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Fundamental study

Global dynamics for non-autonomous reaction-diffusion neural networks with time-varying delays

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ABSTRACT

In this paper, a class of non-autonomous reaction-diffusion neural networks with time-varying delays is considered. Novel methods to study the global dynamical behavior of these systems are proposed. Employing the properties of diffusion operator and the method of delayed inequalities analysis, we investigate global exponential stability, positive invariant sets and global attracting sets of the neural networks under consideration. Furthermore, conditions sufficient for the existence and uniqueness of periodic attractors for periodic neural networks are derived and the existence range of the attractors is estimated. Finally two examples are given to demonstrate the effectiveness of these results.

1. Introduction

Dynamics of autonomous neural networks based on Hopfield architecture has attracted considerable attention due to its important role in designs and applications to optimization, pattern recognition, signal processing and associative memories, and so on. Many important results have been obtained, e.g., in [1–11]. However, as we well know, non-autonomous phenomena often occur in many realistic systems. Particularly when we consider the long-term dynamical behavior of a system, network coefficients are subject to environmental disturbances and frequently vary with time. In this case, non-autonomous neural network model [12–14] can even accurately depict evolutionary processes of networks. Therefore, it is important and, in effect, necessary to study the dynamics of non-autonomous neural networks.

In the past few decades, many scientists were interested in electronic implementation of neural networks. However, strictly speaking, the diffusion effect cannot be avoided when electrons are moving in asymmetric electromagnetic fields. Consequently, diffusion phenomena should be introduced into these systems. The stability of autonomous neural networks with diffusion terms has been considered in [15–18]. But the dynamics of reaction-diffusion neural networks has not yet been fully developed. So, further investigation to these neural networks is significant.

In this article, we consider a class of non-autonomous reaction-diffusion neural networks with time-varying delays

$$\begin{cases} \dfrac{\partial u_i(t,x)}{\partial t} = \displaystyle\sum_{k=1}^{m} \dfrac{\partial}{\partial x_k}\left(D_{ik}\dfrac{\partial u_i(t,x)}{\partial x_k}\right) - a_i(t)u_i(t,x) + \sum_{j=1}^{n} b_{ij}(t)g_j(u_j(t-\tau_{ij}(t),x)) + J_i(t), \\ u_i(t_0+s,x) = \phi_i(s,x), \quad -\tau \le s \le 0,\ x \in \Omega, \\ \dfrac{\partial u_i(t,x)}{\partial n} := \left(\dfrac{\partial u_i(t,x)}{\partial x_1}, \ldots, \dfrac{\partial u_i(t,x)}{\partial x_m}\right)^{\mathrm{T}} = 0, \quad t \ge t_0 \ge 0,\ x \in \partial\Omega,\ i = 1,2,\ldots,n. \end{cases} \tag{1}$$

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where $\tau_{ij}(t)$ is the transmission delay at time t with $0 \leq \tau_{ij}(t) \leq \tau$ and $\tau > 0$ is a constant. $\Omega \subset R^m$ is a bounded domain with smooth boundary $\partial\Omega$ and measure $\mu = \text{mes}\,\Omega > 0$. n corresponds to the number of units in neural networks. x_k $(k = 1, 2, \ldots, m)$ corresponds to the kth coordinate in the space. $u_i(t, x)$ corresponds to the state of the ith unit at time t and in space x. Smooth function $D_{ik} = D_{ik}(t, x) \geq 0$ corresponds to the transmission diffusion operator along the ith unit. $g_j(u_j)$ is the activation function of the jth unit. $a_i(t) \geq 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs. $b_{ij}(t)$ denotes the strength of the jth neuron on ith unit at time $t - \tau_{ij}(t)$. $J_i(t)$ is the external bias on the ith unit at time t; $\phi(s, x) = (\phi_1(s, x), \phi_2(s, x), \ldots, \phi_n(s, x))^{\mathrm{T}}$ is the initial value. We always assume that functions $a_i(t)$, $b_{ij}(t)$ and $J_i(t)$ are continuous for $t \in R$ and g_j is the globally Lipschitz continuous, $i, j \in \mathcal{N} \triangleq \{1, 2, \ldots, n\}$.

The main difficulty for global dynamical behavior analysis of system (1) comes from both the diffusion effect and non-autonomous phenomena. The existing criteria on stability for neural networks [1–11] may be difficult and even ineffective for system (1). Therefore, techniques and methods for asymptotic property analysis of non-autonomous reaction-diffusion neural networks with time-varying delays should be developed and explored. Based on this, novel methods to study the global dynamical behavior of system (1) are proposed in this paper. By the properties of diffusion operators and the method of delayed inequalities analysis, we investigate global exponential stability, positive invariant sets and global attracting sets for non-autonomous reaction-diffusion neural networks (1). So the estimate for attracting sets of neural networks (1) is obtained. Estimates can play an important role in applications such as signal analysis and optimal computation of neural networks. Furthermore, employing the Banach fixed point theorem, we obtain the existence and uniqueness of the periodic attractor and provide the existence range of the periodic attractor for periodic neural networks (1). Finally two examples are given to demonstrate the effectiveness of our results.

This paper is organized as follows. In Section 2, we introduce some notations, definitions and lemmas. Section 3 discusses the global dynamical behaviors of non-autonomous neural networks (1). And the periodic attractor and its existence range are investigated in Section 4. Examples to illustrate the proposed methods are included in Section 5. Conclusions are drawn in Section 6.

