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An approximate method for numerically solving fractional order optimal control problems of general form☆

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A B S T R A C T

In this article, we discuss fractional order optimal control problems (FOCPs) and their solutions by means of rational approximation. The methodology developed here allows us to solve a very large class of FOCPs (linear/nonlinear, time-invariant/time-variant, SISO/MIMO, state/input constrained, free terminal conditions etc.) by converting them into a general, rational form of optimal control problem (OCP). The fractional differentiation operator used in the FOCP is approximated using Oustaloup's approximation into a state-space realization form. The original problem is then reformulated to fit the definition used in general-purpose optimal control problem (OCP) solvers such as RIOTS_95, a solver created as a Matlab toolbox. Illustrative examples from the literature are reproduced to demonstrate the effectiveness of the proposed methodology and a free final time OCP is also solved.

1. Introduction

The idea of fractional derivative dates back to a conversation between two mathematicians: Leibniz and L'Hopital. In 1695, they exchanged about the meaning of a derivative of order 1/2. Their correspondence has been well documented and is stated as the foundation of fractional calculus [1].

Many real-world physical systems display fractional order dynamics, that is their behavior is governed by fractional order differential equations [2]. For example, it has been illustrated that materials with memory and hereditary effects, and dynamical processes, including gas diffusion and heat conduction, in fractal porous media can be more adequately modeled by fractional order models than integer order models [3].

The general definition of an optimal control problem requires the minimization of a criterion function of the states and control inputs of the system over a set of admissible control functions. The system is subject to constrained dynamics and control variables. Additional constraints such as final time constraints can be considered. This paper introduces an original formulation and a general numerical scheme for a potentially almost unlimited class of FOCPs. An FOCP is an optimal control problem in which the criterion and/or the differential equations governing the dynamics of the system contain at least one fractional derivative operator.

Integer order optimal controls (IOOCs) have been discussed for a long time and a large collection of numerical techniques have been developed to solve IOOC problems. However, the number of publications on FOCPs is limited. A general formulation and a solution scheme for FOCPs were first introduced in [4] where fractional derivatives were introduced

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in the Riemann–Liouville sense, and FOCP formulation was expressed using the fractional variational principle and the Lagrange multiplier technique. The state and the control variables were given as a linear combination of test functions, and a virtual work type approach was used to obtain solutions. In [5,6], the FOCPs are formulated using the definition of fractional derivatives in the sense of Caputo, the FDEs are substituted into Volterra-type integral equations and a direct linear solver helps calculating the solution of the obtained algebraic equations. In [7], the fractional dynamics of the FOCPs are defined in terms of the Riemann–Liouville fractional derivatives. The Grunwald and Letnikov formula is used as an approximation and the resulting equations are solved using a direct scheme. Frederico and Torres [8–10], using similar definitions of the FOCPs, formulated a Noether-type theorem in the general context of the fractional optimal control in the sense of Caputo and studied fractional conservation laws in FOCPs. However, none of this work has taken advantage of the colossal research achieved in the numerical solutions of IOOCs.

In this paper, we present a formulation and a numerical scheme for FOCP based on IOOC problem formulation. Therefore, the class of FOCP solvable by the proposed methodology is closely related to the considered IOOC solver RIOTS_95 [11,12]. The fractional derivative operator is approximated in frequency-domain by using Oustaloup's Recursive Approximation which results in a state-space realization. The fractional differential equation governing the dynamics of the system is expressed as an integer order state-space realization. The FOCP can then be reformulated into an IOOC problem, solvable by a wide variety of algorithms from the literature. Three examples are solved to demonstrate the performance of the method. The work described here was first introduced in [13].

2. Fractional optimal control problem formulation

In this section, we briefly give some definitions regarding fractional derivatives allowing us to formulate a general definition of an FOCP.

There are different definitions of the fractional derivative operator. The Left Riemann–Liouville Fractional Derivative (LRLFD) of a function $f(t)$ is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\mathrm{d}\tau, \tag{1} $$

where the order of the derivative α satisfies $n-1 \le \alpha < n$. The Right Riemann–Liouville Fractional Derivative (RRLFD) is defined as

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (t-\tau)^{n-\alpha-1} f(\tau)\mathrm{d}\tau. \tag{2} $$

Another fractional derivative is the left Caputo fractional derivative LCFD defined as

$$ {}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(\tau)\mathrm{d}\tau. \tag{3} $$

The right Caputo fractional derivative RCFD defined as

$$ {}_t^CD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (t-\tau)^{n-\alpha-1} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(\tau)\mathrm{d}\tau. \tag{4} $$

