}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty} &= \sum_{n=0}^{\infty} \frac{a^n e^{in\theta}}{(q;q)_n} \sum_{m=0}^{\infty} \frac{a^m e^{-im\theta}}{(q;q)_m} \\ &= \sum_{n,m=0}^{\infty} \frac{a^{m+n}}{(q;q)_n (q;q)_m} e^{i(n-m)\theta} \\ &= \sum_{k=-\infty}^{+\infty} e^{-ik\theta} \sum_{m=0}^{\infty} \frac{a^{2m-k}}{(q;q)_{m-k} (q;q)_m}. \end{aligned} \tag{5.2}$$

From the Jacobi triple product identity, we have

$$\begin{aligned} (e^{2i\theta}, e^{-2i\theta}; q)_\infty &= (1-e^{-2i\theta})(e^{2i\theta}, q/e^{2i\theta}; q)_\infty \\ &= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{+\infty} (-1)^n q^{\binom{n}{2}} (1-e^{-2i\theta})\, e^{2ni\theta} \\ &= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{+\infty} (-1)^n (1+q^n)\, q^{\binom{n}{2}} e^{2ni\theta}. \end{aligned} \tag{5.3}$$

Thus, we obtain

$$\int_{-\pi}^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty}\, d\theta$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \sum_{m=0}^{\infty} (-1)^n q^{\binom{n}{2}} (1+q^n)$$
$$\times \frac{a^{2m-k}}{(q;q)_{m-k}\,(q;q)_m} \int_{-\pi}^{\pi} e^{(2n-k)\,i\theta}\, d\theta.$$

The following identity is straightforward:

$$\int_{-\pi}^{\pi} e^{(2n-k)\,i\theta}\, d\theta = 2\pi \delta_{k,\,2n}.$$

Hence we have the following evaluation:

$$\int_{-\pi}^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty}\, d\theta$$
$$= \frac{2\pi}{(q;q)_\infty} \sum_{n=-\infty}^{+\infty} \sum_{m=0}^{\infty} (-1)^n (1+q^n)\, q^{\binom{n}{2}} \frac{a^{2m-2n}}{(q;q)_{m-2n}\,(q;q)_m}$$
$$= \frac{2\pi}{(q;q)_\infty} \sum_{N=0}^{\infty} \frac{a^{2N}}{(q;q)_{2N}} \sum_{n=-\infty}^{+\infty} (-1)^n (1+q^n)\, q^{\binom{n}{2}} \begin{bmatrix} 2N \\ n+N \end{bmatrix}$$
$$= \frac{4\pi}{(q;q)_\infty} \sum_{N=0}^{\infty} \frac{a^{2N}}{(q;q)_{2N}} \sum_{n=-\infty}^{+\infty} (-1)^n q^{\binom{n}{2}} \begin{bmatrix} 2N \\ n+N \end{bmatrix}$$
$$= \frac{4\pi}{(q;q)_\infty} \sum_{N=0}^{\infty} \frac{a^{2N}}{(q;q)_{2N}} \delta_{N,\,0}$$
$$= \frac{4\pi}{(q;q)_\infty}.$$

Since the above integral is an even function of θ, we obtain the following theorem.

THEOREM 5.1. *We have*

$$\int_{0}^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty}\, d\theta = \frac{2\pi}{(q;q)_\infty}. \tag{5.4}$$

The above identity is a very special case of the Askey–Wilson integral with other parameters b, c, d reduced to zero. Our aim is to arrive at the general case of the Askey–Wilson integral by three steps of parameter augmentation on the above special case. By adding the parameter b to (5.4) one obtains the identity which is closely related to the orthogonality of the Rogers–Szegö polynomials [10, 19, 32, 33]. These polynomials are a variant form of the q-Hermite polynomials. In the last section, we will revisit some fundamental properties of Rogers–Szegö polynomials by the approach of parameter augmentation.