2. Preliminaries

Let $\mathcal{C} = C([-\tau, 0], (L^2(\Omega))^n)$. Then for $\phi(s, x) \in \mathcal{C}$, we define $[\phi]_\tau^+ \triangleq (\|\phi_1\|_{2\tau}, \|\phi_2\|_{2\tau}, \ldots, \|\phi_n\|_{2\tau})^{\mathrm{T}}$, where $\|\phi_i(s, x)\|_{2\tau} \triangleq \max_{-\tau \leq s \leq 0} \|\phi_i(s, x)\|_2$ and $\|\phi_i(s, x)\|_2 \triangleq (\int_\Omega \phi_i^2(s, x)\mathrm{d}x)^{\frac{1}{2}}$, $i \in \mathcal{N}$. For any real matrices $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{n\times m}$, we write $A \geq B$ if $a_{ij} \geq b_{ij}$, $\forall i \in \mathcal{N}, j \in \{1, 2, \ldots, m\}$.

For $\phi \in \mathcal{C}$ and $t_0 \in R$, there exists a solution $u(t, x) = (u_1(t, x), \ldots, u_n(t, x))^{\mathrm{T}} = u(t; t_0, \phi) \in C([t_0, \infty), (L^2(\Omega))^n)$ for $t \geq t_0$ [19]. It is verified that $u_t(t_0, \phi) \in \mathcal{C}$, where $u_t(t_0, \phi)(s) = u(t + s; t_0, \phi)$ for $s \in [-\tau, 0]$.

Definition 1. A set $S \subset \mathcal{C}$ is called to be a positive invariant set of system (1) if for any initial value $\phi \in S$, $u_t(t_0, \phi) \in S$, $\forall t \geq t_0$, where $u_t(t_0, \phi)(s) = u(t + s; t_0, \phi))$ for $s \in [-\tau, 0]$.

Definition 2. System (1) is said to be globally exponentially stable, if there are constants $\lambda > 0$ and $M \geq 1$ such that for any two solutions $u(t; t_0, \phi)$ and $u(t; t_0, \psi)$ with the initial functions $\phi, \psi \in \mathcal{C}$, respectively, one has

$$\|u(t; t_0, \phi) - u(t; t_0, \psi)\| \leq M\|\phi - \psi\|_\tau \mathrm{e}^{-\lambda(t-t_0)}, \quad \forall t \geq t_0,$$

where, $\|u\| \triangleq (\sum_{i=1}^n \|u_i(t, x)\|_2^2)^{\frac{1}{2}}$, $\|u_i(t, x)\|_2 \triangleq (\int_\Omega u_i^2(t, x)\mathrm{d}x)^{\frac{1}{2}}$, $i \in \mathcal{N}$, $\|\phi\|_\tau \triangleq (\sum_{i=1}^n \|\phi_i(s, x)\|_{2\tau}^2)^{\frac{1}{2}} = |[\phi]_\tau^+|$, $|.|$ is Euclidean norm of R^n.

Definition 3. A set $S \subset \mathcal{C}$ is called a global attracting set of system (1), if for any initial value $\phi \in \mathcal{C}$, the solution $u_t(t_0, \phi)$ converges to S as $t \to \infty$, that is,

$$\text{dist}(u_t(t_0, \phi), S) \to 0, \quad \text{as } t \to \infty,$$

where $\text{dist}(u_t(t_0, \phi), S) = \inf_{\varphi \in S} \|u_t(t_0, \phi) - \varphi\|_\tau$.

Definition 4 (*[20]*). The matrix $A = (a_{ij})_{n\times n}$ is called an M-matrix if the following conditions hold.

(i) $a_{ii} > 0,\ i \in \mathcal{N}$, and $a_{ij} \leq 0,\ i \neq j,\ i, j \in \mathcal{N}$.
(ii) A is inverse-positive; that is, A^{-1} exists and $A^{-1} \geq 0$.

Definition 5. Let $C^n = C([t - \tau, t], R^n)$, $\tau \geq 0$. A function $F(t, x, y) \in C(R^+ \times R^n \times C^n, R^n)$ is called an M-function, if the following conditions hold.

(i) $F(t, x, y^{(1)}) \leq F(t, x, y^{(2)})$, $\quad \forall t \in R^+, x \in R^n, y^{(1)}, y^{(2)} \in C^n$ and $y^{(1)} \leq y^{(2)}$.
(ii) $\forall t \in R^+, y \in C^n, x^{(1)}, x^{(2)} \in R^n, x^{(1)} \leq x^{(2)}$ and there is some $i_0 \in \mathcal{N}$ satisfying $x_{i_0}^{(1)} = x_{i_0}^{(2)}$, then

$$F_{i_0}(t, x^{(1)}, y) \leq F_{i_0}(t, x^{(2)}, y).$$

Lemma 1 (*Generalized Halanay Inequality [20]*). *Assume that*

(i) $x(t) < y(t),\ t \in [t_0 - \tau, t_0]$.
(ii) $D^+y(t) > F(t, y(t), y^s(t)),\ D^+x(t) \leq F(t, x(t), x^s(t)), t \geq t_0 \geq 0$,

where $F(t,x,y)$ is an M-function, $D^+y(t)$ is the upper-right derivation of $y(t)$, $x(t)=(x_1(t),\dots,x_n(t))^{\mathrm{T}}$, $y(t)=(y_1(t),\dots,y_n(t))^{\mathrm{T}}$, $x^s(t)\triangleq(x_1^s(t),\dots,x_n^s(t))^{\mathrm{T}}$, $y^s(t)\triangleq(y_1^s(t),\dots,y_n^s(t))^{\mathrm{T}}$, $x_i^s(t)\triangleq\max_{-\tau\le s\le 0}x_i(t+s)$, $y_i^s(t)\triangleq\max_{-\tau\le s\le 0}y_i(t+s)$, $i\in\mathcal{N}$. Then

$$x(t)<y(t),\quad t\ge t_0.$$

To study the global dynamic behavior of system (1), we suppose

(A_1) The activation function g_j satisfies the global Lipschitz condition, that is, there exists $\sigma_j>0$ such that

$$|g_j(u)-g_j(v)|\le\sigma_j|u-v|,\quad\forall j\in\mathcal{N},\ u,v\in R.$$

(A_2) There exist continuous function $h_i(t)>0$ and constants $\hat a_i>0$, $\hat b_{ij}\ge 0$, $\hat J_i\ge 0$ such that

$$a_i(t)\ge\hat a_i\,h_i(t),\qquad|b_{ij}(t)|\le\hat b_{ij}\,h_i(t),\qquad|J_i(t)|\le\hat J_i\,h_i(t),\quad\forall i,j\in\mathcal{N}.$$

(A_3) $\hat A-\hat B\sigma$ is an M-matrix, where $\hat A=\mathrm{diag}\{\hat a_1,\dots,\hat a_n\}$, $\hat B=(\hat b_{ij})_{n\times n}$, $\sigma=\mathrm{diag}\{\sigma_1,\dots,\sigma_n\}$.