From any of these definitions, we can specify a general FOCP: Find the optimal control $u(t)$ for a fractional dynamical system that minimizes the following performance criterion

$$ J(u) = G(x(a), x(b)) + \int_a^b L(x, u, t)\mathrm{d}t \tag{5} $$

subject to the following system dynamics

$$ {}_aD_t^\alpha x(t) = H(x, u, t) \tag{6} $$

with initial condition

$$ x(a) = x_a \tag{7} $$

and with the following constraints

$$ u_{\min}(t) \le u(t) \le u_{\max}(t), \tag{8} $$

$$ x_{\min}(a) \le x(a) \le x_{\max}(a), \tag{9} $$

$$ L_{ti}^\nu(t, x(t), u(t)) \le 0, \tag{10} $$

$$ G_{ei}^\nu(x(a), x(b)) \le 0, \tag{11} $$

$$ G_{ee}^\nu(x(a), x(b)) = 0 \tag{12} $$

where x is the state variable, $t \in [a, b]$ stands for the time, and F, G and H are arbitrary given nonlinear functions. The subscripts o, ti, ei, and ee on the functions $G(.,.)$ and $L(.,.,.)$ stand for, respectively, objective function, trajectory constraint, endpoint inequality constraint and endpoint equality constraint.

3. Oustaloup recursive approximation of the fractional derivative operator

Oustaloup Recursive Approximation (ORA) was introduced and is now utilized to approximate fractional order transfer functions using a rational transfer function [14,15]. The approximation is given by

$$s^\alpha = \prod_{n=1}^{N} \frac{1 + s/\omega_{z,n}}{1 + s/\omega_{p,n}}. \tag{13}$$

The resulting approximation is only valid within a frequency range $[\omega_l; \omega_h]$. The number of poles and zeros N has to be decided beforehand, and the performance of the approximation are strongly dependent on its approximation parameter choice: small values of N cause low order, simpler approximations. Consequently, the Bode diagram exhibits undulations in both phase and gain responses around the real response. Such undulations can easily be removed by increasing the value of N, at the cost of higher order and increased amount of calculations. Frequencies of poles and zeros in (13) are obtained given α, N, ω_l and ω_h by [3]:

$$\omega_{z,1} = \omega_l \sqrt{\eta}, \tag{14}$$

$$\omega_{p,n} = \omega_{z,n}\varepsilon; \quad n \in [1; N], \tag{15}$$

$$\omega_{z,n+1} = \omega_{p,n}\eta; \quad n \in [1; N-1], \tag{16}$$

$$\varepsilon = (\omega_h/\omega_l)^{\alpha/N}, \tag{17}$$

$$\eta = (\omega_l/\omega_h)^{(1-\alpha)/N}. \tag{18}$$

When $\alpha < 0$, inverting (13) helps obtaining the approximation. For $|\alpha| > 1$, our definition does not hold anymore. A practical solution is to separate the fractional orders of s in the following way:

$$s^\alpha = s^n s^\delta; \qquad \upsilon = n + \delta; \quad n \in \mathbb{Z}; \;\; \delta \in [0, 1]. \tag{19}$$

Under such condition, only s^δ needs to be approximated. Discrete approximation for the fractional differentiation operator can be found in [16].

For FOCP, such a definition of ORA as a zero-pole transfer function is not helpful. Instead, a state-space realization of the approximation is required. The first step towards a state-space realization is to expand the transfer function given in (13).

$$s^\alpha = \frac{\sum_{i=0}^{N} a_i s^i}{\sum_{j=0}^{N} b_j s^j} \tag{20}$$

where

$$a_i = \sum_{k=i}^{N} \prod_{l=0}^{k} \frac{1}{\omega(z, l)} \tag{21}$$

and

$$b_j = \sum_{k=j}^{N} \prod_{l=0}^{k} \frac{1}{\omega(p, l)}. \tag{22}$$

Eq. (20) can further be modified to match the following definition

$$s^\alpha = \frac{\sum_{i=0}^{N-1} c_i s^i}{\sum_{j=0}^{N} b_j s^j} + d \tag{23}$$

with $b_N = 1$. It is finally possible to approximate the operator s^α using a state-space definition

$$_aD_t^\alpha x(t) \approx \begin{Bmatrix} \dot{z} = Az + Bu \\ x = Cz + Du \end{Bmatrix} \tag{24}$$

with

$$A = \begin{bmatrix} -b_{N-1} & -b_{N-2} & \cdots & -b_1 & -b_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{25}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{26}$$

$$C = \begin{bmatrix} c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{bmatrix} \tag{27}$$