For notational simplicity, we adopt the following notation [15]:

$$h(\cos\theta; r) = (re^{i\theta}, re^{-i\theta}; q)_\infty,$$

$$h(\cos\theta; a_1, a_2, ..., a_m) = h(\cos\theta; a_1)\, h(\cos\theta; a_2) \cdots h(\cos\theta; a_m).$$

Hence (5.4) becomes

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; a)}\, d\theta = \frac{2\pi}{(q; q)_\infty}. \tag{5.5}$$

Taking the action of $T(bD_q)$ on both sides of (5.5), we obtain

$$\int_0^\pi h(\cos 2\theta; 1)\, T(bD_q) \left\{\frac{1}{h(\cos\theta; a)}\right\} d\theta = \frac{2\pi}{(q; q)_\infty},$$

where

$$\begin{aligned} T(bD_q) \left\{\frac{1}{h(\cos\theta; a)}\right\} &= T(bD_q) \left\{\frac{1}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty}\right\} \\ &= \frac{(ab; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}; q)_\infty} \\ &= \frac{(ab; q)_\infty}{h(\cos\theta; a, b)}. \end{aligned}$$

It follows that

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; a, b)}\, d\theta = \frac{2\pi}{(q, ab; q)_\infty}. \tag{5.6}$$

Taking the action of $T(cD_q)$ on both sides of (5.6), we obtain

$$\begin{aligned} &\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; b)}\, T(cD_q) \left\{\frac{1}{h(\cos\theta; a)}\right\} d\theta \\ &\quad = \frac{2\pi}{(q; q)_\infty}\, T(cD_q) \left\{\frac{1}{(ab; q)_\infty}\right\}, \end{aligned}$$

where

$$T(cD_q)\left\{\frac{1}{h(\cos\theta;a)}\right\}=\frac{(ac;q)_\infty}{h(\cos\theta;a,c)},$$

$$T(cD_q)\left\{\frac{1}{(ab;q)_\infty}\right\}=\frac{1}{(ab,bc;q)_\infty}.$$

Thus, we have

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c)}\,d\theta=\frac{2\pi}{(q,ab,ac,bc;q)_\infty}. \tag{5.7}$$

One more action of $T(dD_q)$ on the above identity leads to the Askey–Wilson integral,

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c)}\,T(dD_q)\left\{\frac{1}{h(\cos\theta;a)}\right\}d\theta$$
$$=\frac{2\pi}{(q,bc;q)_\infty}\,T(dD_q)\left\{\frac{1}{(ab,ac;q)_\infty}\right\}, \tag{5.8}$$

where

$$T(dD_q)\left\{\frac{1}{h(\cos\theta;a)}\right\}=\frac{(ad;q)_\infty}{h(\cos\theta;a,d)},$$

$$T(dD_q)\left\{\frac{1}{(ab,ac;q)_\infty}\right\}=\frac{(abcd;q)_\infty}{(ab,ac,bd,cd;q)_\infty}.$$

The above identity (5.8) is just the Askey–Wilson integral [7, 9, 18, 19, 21, 25, 34]:

THEOREM 5.2. (Askey–Wilson). *We have*

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}\,d\theta=\frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{5.9}$$

where $\max\{|a|,|b|,|c|,|d|\}<1$.

5.2. *The Ismail–Stanton–Viennot Integral*

One naturally wonders what would happen if one tried to add another parameter, say f, to the Askey–Wilson integral. It is interesting that such a consideration of parameter augmentation on the Askey–Wilson integral leads to an integral formula obtained by Ismail–Stanton–Viennot [19]. Here we give a proof in a few lines:

Proof of the Ismail–Stanton–Viennot Integral. Taking the action of $T(fD_q)$ on the Askey–Wilson integral, one obtains

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; a, b, c, d, f)}\, d\theta = \frac{2\pi}{(q, af, bc, bd, cd; q)_\infty} T(fD_q)\left\{\frac{(abcd; q)_\infty}{(ab, ac, ad; q)_\infty}\right\}. \tag{5.10}$$