3. Global exponential stability and attracting sets

In this section, we shall investigate the global exponential stability, positive invariant sets and global attracting sets of non-autonomous system (1).

Theorem 1. *Assume that conditions (A_1)–(A_3) are satisfied. Let $\hat J=(\hat J_1,\dots,\hat J_n)^{\mathrm{T}}$ and $\mu=\mathrm{mes}\,\Omega$, $\hat g=(|g_1(0)|,\dots,|g_n(0)|)^{\mathrm{T}}$, $I=\hat B\hat g+\hat J$, $S=\{\phi\in\mathcal{C}\,|\,[\phi]_\tau^+\le(\hat A-\hat B\sigma)^{-1}I\mu\}$. Then S is a positive invariant set of system (1).*

Proof. Without loss of generality, we let $\hat J>0$. Since $\hat A-\hat B\sigma$ is an M-matrix, from the Definition 4, we have $(\hat A-\hat B\sigma)^{-1}\ge 0$, and $N=(N_1,\dots,N_n)^{\mathrm{T}}\triangleq(\hat A-\hat B\sigma)^{-1}I\mu>0$. We now prove for $\phi\in\mathcal{C}$, when $[\phi]_\tau^+\le N$,

$$[u(t,x)]^+\triangleq(\|u_1(t,x)\|_2,\|u_2(t,x)\|_2,\dots,\|u_n(t,x)\|_2)^{\mathrm{T}}\le N,\quad\forall t\ge t_0,\tag{2}$$

where $u(t,x)=u(t;t_0,\phi)$ is the solution of system (1) with the initial functions $\phi\in\mathcal{C}$.

First, we shall prove that for $p>1$, $[\phi]_\tau^+<pN$ implies

$$[u(t,x)]^+<pN,\quad t\ge t_0.\tag{3}$$

If not, there must be l and $t_1>t_0$ such that

$$\|u_l(t_1,x)\|_2=pN_l,\qquad\|u_l(t,x)\|_2<pN_l,\quad t_0-\tau\le t<t_1,\tag{4}$$

and

$$\|u_i(t,x)\|_2\le pN_i,\quad\forall i\in\mathcal{N},\quad t_0-\tau\le t\le t_1.\tag{5}$$

Since $u_i(t,x)$ satisfies

$$\begin{aligned}\frac{\partial u_i}{\partial t}&=\sum_{k=1}^m\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right)-a_i(t)u_i+\sum_{j=1}^n b_{ij}(t)g_j(u_j(t-\tau_{ij}(t),x))+J_i(t),\\&=\sum_{k=1}^m\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right)-a_i(t)u_i+\sum_{j=1}^n b_{ij}(t)[g_j(u_j(t-\tau_{ij}(t),x))-g_j(0)]+\sum_{j=1}^n b_{ij}(t)g_j(0)+J_i(t),\quad\forall i\in\mathcal{N},\end{aligned}$$

multiply both sides of the equation above with $u_i(t,x)$, and integrate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_i^2\mathrm{d}x&=\sum_{k=1}^m\int_\Omega u_i\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}x-a_i(t)\int_\Omega u_i^2\mathrm{d}x+\sum_{j=1}^n b_{ij}(t)\int_\Omega u_i[g_j(u_j(t-\tau_{ij}(t),x))-g_j(0)]\mathrm{d}x\\&\quad+\left[\sum_{j=1}^n b_{ij}(t)g_j(0)+J_i(t)\right]\int_\Omega u_i\mathrm{d}x,\quad\forall i\in\mathcal{N}.\end{aligned}\tag{6}$$

From the boundary condition, we get

$$\begin{aligned}\sum_{k=1}^m\int_\Omega u_i\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}x&=\int_\Omega u_i\nabla\cdot\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right)_{k=1}^m\mathrm{d}x\\&=\int_\Omega\nabla\cdot\left(u_i\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right)_{k=1}^m\right)\mathrm{d}x-\int_\Omega\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right)_{k=1}^m\cdot\nabla u_i\mathrm{d}x\\&=\int_{\partial\Omega}\left(u_iD_i\frac{\partial u_i}{\partial n}\right)\cdot\mathrm{d}s-\sum_{k=1}^m\int_\Omega D_{ik}\left(\frac{\partial u_i}{\partial x_k}\right)^2\mathrm{d}x\\&=-\sum_{k=1}^m\int_\Omega D_{ik}\left(\frac{\partial u_i}{\partial x_k}\right)^2\mathrm{d}x,\quad\forall i\in\mathcal{N},\end{aligned}\tag{7}$$

where $\nabla=(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m})^{\mathrm{T}}$ is the gradient operator, $(D_{ik}\frac{\partial u_i}{\partial x_k})_{k=1}^m=(D_{i1}\frac{\partial u_i}{\partial x_1},\dots,D_{im}\frac{\partial u_i}{\partial x_m})^{\mathrm{T}}$ and $D_i=\mathrm{diag}\,\{D_{i1},\dots,D_{im}\}$.

