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Hamiltonian square roots of skew-Hamiltonian matrices revisited

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Abstract

Recently, H. Fassbender et al. [Linear Algebra Appl. 287 (1999) 125] proved the following theorem: Every real skew-Hamiltonian matrix W has a real Hamiltonian square root H, i.e., $H^2 = W$. We prove an analog of this theorem for complex matrices. Our approach may be of independent interest, namely, we use the polar decomposition of a nonsingular operator acting in a space with the symplectic inner product.

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1. Introduction

Let $M_n(\mathbb{C})$ and $M_n(\mathbb{R})$ be the sets of $n \times n$ complex and real matrices, respectively. Define

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n. A $2n \times 2n$ matrix H of the form

$$H = \begin{pmatrix} E & F \\ G & -E^{\mathrm{T}} \end{pmatrix}$$

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is said to be *Hamiltonian* if $E, F, G \in M_n(\mathbb{C})$, with $F^{\mathrm{T}} = F$ and $G^{\mathrm{T}} = G$. Equivalently, one may characterize Hamiltonian matrices by the relation

$$(JH)^{\mathrm{T}} = JH. \tag{1}$$

A $2n \times 2n$ matrix W of the form

$$W = \begin{pmatrix} A & B \\ C & A^{\mathrm{T}} \end{pmatrix}$$

is said to be *skew-Hamiltonian* if $A, B, C \in M_n(\mathbb{C})$, with $B^{\mathrm{T}} = -B$ and $C^{\mathrm{T}} = -C$. An equivalent characterization of W is by the relation

$$(JH)^{\mathrm{T}} = -JH. \tag{2}$$

(The term 'skew-Hamiltonian' is explained by the negative sign in (2) compared to (1). Still, this is a somewhat misleading term. Let

$$x = (\xi_1, \ldots, \xi_{2n})^{\mathrm{T}} \quad \text{and} \quad y = (\eta_1, \ldots, \eta_{2n})^{\mathrm{T}}$$

be arbitrary vectors in $\mathbb{C}^{2n}$, and

$$(x, y) = \xi_1\eta_1 + \cdots + \xi_{2n}\eta_{2n}$$

their Euclidean inner product. Define a new scalar product in $\mathbb{C}^{2n}$ by

$$\langle x, y\rangle = (Jx, y). \tag{3}$$

Then skew-Hamiltonian matrices are *symmetric* operators with respect to (3), and Hamiltonian matrices are *skew-symmetric* operators.)

A $2n \times 2n$ matrix S is said to be *symplectic* if

$$S^{\mathrm{T}} J S = J. \tag{4}$$

(Symplectic matrices are orthogonal operators in $\mathbb{C}^{2n}$ equipped with scalar product (3).)

The following fact was proved in a recent paper [1].

Theorem 1. *Every real skew-Hamiltonian matrix W can be brought into 'skew-Hamiltonian Jordan form' by a symplectic similarity transformation. This means that there exists a real symplectic matrix S such that*

$$S^{-1} W S = \begin{pmatrix} K & 0 \\ 0 & K^{\mathrm{T}} \end{pmatrix}, \tag{5}$$

where $K \in M_n(\mathbb{R})$ is in real Jordan form.

An important implication of this assertion is as follows.

Theorem 2. *Every real skew-Hamiltonian matrix W has a real Hamiltonian square root H, i.e., $H^2 = W$.*

In "Note added in Proof", Fassbender et al. [1] mention that since acceptance of their paper, they obtained similar results for complex Hamiltonian and skew-Hamiltonian matrices; see their technical report [2].

Our intention in this paper is to prove the following assertion, which is a complex counterpart of Theorem 1.

Theorem 3. *Every skew-Hamiltonian matrix $W \in M_{2n}(\mathbb{C})$ can be brought into skew-Hamiltonian Jordan form* (5), *where K is in complex Jordan form, by a symplectic similarity transformation.*

As in [1], Theorem 3 entails:

Theorem 4. *Every skew-Hamiltonian matrix $W \in M_{2n}(\mathbb{C})$ has a Hamiltonian square root H.*

To prove Theorem 3, we use an entirely different approach from that in [1]. We first show in Section 2 that every skew-Hamiltonian matrix W is *similar* to a Jordan matrix L of form (5). To this end, a well-known property of W is used, namely, that W has a double spectrum. A short proof of this property is given in Section 2. The second main ingredient of the entire proof is the following fact shown in Section 4.