By the Leibniz formula, it follows that

$$\begin{aligned}
&T(fD_q)\left\{\frac{(abcd; q)_\infty}{(ab, ac, ad; q)_\infty}\right\}\\
&\quad=\sum_{n=0}^\infty \frac{(dc; q)_n\, b^n}{(q; q)_n} \sum_{k=0}^\infty \frac{f^k}{(q; q)_k} D_q^k \left\{\frac{a^n}{(ac, ad; q)_\infty}\right\}\\
&\quad=\sum_{n=0}^\infty \frac{(dc; q)_n\, b^n}{(q; q)_n} \sum_{k=0}^\infty \frac{f^k}{(q; q)_k} \sum_{j=0}^k q^{j(j-k)} \begin{bmatrix} k\\ j\end{bmatrix} D_q^j\\
&\qquad\times\left\{\frac{1}{(ac, ad; q)_\infty}\right\} D_q^{k-j}(aq^j)^n\\
&\quad=\sum_{n=0}^\infty \frac{(dc; q)_n\, b^n}{(q; q)_n} \sum_{j=0}^\infty \frac{(fD_q)^j}{(q; q)_j} \left\{\frac{1}{(ac, ad; q)_\infty}\right\} \sum_{m=0}^n q^{j(n-m)} a^{n-m} \begin{bmatrix} n\\ m\end{bmatrix} f^m\\
&\quad=\sum_{n=0}^\infty \frac{(dc; q)_n\, b^n}{(q; q)_n} \sum_{m=0}^n a^{n-m} \begin{bmatrix} n\\ m\end{bmatrix} f^m T(fq^{n-m}D_q)\left\{\frac{1}{(ac, ad; q)_\infty}\right\}\\
&\quad=\sum_{m=0}^\infty \frac{(bf)^m}{(q; q)_m} \sum_{k=0}^\infty \frac{(dc; q)_{k+m}}{(q; q)_k} (ab)^k\, T(fq^kD_q)\left\{\frac{1}{ac, ad; q)_\infty}\right\}\\
&\quad=\sum_{m=0}^\infty \frac{(bf)^m}{(q; q)_m} \sum_{k=0}^\infty \frac{(dc; q)_{k+m}}{(q; q)_k} (ab)^k \frac{(acdfq^k; q)_\infty}{(ac, ad, fcq^k, fdq^k; q)_\infty}\\
&\quad=\frac{(acdf; q)_\infty}{(ac, ad, cf, df; q)_\infty} \sum_{k=0}^\infty \frac{(cf, df, cd; q)_k}{(q, acdf; q)_k} (ab)^k \sum_{m=0}^\infty \frac{(cdq^k; q)_m}{(q; q)_m} (bf)^m\\
&\quad=\frac{(acdf; q)_\infty}{(ac, ad, cf, df; q)_\infty} \sum_{k=0}^\infty \frac{(cf, df, cd; q)_k}{(q, acdf; q)_k} (ab)^k \frac{(bcdfq^k; q)_\infty}{(bf; q)_\infty}\\
&\quad=\frac{(acdf, bcdf; q)_\infty}{(ac, ad, bf, cf, df; q)_\infty}\, {}_3\phi_2\left(\begin{matrix} cf, df, cd\\ acdf, bcdf\end{matrix}; q, ab\right).
\end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11), we are led to the Ismail–Stanton–Viennot integral [19]:

THEOREM 5.3. *We have*

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; a, b, c, d, f)}\, d\theta$$

$$= \frac{2\pi(acdf, bcdf; q)_\infty}{(q, ac, ad, af, bc, bd, bf, cd, cf, df; q)_\infty}\, {}_3\phi_2\left(\begin{matrix} cf, df, cd \\ acdf, bcdf \end{matrix}; q, ab\right), \tag{5.12}$$

where $\max\{|a|, |b|, |c|, |d|, |f|\} < 1$.