By the conditions (A_1)–(A_2), Eqs. (6) and (7) and Schwarz inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_i\|_2^2 &\le -\hat{a}_i h_i(t)\|u_i\|_2^2 + \sum_{j=1}^{n}\hat{b}_{ij}h_i(t)\int_\Omega |u_i|\sigma_j|u_j(t-\tau_{ij}(t),x)|\mathrm{d}x + \left[\sum_{j=1}^{n}\hat{b}_{ij}|g_j(0)| + \hat{J}_i\right]h_i(t)\int_\Omega |u_i|\mathrm{d}x \\ &\le -\hat{a}_i h_i(t)\|u_i\|_2^2 + \sum_{j=1}^{n}\hat{b}_{ij}h_i(t)\sigma_j\|u_i\|_2\|u_j(t-\tau_{ij}(t),x)\|_2 + I_i h_i(t)\mu\|u_i\|_2,\end{aligned}$$

where $I_i = \sum_{j=1}^{n}\hat{b}_{ij}|g_j(0)| + \hat{J}_i$, $i \in \mathcal{N}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_i\|_2 \le -\hat{a}_i h_i(t)\|u_i\|_2 + h_i(t)\left[\sum_{j=1}^{n}\hat{b}_{ij}\sigma_j\|u_j(t-\tau_{ij}(t),x)\|_2 + I_i\mu\right], \quad \forall i \in \mathcal{N},\ t \ge t_0. \tag{8}$$

From $[\phi]_\tau^+ < pN$, (4), (5) and (8), we obtain by Gronwall's inequality

$$\begin{aligned}\|u_l(t_1,x)\|_2 &\le \mathrm{e}^{-\int_{t_0}^{t_1}\hat{a}_l h_l(s)\mathrm{d}s}\|\phi_l\|_{2\tau} + \int_{t_0}^{t_1}\mathrm{e}^{-\int_s^{t_1}\hat{a}_l h_l(\xi)\mathrm{d}\xi}h_l(s)\left[\sum_{j=1}^{n}\hat{b}_{lj}\sigma_j\|u_j(s-\tau_{lj}(s),x)\|_2 + I_l\mu\right]\mathrm{d}s \\ &< \mathrm{e}^{-\int_{t_0}^{t_1}\hat{a}_l h_l(s)\mathrm{d}s}pN_l + \frac{1}{\hat{a}_l}(1-\mathrm{e}^{-\int_{t_0}^{t_1}\hat{a}_l h_l(\xi)\mathrm{d}\xi})\left[\sum_{j=1}^{n}\hat{b}_{lj}\sigma_j pN_j + I_l\mu\right] \\ &= \mathrm{e}^{-\int_{t_0}^{t_1}\hat{a}_l h_l(s)\mathrm{d}s}\left[pN_l - \frac{1}{\hat{a}_l}\left(\sum_{j=1}^{n}\hat{b}_{lj}\sigma_j pN_j + I_l\mu\right)\right] + \frac{1}{\hat{a}_l}\left(\sum_{j=1}^{n}\hat{b}_{lj}\sigma_j pN_j + I_l\mu\right).\end{aligned} \tag{9}$$

Since $\hat{A} - \hat{B}\sigma$ is an M-matrix and $N = (\hat{A} - \hat{B}\sigma)^{-1}I\mu$, one can get $\hat{A}N = \hat{B}\sigma N + I\mu$, or

$$\hat{a}_i N_i = \sum_{j=1}^{n}\hat{b}_{ij}\sigma_j N_j + I_i\mu, \quad \forall i \in \mathcal{N},$$

yielding

$$\hat{a}_i pN_i \ge \sum_{j=1}^{n}\hat{b}_{ij}\sigma_j pN_j + I_i\mu, \quad \forall i \in \mathcal{N}, p > 1. \tag{10}$$

Noting that $\mathrm{e}^{-\int_{t_0}^{t_1}\hat{a}_l h_l(s)\mathrm{d}s} \le 1$, from (9) and (10), we obtain

$$\|u_l(t_1,x)\|_2 < \left[pN_l - \frac{1}{\hat{a}_l}\left(\sum_{j=1}^{n}\hat{b}_{lj}\sigma_j pN_j + I_l\mu\right)\right] + \frac{1}{\hat{a}_l}\left(\sum_{j=1}^{n}\hat{b}_{lj}\sigma_j pN_j + I_l\mu\right) = pN_l,$$

which contradicts the equality in (4). This shows (3). Let $p \to 1$ in (3), then (2) is true and the proof is completed. □

Remark 1. From the proof of Theorem 1 and (3), it is easy to conclude that for arbitrary $\alpha \ge 1$, $S_1 = \{\phi \in \mathcal{C} | [\phi]_\tau^+ \le \alpha N\}$ is a positive invariant set of system (1).

Theorem 2. *Suppose that the conditions (A_1)–(A_3) hold. In addition, $h_i(t) \triangleq h(t) > 0, i \in \mathcal{N}$ and $h(t)$ is a continuous and ω-periodic function, i.e.,*

$$h(t+\omega) = h(t), \quad \omega > 0.$$

Then system (1) is globally exponentially stable and the exponential convergence rate is equal to $\frac{\delta\rho}{\omega}$, where $\rho \triangleq \int_0^\omega h(t)\mathrm{d}t > 0$ and δ satisfying (16).

Proof. For any $\phi, \psi \in \mathcal{C}$, we denote

$$y(t,x) = u(t,x) - v(t,x), \tag{11}$$

where $u(t,x) = u(t;t_0,\phi)$ and $v(t,x) = v(t;t_0,\psi)$ are the solutions of (1) with the initial functions $\phi, \psi \in \mathcal{C}$, respectively. Then from system (1), $y(t,x)$ must satisfy

$$\begin{cases}\dfrac{\partial y_i}{\partial t} = \displaystyle\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i}{\partial x_k}\right) - a_i(t)y_i + \sum_{j=1}^{n}b_{ij}(t)\bar{g}_j(y_j(t-\tau_{ij}(t),x)), \\ y_i(t_0+s,x) = \phi_i(s,x) - \psi_i(s,x), \quad -\tau \le s \le 0,\ x \in \Omega, \\ \dfrac{\partial y_i}{\partial n} := \left(\dfrac{\partial y_i}{\partial x_1},\ldots,\dfrac{\partial y_i}{\partial x_m}\right)^{\mathrm{T}} = 0, \quad t \ge t_0 \ge 0,\ x \in \partial\Omega,\ i \in \mathcal{N}\end{cases} \tag{12}$$

where $\bar{g}_j(y_j(t-\tau_{ij}(t),x)) = g_j(u_j(t-\tau_{ij}(t),x)) - g_j(v_j(t-\tau_{ij}(t),x)),\ j \in \mathcal{N}$.