$$D = d. \tag{28}$$

4. Fractional optimal control problem reformulation

With our state-space approximation of the fractional derivative operator, it is now possible to reformulate the FOCP described in Eqs. (5)–(12). Find the optimal control $u(t)$ for a dynamical system that minimizes the performance criterion

$$J(u) = G(Cz(a) + Du(a), Cz(b) + Du(b)) + \int_a^b L(Cz + Du, u, t)\mathrm{d}t \tag{29}$$

subject to the following dynamics

$$\dot{z(t)} = Az + B(H(Cz + Du, u, t)) \tag{30}$$

with initial condition

$$z(a) = x_a w/(Cw) \tag{31}$$

and with the following constraints

$$u_{\min}(t) \le u(t) \le u_{\max}(t) \tag{32}$$

$$x_{\min}(a) \le Cz(a) + Du(a) \le x_{\max}(a) \tag{33}$$

$$L_{ti}^{\nu}(t, Cz(t) + Du(t), u(t)) \le 0 \tag{34}$$

$$G_{ei}^{\nu}(Cz(a) + Du(a), Cz(b) + Du(b)) \le 0 \tag{35}$$

$$G_{ee}^{\nu}(Cz(a) + Du(a), Cz(b) + Du(b)) = 0 \tag{36}$$

where z is the state vector, w is a vector of size N, $t \in [a, b]$ stands for again the time, and F, G and H are arbitrary nonlinear functions. The subscripts o, ti, ei, and ee on the functions $G(., .)$ and $L(., ., .)$ stand for, respectively, objective function, trajectory constraint, endpoint inequality constraint and endpoint equality constraint.

The choice for the vector w is indeed important as it can improve the convergence of the optimization. Since B has the form given in (26), our method here is to choose w as

$$w = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}. \tag{37}$$

The state $x(t)$ of the initial FOCP can be retrieved from

$$x(t) = Cz(t) + Du(t). \tag{38}$$

The choice of $[\omega_l; \omega_h]$ needs to be carefully taken into consideration as a narrow bandwidth may lead to inaccurate results because of possible missing dynamics, and a large bandwidth would create a large computational burden as N would increase. The choice of N is not considered here as we use the rule of thumb $N = \log(\omega_h) - \log(\omega_l)$.

This framework allows us to approximately solve a large variety of FOCPs thanks to the link we created with the traditional optimal control problems. In fact, the proposed conversion allows us to apply any readily-available IOOC solver to find an approximate solution of almost any given FOCP problem. For this paper, we decide to use the RIOTS_95 Matlab Toolbox to be briefly introduced in the next section.

5. RIOTS_95 Matlab toolbox: A brief introduction

The acronym RIOTS means "recursive integration optimal trajectory solver". It is a Matlab toolbox developed to solve a large class of optimal control problems. From [11], the following general class of IOCPs can be solved using the RIOTS_95:

$$\min_{(u,\xi)\in L^2 N_\infty[t_0,t_f]\times\mathbb{R}^K} J(u,\xi) \tag{39}$$

where

$$J(u,\xi) = g_0(\xi, \mathbf{x}(t_f)) + \int_{t_0}^{t_f} l_0(\mathbf{x}, t, u)\mathrm{d}t \tag{40}$$

subject to the following conditions and constraints

$$\begin{aligned}
&\dot{\mathbf{x}} = h(\mathbf{x}, t, \tau),\\
&\mathbf{x}(t_0) = \xi, \quad t \in [t_0, t_f],\\
&u^{(j)}_{\min}(t) \le u^{(j)}(t) \le u^{(j)}_{\max}(t), \quad j = 1, \ldots, N, t \in [t_0, t_f],\\
&\xi^{(j)}_{\min}(t) \le \xi^{(j)}(t) \le \xi^{(j)}_{\max}(t), \quad j = 1, \ldots, K, t \in [t_0, t_f],\\
&l_{ti}(\mathbf{x}(t), t, \tau(t)) \le 0, \quad t \in [t_0, t_f],\\
&g_{ei}(\xi, \mathbf{x}(t_f)) \le 0, \quad g_{ee}(\xi, \mathbf{x}(t_f)) = 0
\end{aligned}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, l : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and we have used the notation $\mathbf{q} = \{1, \cdot, q\}$ and $L^m_\infty[t_0, t_f]$ is the space of Lebesgue measurable, essentially bounded functions $[t_0, t_f] \to \mathbb{R}^m$.

The subscripts *o*, *ti*, *ei*, and *ee* on the functions $g(.,.)$ and $l(.,.,.)$ stand for, respectively, objective function, trajectory constraint, endpoint inequality constraint and endpoint equality constraint.