5.3. *The Nassrallah–Rahman Integral*

We point out that the well-known Nassrallah–Rahman integral originally obtained by the integral representation of the Sears transformation and the Bailey ${}_8\phi_7$ transformation can easily be derived from the above Ismail–Stanton–Viennot integral by an application of the q-Gauss summation formula and the q-Pfaff–Saalschütz formula.

Replacing f by fq^n in (5.12), multiplying

$$\frac{(abcd)^n}{(q, abcdf^2; q)_n}$$

on both sides, and taking summation over n, we get

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; a, b, c, d, f)}\, {}_2\phi_1\left(\begin{matrix} fe^{i\theta}, fe^{-i\theta} \\ abcdf^2 \end{matrix}; q, abcd\right) d\theta \tag{5.13}$$

$$= \frac{2\pi(acdf, bcdf; q)_\infty}{(q, ac, ad, af, bc, bd, bf, cd, cf, df; q)_\infty} \sum_{n=0}^\infty \frac{(cf, df, cd; q)_n}{(q, acdf, bcdf; q)_n} (ab)^n$$

$$\times\, {}_3\phi_2\left(\begin{matrix} q^{-n}, af, bf \\ abcdf^2, q^{1-n}/cd \end{matrix}; q, q\right). \tag{5.14}$$

By the q-Gauss summation formula (3.6), we have

$${}_2\phi_1\left(\begin{matrix} fe^{i\theta}, fe^{-i\theta} \\ abcdf^2 \end{matrix}; q, abcd\right) = \frac{h(\cos\theta; abcdf)}{(abcd, abcdf^2; q)_\infty}. \tag{5.15}$$

By the q-Pfaff–Saalschütz formula (3.4), we have

$${}_3\phi_2\left(\begin{matrix} q^{-n}, af, bf \\ abcdf^2, q^{1-n}/cd \end{matrix}; q, q\right) = \frac{(bcdf, acdf; q)_n}{(abcdf^2, cd; q)_n}. \tag{5.16}$$

Substituting (5.15) and (5.16) into (5.14), it follows that

$$\int_0^\pi \frac{h(\cos 2\theta; 1)\, h(\cos\theta; abcdf)}{h(\cos\theta; a, b, c, d, f)}\, d\theta$$

$$= \frac{2\pi(acdf, bcdf, abcd, abcdf^2; q)_\infty}{(q, ac, ad, af, bc, bd, bf, cd, cf, df; q)_\infty}\, {}_2\phi_1\left(\begin{matrix} cf, df \\ abcdf^2 \end{matrix}; q, ab\right). \quad (5.17)$$

Applying the q-Gauss summation formula again, we have

$$ {}_2\phi_1\left(\begin{matrix} cf, df \\ abcdf^2 \end{matrix}; q, ab\right) = \frac{(abdf, abcf; q)_\infty}{(abcdf^2, ab; q)_\infty}. \quad (5.18)$$

Substituting (5.18) into (5.17), we are led to the Nassrallah–Rahman integral [23]:

THEOREM 5.4. *We have*

$$\int_0^\pi \frac{h(\cos 2\theta; 1)\, h(\cos\theta; abcdf)}{h(\cos\theta; a, b, c, d, f)}\, d\theta$$

$$= \frac{2\pi(abcd, abcf, abdf, acdf, bcdf; q)_\infty}{(q, ab, ac, ad, af, bc, bd, bf, cd, cf, df; q)_\infty}, \quad (5.19)$$

where $\max\{|a|, |b|, |c|, |d|, |f|\} < 1$.