From condition (A_1), we have

$$|\bar{g}_j(y_j(t-\tau_{ij}(t),x))| \le \sigma_j|u_j(t-\tau_{ij}(t),x)-v_j(t-\tau_{ij}(t),x)| = \sigma_j|y_j(t-\tau_{ij}(t),x)|, \quad j\in\mathcal{N}.$$

Similar to the proof of the inequality (8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y_i\|_2 \le -\hat{a}_i h_i(t)\|y_i\|_2 + h_i(t)\sum_{j=1}^{n}\hat{b}_{ij}\sigma_j\|y_j(t-\tau_{ij}(t),x)\|_2, \quad \forall i\in\mathcal{N},\ t\ge t_0. \tag{13}$$

To employ Lemma 1, we define

$$F(t,x,y) \triangleq -h(t)\hat{A}x + h(t)\hat{B}\sigma y \in C(R^+\times R^n\times C^n, R^n).$$

Obviously, $F(t,x,y)$ satisfies the condition (i) of the Definition 5; On the other hand, for any $t\in R^+$, $y\in C^n$, $x^{(1)}, x^{(2)}\in R^n$, $x^{(1)}\le x^{(2)}$ and there is some $i_0\in\mathcal{N}$ satisfying $x^{(1)}_{i_0}=x^{(2)}_{i_0}$, then

$$F_{i_0}(t,x^{(1)},y) = -h(t)\hat{a}_{i_0}x^{(1)}_{i_0} + h(t)\sum_{j=1}^{n}\hat{b}_{i_0j}\sigma_j y_j = -h(t)\hat{a}_{i_0}x^{(2)}_{i_0} + h(t)\sum_{j=1}^{n}\hat{b}_{i_0j}\sigma_j y_j = F_{i_0}(t,x^{(2)},y).$$

So, $F(t,x,y)$ is an M-function.

By (13), one can get

$$D^+([y(t,x)]^+) \le -h(t)\hat{A}[y(t,x)]^+ + h(t)\hat{B}\sigma([y(t,x)]^+)^s = F(t,[y(t,x)]^+,([y(t,x)]^+)^s), \quad t\ge t_0, \tag{14}$$

where $([y(t,x)]^+)^s = (\|y_1(t,x)\|_2^s,\ldots,\|y_n(t,x)\|_2^s)^{\mathrm{T}}$, $\|y_i(t,x)\|_2^s = \max_{-\tau\le s\le 0}\|y_i(t+s,x)\|_2$, $\forall i\in\mathcal{N}$.

Since $h(t)$ is a continuous and ω-periodic function with $h(t)>0$, we can get $\rho=\int_0^\omega h(s)\mathrm{d}s>0$ and

$$\int_{t_0}^{t} h(s)\mathrm{d}s \ge \left(\frac{t-t_0}{\omega}-1\right)\int_0^\omega h(s)\mathrm{d}s = \left(\frac{t-t_0}{\omega}-1\right)\rho, \quad \forall t\ge t_0; \tag{15}$$

$$\int_{t-\tau}^{t} h(s)\mathrm{d}s \le \left(\frac{\tau}{\omega}+1\right)\int_0^\omega h(s)\mathrm{d}s = \left(\frac{\tau}{\omega}+1\right)\rho \triangleq \eta, \quad \forall t\ge t_0.$$

Since $\hat{A}-\hat{B}\sigma$ is an M-matrix [20], there exists an $r=(r_1,\ldots,r_n)^{\mathrm{T}}>0$ such that

$$(-\hat{A}+\hat{B}\sigma)r<0.$$

Then there must exist a $\delta>0$ such that

$$\delta r-\hat{A}r+\hat{B}\sigma r\mathrm{e}^{\eta\delta}<0. \tag{16}$$

For any $\varepsilon>0$, we define

$$q(t) = Rr(\|\phi-\psi\|_\tau+\varepsilon)\mathrm{e}^{-\delta\int_{t_0}^{t}h(s)\mathrm{d}s},$$

where R is a positive constant satisfying $Rr_i\ge 1$, $i\in\mathcal{N}$. Then

$$\begin{aligned} D^+q(t) &= -\delta rRh(t)(\|\phi-\psi\|_\tau+\varepsilon)\mathrm{e}^{-\delta\int_{t_0}^{t}h(s)\mathrm{d}s}\\ &> (-\hat{A}r+\hat{B}\sigma r\mathrm{e}^{\eta\delta})Rh(t)(\|\phi-\psi\|_\tau+\varepsilon)\mathrm{e}^{-\delta\int_{t_0}^{t}h(s)\mathrm{d}s}\\ &\ge -h(t)\hat{A}q(t)+h(t)\hat{B}\sigma rR(\|\phi-\psi\|_\tau+\varepsilon)\mathrm{e}^{-\delta\int_{t_0}^{t-\tau}h(s)\mathrm{d}s}\\ &= -h(t)\hat{A}q(t)+h(t)\hat{B}\sigma q^s(t)\\ &= F(t,q(t),q^s(t)), \quad t\ge t_0, \end{aligned} \tag{17}$$

where, $q^s(t)=(q_1^s(t),\ldots,q_n^s(t))^{\mathrm{T}}$, $q_i^s(t)=\max_{-\tau\le s\le 0}q_i(t+s)=\max_{-\tau\le s\le 0}r_iR(\|\phi-\psi\|_\tau+\varepsilon)\mathrm{e}^{-\delta\int_{t_0}^{t+s}h(\xi)\mathrm{d}\xi}$, $i\in\mathcal{N}$.