Theorem 5. *If skew-Hamiltonian matrices W_1 and W_2 are similar, then they are symplectically similar, i.e., there exists a symplectic matrix S such that*

$$W_2 = S^{-1} W_1 S. \tag{6}$$

The proof of Theorem 5 uses heavily the argumentation in [3, Chapter XI, Theorem 3]. In particular, we develop in Section 3 an analog of the polar decomposition for the skew-symmetric scalar product (3). This 'symplectic' polar decomposition is of interest in its own right.

2. The Jordan form of a skew-Hamiltonian matrix

In this section, we prove:

Theorem 6. *Let $W \in M_{2n}(\mathbb{C})$ be a skew-Hamiltonian matrix. Then, among the Jordan forms of W, there exists a skew-Hamiltonian form shown in* (5), *where K is in complex Jordan form.*

Note that the theorem will be proved if we show that each $m \times m$ Jordan block $J_m(\lambda_i)$ has an even number of appearances in the Jordan form of W. It is this property that is meant when we say that W has a double spectrum. For a matrix with this property, the Jordan blocks can obviously be arranged so that a skew-Hamiltonian Jordan form (5) is obtained.

The theorem is an immediate consequence of the four propositions below.

Proposition 1. *A skew-Hamiltonian matrix $W \in M_{2n}(\mathbb{C})$ can be represented as the product*

$$W = JN, \tag{7}$$

where N is a skew-symmetric matrix.

Proof. Set $N = -JW$, and use (2). □

Proposition 2. *Let*

$$Z = N_1 N_2, \tag{8}$$

where N_2 is skew-symmetric and N_1 is nonsingular. Then rank Z *is an even integer.*

Proof. By hypothesis, $\operatorname{rank}(Z) = \operatorname{rank}(N_2)$. It is a well-known fact that the rank of a skew-symmetric matrix N_2 is an even integer (see [4, p. 217, Problem 26]). □

Proposition 3. *Assume that in* (8), *both N_1 and N_2 are skew-symmetric matrices and N_1 is nonsingular. Then for every $\lambda \in \mathbb{C}$ and every nonnegative integer m,*

$$r_m(\lambda) = \operatorname{rank}(Z - \lambda I)^m \tag{9}$$

is an even integer.

Proof. For the skew-symmetric matrix N_1 to be nonsingular, its size must be an even integer $2l$. This proves the proposition for $m = 0$, since $(Z - \lambda I)^0 = I_{2l}$.

Define $N_2(\lambda) = N_2 - \lambda N_1^{-1}$. For $m = 1$, we have

$$Z - \lambda I = N_1 N_2 - \lambda I = N_1\left(N_2 - \lambda N_1^{-1}\right) = N_1 N_2(\lambda).$$

The product on the right-hand side is that of two skew-symmetric matrices with the first factor N_1 being nonsingular. By Proposition 2, $r_1(\lambda)$ is an even integer.

For $m > 1$, write

$$\begin{aligned}(Z - \lambda I)^m &= (Z - \lambda I)(Z - \lambda I) \cdots (Z - \lambda I) \\ &= N_1\left[N_2(\lambda) N_1 N_2(\lambda) \cdots N_1 N_2(\lambda)\right].\end{aligned}$$

Then the product inside the square brackets, being a palindromic product of an odd number of skew-symmetric matrices, is clearly also skew-symmetric. Invoking again Proposition 2, we conclude that $r_m(\lambda)$ is an even integer. □

Proposition 4. *Let λ be an eigenvalue of a matrix $A \in M_n(\mathbb{C})$. Then, in the Jordan form of A, the number $S_m(\lambda)$ of $m \times m$ Jordan blocks corresponding to λ is given by*

$$S_m(\lambda) = r_{m-1}(\lambda) + r_{m+1}(\lambda) - 2r_m(\lambda), \quad m = 1, 2, \ldots, \tag{10}$$

where $r_0(\lambda) = n$.

This is a well-known fact (see, for example, [6, Problem 6.4.81]) or [4, pp. 126–127]).