6. THE q-INTEGRAL FORM OF THE SEARS TRANSFORMATION

In this section, we point out the relationship between the Andrews–Askey integral [6] and the q-integral form of the Sears transformation [15] in the light of parameter augmentation. The following is the Andrews–Askey integral which can be derived from Ramanujan's sum of ${}_1\psi_1$:

THEOREM 6.1. *We have*

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty}\, d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}. \quad (6.1)$$

Dividing both sides of (6.1) by $(abcd; q)_\infty$, we obtain

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt, abcd; q)_\infty}\, d_q t = \frac{d(1-q)(q, dq/c, c/d; q)_\infty}{(ac, ad, bc, bd; q)_\infty}.$$

Taking the action of $T(eD_q)$ on both sides of the above identity, the q-integral form of Sears transformation can be immediately obtained from the following relations:

$$T(eD_q)\left\{\frac{1}{(at, abcd; q)_\infty}\right\} = \frac{(abcdet; q)_\infty}{(at, et, abcd, ebcd; q)_\infty},$$

$$T(eD_q)\left\{\frac{1}{(ca, da; q)_\infty}\right\} = \frac{(acde; q)_\infty}{(ca, da, ce, de; q)_\infty}.$$

THEOREM 6.2. *We have*

$$\int_c^d \frac{(qt/c, qt/d, abcdet; q)_\infty}{(at, bt, et; q)_\infty} d_q t$$
$$= \frac{d(1-q)(q, dq/c, c/d, abcd, bcde, acde; q)_\infty}{(ac, ad, bc, bd, ce, de; q)_\infty}. \tag{6.2}$$

7. AN EXTENSION OF THE ASKEY–ROY INTEGRAL: GASPER'S FORMULA

We observe that a recent integral formula discovered by Gasper [14] and proved also by Rahman and Suslov [26] can be derived from the Askey–Roy integral in one step of parameter augmentation. The Askey–Roy integral [26] is given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_\infty} d\theta$$
$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho; q)_\infty}{(q, ac, ad, bc, bd; q)_\infty}. \tag{7.1}$$

Dividing both sides of the above equation by $(abcd; q)_\infty$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, abcd; q)_\infty} d\theta$$
$$= \frac{(\rho c/d, dq/\rho c, \rho, q/\rho; q)_\infty}{(q, ac, ad, bc, bc, bd; q)_\infty}. \tag{7.2}$$

Taking the action of $T(fD_q)$ on both sides of (7.2), and applying the relations

$$T(fD_q)\left\{\frac{1}{(ae^{i\theta}, abcd; q)_\infty}\right\} = \frac{(abcdfe^{i\theta}; q)_\infty}{(ae^{i\theta}, fe^{i\theta}, abcd, bcdf; q)_\infty},$$

$$T(fD_q)\left\{\frac{1}{(ac, ad; q)_\infty}\right\} = \frac{(acdf; q)_\infty}{(ac, ad, cf, df; q)_\infty},$$

we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, fe^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_\infty}\, d\theta$$

$$= \frac{(\rho c/d, dq/\rho c, \rho, q/\rho, abcd, bcdf, acdf; q)_\infty}{(q, ac, ad, bc, bd, cf, df; q)_\infty}, \tag{7.3}$$

where $\max\{|a|, |b|, |c|, |d|\} < 1$, $cd\rho \neq 0$. This is exactly the formula recently discovered by Gasper [14].

8. THE ROGERS–SZEGÖ POLYNOMIALS

The Rogers–Szegö polynomials play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey–Wilson integral [8, 18, 19]. We observe that some important results on the Rogers–Szegö polynomials or the equivalent forms on q-Hermite polynomials naturally fall into the framework of parameter augmentation such as Mehler's formula, and the linearization formula and its inverse [4, 8, 11, 18, 19, 22]. The Rogers–Szegö polynomial is defined by

$$h_n(x \mid q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{8.1}$$

which has the following generating function:

$$\sum_{n=0}^{\infty} h_n(x \mid q)\frac{t^n}{(q; q)_n} = \frac{1}{(t, xt; q)_\infty}. \tag{8.2}$$