Furthermore, when $t\in[t_0-\tau,t_0]$, we have

$$\begin{aligned} \|y_i(t,x)\|_2 &\le \max_{-\tau\le s\le 0}\|y_i(t_0+s,x)\|_2 = \max_{-\tau\le s\le 0}\|\phi_i(s,x)-\psi_i(s,x)\|_2 = \|\phi_i-\psi_i\|_{2\tau}\\ &\le \|\phi-\psi\|_\tau < r_iR(\|\phi-\psi\|_\tau+\varepsilon)\mathrm{e}^{-\delta\int_{t_0}^{t}h(s)\mathrm{d}s} = q_i(t), \quad i\in\mathcal{N}. \end{aligned}$$

Then

$$[y(t,x)]^+ < q(t), \quad t\in[t_0-\tau,t_0]. \tag{18}$$

By (14), (17) and (18) and Lemma 1, we obtain

$$[y(t,x)]^+ < q(t) = rR(\|\phi-\psi\|_\tau+\varepsilon)\mathrm{e}^{-\delta\int_{t_0}^{t}h(s)\mathrm{d}s}, \quad t\ge t_0.$$

Let $\varepsilon\to 0$, then

$$[y(t,x)]^+ \le rR\|\phi-\psi\|_\tau\mathrm{e}^{-\delta\int_{t_0}^{t}h(s)\mathrm{d}s}, \quad t\ge t_0.$$

So,

$$\|y(t,x)\| = \left(\sum_{i=1}^{n}\|y_i(t,x)\|_2^2\right)^{\frac{1}{2}} = |[y(t,x)]^+| \le |r|R\|\phi-\psi\|_\tau e^{-\delta\int_{t_0}^{t}h(s)\mathrm{d}s}, \quad t \ge t_0. \tag{19}$$

By using (11), (15) and (19), for any $\phi, \psi \in \mathcal{C}$, we have

$$\|u(t;t_0,\phi)-u(t;t_0,\psi)\| \le |r|Re^{\delta\rho(\frac{t_0}{\omega}+1)}\|\phi-\psi\|_\tau e^{-\frac{\delta\rho}{\omega}t}, \quad t\ge t_0. \tag{20}$$

Therefore, system (1) is globally exponentially stable. Furthermore, its exponential convergence rate is equal to $\frac{\delta\rho}{\omega}$. The proof is completed. □

Using Theorems 1 and 2, we easily obtain the follow theorem.

Theorem 3. *Suppose all conditions in Theorem 2 are satisfied. Then* $S=\{\phi\in\mathcal{C}|[\phi]_\tau^+ \le (\hat{A}-\hat{B}\sigma)^{-1}I\mu\}$ *is a globally attracting set of system* (1).

Proof. By Theorem 2, it follows that for any $\phi, \psi \in C$, the inequality (20) holds. And by Theorem 1, S is a positive invariant set of system (1). Then, $\psi \in S$ yields

$$u_t(t_0,\psi)\in S, \quad t\ge t_0. \tag{21}$$

This together with (20) yields that

$$\begin{aligned}
\mathrm{dist}(u_t(t_0,\phi),S) &= \inf_{\varphi\in S}\|u_t(t_0,\phi)-\varphi\|_\tau \\
&\le \|u_t(t_0,\phi)-u_t(t_0,\psi)\|_\tau \\
&= \left[\sum_{i=1}^{n}\max_{-\tau\le s\le 0}\|u_i(t+s;t_0,\phi)-u_i(t+s;t_0,\psi)\|_2^2\right]^{\frac{1}{2}} \\
&\le \left[n\max_{-\tau\le s\le 0}\|u(t+s;t_0,\phi)-u(t+s;t_0,\psi)\|^2\right]^{\frac{1}{2}} \\
&\le \sqrt{n}\,|r|Re^{\delta\rho(\frac{t_0}{\omega}+1)}\|\phi-\psi\|_\tau e^{-\frac{\delta\rho}{\omega}(t-\tau)}, \quad t\ge t_0.
\end{aligned} \tag{22}$$

So, we have

$$\mathrm{dist}(u_t(t_0,\phi),S)\to 0, \quad \text{as } t\to\infty.$$

The proof is completed. □

4. Periodic attractor and its existence range

In this section, we assume that system (1) of neural networks be ω-periodic, i.e.,

$$\begin{aligned}
&a_i(t+\omega)=a_i(t), \quad J_i(t+\omega)=J_i(t), \quad b_{ij}(t+\omega)=b_{ij}(t), \quad \tau_{ij}(t+\omega)=\tau_{ij}(t), \quad \forall i,j\in\mathcal{N}, \\
&D_{ik}(t+\omega,x)=D_{ik}(t,x), \quad \forall i\in\mathcal{N}, \ k\in\{1,2,\ldots,m\}.
\end{aligned}$$

Theorem 4. *Let system* (1) *be* ω*-periodic. Suppose all conditions in Theorem 2 are satisfied. Then system* (1) *has uniquely one* ω*-periodic attractor, which is globally exponentially stable and lies in* $S=\{\phi\in\mathcal{C}|[\phi]_\tau^+\le(\hat{A}-\hat{B}\sigma)^{-1}I\mu\}$.