Proof of Theorem 6. The required assertion is immediate from Propositions 1, 3, and 4. □

3. The symplectic polar decomposition

In this section, we regard $\mathbb{C}^{2n}$ as the scalar product space with the scalar product defined by (3). For a matrix $Z \in M_{2n}(\mathbb{C})$, we define the *symplectic adjoint* matrix Z^J by the relation

$$\langle Zx, y\rangle = \left\langle x, Z^J y\right\rangle \quad \forall x, y \in \mathbb{C}^{2n}.$$

This implies that

$$Z^J = J^{\mathrm{T}} Z^{\mathrm{T}} J = -J Z^{\mathrm{T}} J.$$

In particular,

$$Z^J = -Z, \qquad Z^J = Z, \quad \text{or} \quad Z^J = Z^{-1}, \tag{11}$$

if Z is, respectively, a Hamiltonian, skew-Hamiltonian or symplectic matrix.

Lemma 1. *For every $Z \in M_{2n}(\mathbb{C})$, both matrices*

$$ZZ^J = -ZJZ^{\mathrm{T}}J \quad \textit{and} \quad Z^J Z = -JZ^{\mathrm{T}}JZ \tag{12}$$

are skew-Hamiltonian.

Proof. It is easy to verify that (2) is fulfilled for both matrices in (12). □

Theorem 7. *Every nonsingular matrix $Z \in M_{2n}(\mathbb{C})$ can be represented as*

$$Z = WS, \tag{13}$$

where W and S are, respectively, a skew-Hamiltonian and symplectic matrix.

Proof. Suppose that (13) is valid for a given matrix Z. Then

$$Z^J = S^J W^J = S^{-1} W$$

and

$$ZZ^J = W^2.$$

Thus, W must be a square root of ZZ^J. Now, for any nonsingular $Z \in M_{2n}(\mathbb{C})$, we *define* W as a square root of ZZ^J that is a polynomial in ZZ^J. (The existence of such a root is proved, for example, in [5, p. 471].) Then W is skew-Hamiltonian (since ZZ^J is skew-Hamiltonian). Set

$$S = W^{-1}Z. \tag{14}$$

We have

$$SS^J = W^{-1}ZZ^J W^{-1} = W^{-1}\big(W^2\big)W^{-1} = I_{2n}.$$

Hence, S is symplectic, and (14) gives us the required decomposition of Z. □

Remark. In the same way, we can prove that every nonsingular matrix $Z \in M_{2n}(\mathbb{C})$ admits another polar decomposition

$$Z = \tilde{S}\tilde{W},$$

where $\tilde{W}$ and $\tilde{S}$ are again a skew-Hamiltonian and symplectic matrix, respectively. This time, $\tilde{W}$ is a square root of the matrix $Z^J Z$.

4. Proof of Theorem 5

Let $W_1, W_2 \in M_{2n}(\mathbb{C})$ be given skew-Hamiltonian matrices, and let Z be a nonsingular matrix such that

$$W_2 = Z^{-1}W_1 Z. \tag{15}$$

Then

$$W_2 = W_2^J = Z^J W_1\big(Z^J\big)^{-1}.$$

Combining this with (15), we obtain

$$\big(ZZ^J\big)W_1 = W_1\big(ZZ^J\big),$$

i.e., ZZ^J commutes with W_1. If (13) is the symplectic polar decomposition of Z, then W, being a polynomial in ZZ^J, also commutes with W_1. As a result, we have

$$W_2 = Z^{-1}W_1 Z = S^{-1}W^{-1}(W_1 W)S = S^{-1}W^{-1}(WW_1)S = S^{-1}W_1 S,$$

which proves Theorem 5. □

5. Proof of Theorems 3 and 4

Theorem 3 is an immediate corollary of Theorem 5: just apply the latter to a given skew-Hamiltonian matrix W and its skew-Hamiltonian Jordan form (5).

Theorem 4 is obtained from Theorem 3 in the same way as Theorem 2 is derived from Theorem 1. For completeness, we remind the reader the clever trick used by Fassbender et al. [1] to do this derivation. Let (5) be the skew-Hamiltonian Jordan form of W. Express K as a product $K = XY$ of two complex symmetric matrices X and Y. Then W is the square of the complex Hamiltonian matrix

$$S\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} S^{-1}.$$

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