The q-Hermite polynomials $H_n(x \mid q)$ is often defined by its generating function [11]:

$$\sum_{n=0}^{\infty} H_n(x \mid q)\frac{t^n}{(q; q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1 - 2xtq^n + t^2q^{2n})}.$$

The Rogers–Szegö polynomials and the q-Hermite polynomials are related by

$$H_n(\cos\theta \mid q) = e^{-in\theta} h_n(e^{2i\theta} \mid q).$$

The polynomials $h_n(x \mid q)$ can easily be represented by the augmentation operator as follows:

$$h_n(x \mid q) = T(D_q)\, x^n. \tag{8.3}$$

Using the above operator definition of the Rogers–Szegö polynomial and our augmentation argument, it is easy to derive Mehler's formula.

THEOREM 8.1. *We have*

$$\sum_{n=0}^{\infty} h_n(x \mid q)\, h_n(y \mid q) \frac{t^n}{(q;q)_n} = \frac{(xyt^2;q)_\infty}{(t, xt, yt, xyt; q)_\infty}. \tag{8.4}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n(x \mid q)\, h_n(y \mid q) \frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} h_n(y \mid q)\, T(D_q) \left\{ \frac{(xt)^n}{(q;q)_n} \right\} \\
&= T(D_q) \left\{ \sum_{n=0}^{\infty} h_n(y \mid q) \frac{(xt)^n}{(q;q)_n} \right\} \\
&= T(D_q) \left\{ \frac{1}{(xt, xyt; q)_\infty} \right\} \\
&= \frac{(xyt^2;q)_\infty}{(t, xt, yt, xyt; q)_\infty},
\end{aligned}$$

as desired. ∎

The above formula is a special case of the following identity due to Rogers [27, 28] which implies the linearization formula for Rogers–Szegö polynomials, and accordingly for the q-Hermite polynomials. A simple proof of the Rogers formula is given by Bressoud [11] based the recurrence relation. As is shown below, the Rogers formula becomes apparent from the point view of parameter augmentation,.

THEOREM 8.2. *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{m+n}(x \mid q)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m} = (tsx;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_n(x \mid q)\,h_m(x \mid q)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}. \tag{8.5}$$

Proof. The left-hand side of (8.5) equals

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(D_q)\,x^{m+n}\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}
&= T(D_q)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(tx)^n}{(q;q)_n}\,\frac{(sx)^m}{(q;q)_m}\\
&= T(D_q)\,\frac{1}{(tx, sx;q)_\infty}\\
&= (tsx;q)_\infty\,(tx, sx, t, s;q)_\infty\\
&= (tsx;q)_\infty\,\frac{1}{(t, tx;q)_\infty}\,\frac{1}{(s, sx;q)_\infty}\\
&= (tsx;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_n(x \mid q)\,h_m(x \mid q)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m},
\end{aligned}$$

as required. ∎

Substituting the expansion

$$\frac{1}{(tsx;q)_\infty} = \sum_{n=0}^{\infty}\frac{(ts)^n\,x^n}{(q;q)_n}$$

in (8.5) and comparing the coefficients of $t^n s^m$, we get the linearization formula for $h_n(x \mid q)$:

THEOREM 8.3. *We have*

$$h_n(x \mid q)\,h_m(x \mid q) = \sum_{k=0}^{\min\{m,n\}} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix} (q;q)_k\,x^k h_{n+m-2k}(x \mid q). \tag{8.6}$$

A combinatorial proof of the above formula (8.6) is given by Ismail, Stanton, and Viennot [19]. Similarly, using the expansion

$$(tsx; q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} x^n t^n s^n$$

in (8.5), we are led to the inverse relation of the linearization formula obtained by Askey and Ismail [8]:

THEOREM 8.4. *We have*

$$h_{m+n}(x \mid q) = \sum_{k=0}^{\min\{m, n\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (q; q)_k \, q^{\binom{k}{2}} (-x)^k \, h_{n-k}(x \mid q) \, h_{m-k}(x \mid q). \tag{8.7}$$

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