Proof. For any $\phi, \psi \in \mathcal{C}$, let $u(t;t_0,\phi)$ and $u(t;t_0,\psi)$ be the solutions of system (1) with the initial functions $\phi, \psi \in C$, respectively. Owing to $S\subset\mathcal{C}$ is a positive invariant set of system (1), we may define $\Upsilon: S\to S$ by

$$\Upsilon\phi = u(t+\omega;t_0,\phi), \quad \text{for } t\in[t_0-\tau,t_0].$$

Now $u(t+\omega;t_0,\phi)$ is a solution of system (1) for $t\ge t_0$ and its initial function is $\Upsilon\phi$. Hence,

$$u(t+\omega;t_0,\phi)=u(t;t_0,\Upsilon\phi), \quad \forall t\ge t_0-\tau,$$

by uniqueness. Then,

$$u(t+2\omega;t_0,\phi)=u(t+\omega;t_0,\Upsilon\phi), \quad \forall t\ge t_0-\tau.$$

Next,

$$\Upsilon^2\phi = u(t+\omega;t_0,\Upsilon\phi), \quad \text{for } t\in[t_0-\tau,t_0].$$

Thus,

$$\Upsilon^2\phi = u(t+2\omega;t_0,\phi), \quad \text{for } t\in[t_0-\tau,t_0].$$

In general,

$$\Upsilon^k \phi = u(t + k\omega; t_0, \phi), \quad \text{for } t \in [t_0 - \tau, t_0]. \tag{23}$$

From (20) and (23), there is a positive integer m_0 such that

$$\| \Upsilon^{m_0} \phi - \Upsilon^{m_0} \psi \| \le \frac{1}{2} \|\phi - \psi\|_\tau.$$

Then the operator Υ satisfies all conditions of the general Banach fixed point theorem [21, pp.724]. Therefore, Υ has a fixed point $\phi^* \in S$, that is,

$$u(t + \omega; t_0, \phi^*) = \Upsilon \phi^* = \phi^*, \quad \text{for } t \in [t_0 - \tau, t_0].$$

Thus $u(t; t_0, \phi^*)$ and $u(t + \omega; t_0, \phi^*)$ are both solutions of system (1) with the same initial function and so, by uniqueness, they are equal. This implies that $u(t; t_0, \phi^*) \in S$ is ω-periodic. From (20), it is easy to prove that ω-periodic solution of the ω-periodic system (1) is unique. Then from Theorem 2, system (1) has uniquely one globally exponentially stable ω-periodic attractor $\phi^* \in S$. The proof is completed. □

5. Examples

Example 1. Consider a system of non-autonomous reaction-diffusion neural networks with time-varying delays

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = \Delta u_1(t,x) - (4 + e^{-t}) u_1(t,x) - \sin(t^2) g_1\left(u_1\left(t - \dfrac{1}{2}, x\right)\right) + 4 g_2(u_2(t - |\cos t|, x)) + \arctan(t), \\ \dfrac{\partial u_2(t,x)}{\partial t} = \Delta u_2(t,x) - (5 + e^{-t}) u_2(t,x) + 2\cos(t^2) g_1\left(u_1\left(t - \dfrac{3}{4}, x\right)\right) - 2 g_2(u_2(t-1, x)) + \arctan(t), \\ u_i(t_0 + s, x) = \phi_i(s,x), \quad -1 \le s \le 0, \ x \in \Omega = \{(x_1,x_2,x_3) \in R^3 | x_1^2 + x_2^2 + x_3^2 = 1\}, \\ \dfrac{\partial u_i}{\partial n} := \left(\dfrac{\partial u_i}{\partial x_1}, \dfrac{\partial u_i}{\partial x_2}, \dfrac{\partial u_i}{\partial x_3}\right) = 0, \quad t \ge t_0 \ge 0, \ x \in \partial\Omega, i = 1, 2. \end{cases} \tag{24}$$

where sigmoid function $g_1(s) = g_2(s) = \tanh(s)$. It is easy to check that the conditions (A_1)–(A_3) are satisfied and we may take $\tau = 1, \sigma_1 = \sigma_2 = 1, h_1(t) = h_2(t) \equiv 1, \hat{a}_1 = 4, \hat{a}_2 = 5, \hat{b}_{11} = 1, \hat{b}_{12} = 4, \hat{b}_{21} = 2, \hat{b}_{22} = 2, \hat{J}_1 = \pi/2, \hat{J}_2 = \pi/2$. Then

$$\hat{A} = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I = \hat{B}\hat{g} + \hat{J} = \hat{J} = \begin{pmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{pmatrix}, \qquad \mu = \text{mes}\,\Omega = \frac{4\pi}{3},$$

$$\hat{A} - \hat{B}\sigma = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix} \text{ is an } M\text{-matrix}, \qquad N = (\hat{A} - \hat{B}\sigma)^{-1} I \mu = \begin{pmatrix} \frac{14\pi^2}{3} \\ \frac{10\pi^2}{3} \end{pmatrix},$$

$$S_1 = \left\{ \phi \in \mathcal{C} \mid \|\phi_1\|_{2\tau} \le \frac{14\pi^2}{3}, \ \|\phi_2\|_{2\tau} \le \frac{10\pi^2}{3} \right\}.$$

It follows from Theorems 1–3 that system (24) is globally and exponentially stable, and S_1 is a positive invariant and global attracting set of (24).

Example 2. Consider a system of 2π-periodic cellular neural networks with delays

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = \Delta u_1(t,x) - (4 + \sin t)\, u_1(t,x) - u_1\left(t - \dfrac{1}{2}, x\right) + 3 u_2(t-1, x) + 1, \\ \dfrac{\partial u_2(t,x)}{\partial t} = \Delta u_2(t,x) - 4 u_2(t,x) + u_1(t-1, x) - 2 u_2(t - |\sin t|, x) + |\cos t|, \\ u_i(t_0 + s, x) = \phi_i(s,x), \quad -1 \le s \le 0, \ x \in \Omega = \{(x_1,x_2,x_3) \in R^3 | x_1^2 + x_2^2 + x_3^2 = 1\}, \\ \dfrac{\partial u_i}{\partial n} := \left(\dfrac{\partial u_i}{\partial x_1}, \dfrac{\partial u_i}{\partial x_2}, \dfrac{\partial u_i}{\partial x_3}\right) = 0, \quad t \ge t_0 \ge 0, \ x \in \partial\Omega, i = 1, 2. \end{cases} \tag{25}$$

Take the parameters in Theorem 4 as follows: $\tau = 1, \sigma_1 = \sigma_2 = 1, h_1(t) = h_2(t) \equiv 1, \hat{a}_1 = 3, \hat{a}_2 = 4, \hat{b}_{11} = 1, \hat{b}_{12} = 3, \hat{b}_{21} = 1, \hat{b}_{22} = 2, \hat{J}_1 = 1, \hat{J}_2 = 1$. We can verify that

$$\hat{A} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I = \hat{B}\hat{g} + \hat{J} = \hat{J} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mu = \text{mes}\,\Omega = \frac{4\pi}{3},$$

$$\hat{A} - \hat{B}\sigma = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \text{ is an } M\text{-matrix}, \qquad N = (\hat{A} - \hat{B}\sigma)^{-1} I \mu = \begin{pmatrix} \frac{20\pi}{3} \\ 4\pi \end{pmatrix},$$

$$S_2 = \left\{ \phi \in \mathcal{C} \mid \|\phi_1\|_{2\tau} \le \frac{20\pi}{3}, \|\phi_2\|_{2\tau} \le 4\pi \right\}.$$

It follows from Theorem 4 that system (25) has exactly one globally exponentially stable 2π-periodic attractor in S_2.

Remark 2. To the best of our knowledge, the above global dynamical behaviors of systems (24) and (25) may not be obtained in earlier literature because of non-autonomous characteristic of the system.

6. Conclusions

In this paper, global dynamics for a class of non-autonomous reaction-diffusion neural networks with time-varying delays is investigated by employing the properties of a diffusion operator and the method of delayed inequalities analysis. In particular, we have estimated the existence range of the attracting sets and the periodic attractors which were obtained in [8,13], etc.. And in many applications, this estimate is of great interest. Furthermore, from Theorems 1–4, we conclude if reaction-diffusion terms satisfy weaker conditions, the main effect for the stability of solutions of neural networks model (1) just comes from network parameters. In additional, the given algebra criteria in Theorems 1–4 can be easily checked in practice, and it will bring some convenience for those who design and verify these neural networks.

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References

[1] Y. Zhang, Global exponential convergence of recurrent neural networks with variable delays, Theoretical Computer Science 312 (2004) 281–293.
[2] H.T. Lu, R.M. Shen, F.L. Chung, Absolute exponential stability of a class of recurrent neural networks with multiple and variable delays, Theoretical Computer Science 344 (2005) 103–119.
[3] Q. Zhang, X. Wei, J. Xu, Delay-dependent exponential stability of cellular neural networks with time-varying delays, Chaos, Solitons & Fractals 23 (2005) 1363–1369.
[4] J. Liang, J. Cao, Exponential stability of continuous-time and discrete-time bidirectional associative memory networks with delays, Chaos, Solitons & Fractals 22 (2004) 773–785.
[5] P. Van DenDriessche, X.F. Zou, Global attractivity in delayed Hopfield neural networks models, SIAM Journal of Applied Mathematics 58 (1998) 1878–1890.
[6] D.Y. Xu, H.Y. Zhao, H. Zhu, Global dynamics of Hopfield neural networks involving variable delays, Computers and Mathematics with Applications 42 (2001) 39–45.
[7] X. Li, L. Huang, H. Zhu, Global stability of cellular neural networks with constant and variable delays, Nonlinear Analysis 53 (2003) 319–333.
[8] Y. Zhang, Global exponential stability and periodic solutions of delay Hopfield neural networks, International Journal of Systems Science 27 (1996) 227–231.
[9] Z.G. Zeng, J. Wang, Global exponential stability of recurrent neural networks with time-varying delays in the presence of strong external stimuli, Neural Networks 19 (2006) 1528–1537.
[10] M. Forti, A. Tesi, New conditions for global stability of neural networks with application to linear and quadratic programming problems, IEEE Transactions on Circuits and Systems I 42 (1995) 354–366.
[11] Z.G. Zeng, J. Wang, Improved conditions for global exponential stability of recurrent neural networks with time-varying delays, IEEE Transactions on Neural Networks 17 (2006) 623–635.
[12] D.Y. Xu, H.Y. Zhao, Invariant and attracing sets of Hopfield neural networks with delay, International Journal of Systems Science 32 (2001) 863–866.
[13] S. Mohamad, K. Gopalsamy, Neuronal dynamics in time varying environments: Continuous and discrete time models, Discrete and Continuous Dynamical Systems 6 (2000) 841–860.
[14] H. Jiang, Z. Li, Z. Teng, Boundedness and stability for nonautonomous cellular neural networks with delay, Physics Letters A 306 (2003) 313–325.
[15] Q. He, L. Kang, Existence and stability of global solution for generalized Hopfield neural network system, Neural Parallel & Scientific Computation 2 (1994) 165–176.
[16] X. Liao, Y. Fu, J. Gao, X. Zhao, Stability of Hopfield neural networks with reaction–diffusion terms, Acta Electron Sinica 28 (2000) 78–80 (in Chinese).
[17] X. Liao, S. Yang, S. Cheng, Y. Zhen, Stability of generalized Hopfield neural networks with reaction–diffusition terms, Science in China (Series E) 32 (2002) 87–94 (in Chinese).
[18] L.S. Wang, D.Y. Xu, Global exponential stability of reaction-diffusion Hopfield neural networks with variable delays, Science in China (Series E) 33 (2003) 488–495 (in Chinese).
[19] J. Wu, Theory and Application of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.
[20] X. Liao, Theory and Application of Stability for Dynamical Systems, Defence Industry Publishing House, BeiJing, 2000.
[21] E. Zeidler, Nonlinear Functional Analysis and its Application, I: Fixed-point Theorems, Springer-Verlag, New York, 1986.