RIOTS_95 is based on numerical methods for which the theory is developed and summarized in [17]. The author used the approach of consistent approximations [18]. In this approach, a solution to the optimal control problem defined in (39) is obtained as "an accumulation point of the solutions to a sequence of discrete-time optimal control problems" [11]. To solve this optimal control problem, the system's dynamics is discretized according to a given sample time using Runge–Kutta integration method. The control sequence is represented as finite dimensional B-splines. The integration is done using a mesh that determines the spline breakpoints. The obtained solution is then eventually used to reformulate the problem by iterating a new discretization resulting in a more accurate approximation of the given optimal control problem. The method requires only a few of those iterations to converge to acceptable results.

6. Illustrative examples

In this section, we demonstrate the capability of the introduced approach. First we solve two widely used examples from the literature and then we introduce a new problem that none of the previously introduced methodologies attempted to solve. For each problem, we examine the solution for different values of α. For this purpose, α was taken as 0.6, 0.7, 0.8, 0.9, and 1. Problems are first stated in the traditional FOCP framework and then reformulated via our introduced methodology. Results of these studies are given at the end of each subsection. Readers interested in reproducing the results obtained here can find the code at Matlab Central [19].

6.1. Linear time-invariant problem

Our first example was first introduced in [4]. It is a linear time-invariant (LTI) fractional order optimal control problem stated as follows. Find the control $u(t)$, which minimizes the quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] \mathrm{d}t \tag{41}$$

subject to the following dynamics

$${}_0D^\alpha_t x = -x + u \tag{42}$$

with free terminal condition and the initial condition

$$x(0) = 1. \tag{43}$$

According to [20], the analytical solution of the problem defined above for $\alpha = 1$ is

$$x(t) = \cosh(\sqrt{2}t) + \beta\sinh(\sqrt{2}t) \tag{44}$$

$$u(t) = (1 + \sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2} + \beta)\sinh(\sqrt{2}t) \tag{45}$$

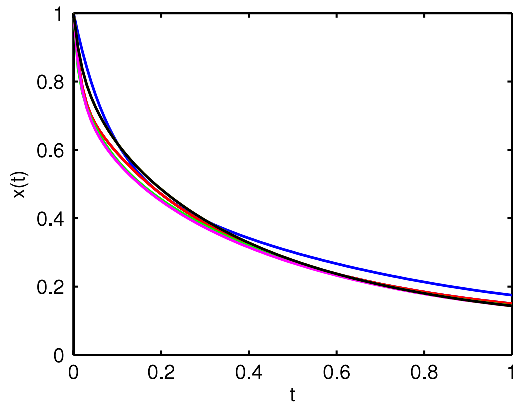

Fig. 1. State $x(t)$ as a function of t for the LTI problem for $\alpha = 0.9$ and different values of N (blue: 1, red: 2, green: 3, magenta: 4, black: 5). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

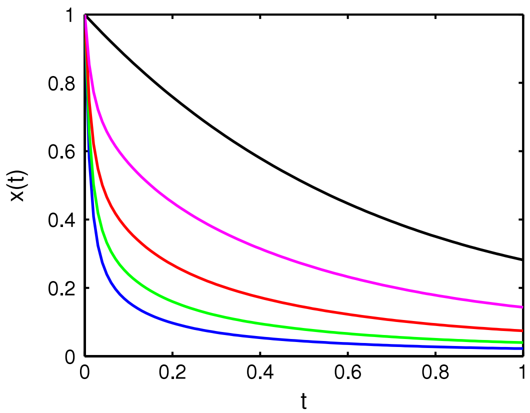

Fig. 2. State $x(t)$ as a function of t for the LTI problem for different values of α (blue: $\alpha = 0.6$, green: $\alpha = 0.7$, red: $\alpha = 0.8$, magenta: $\alpha = 0.9$, black: $\alpha = 1$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

where

$$\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2}t)}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2}t)} \approx -0.98.$$

Using the proposed methodology, we reformulate the problem defined by (41)–(43). Find the control $u(t)$, which minimizes the quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 (Cz(t) + Du(t))^2 + u^2(t)\mathrm{d}t \tag{46}$$

subject to the following dynamics

$$\dot{z} = Az + B(-(Cz + Du) + u) \tag{47}$$

and the initial condition

$$z(0) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}. \tag{48}$$

The bandwidth for the ORA is [0.01, 100] rad/s and the approximation order is $N = 4$. These values of ω_l and ω_h were obtained empirically. However, after extensive trial and error, reasons for this choice appears to be related to the dynamics of the system as well as the discretization of the control horizon necessary for RIOTS_95.

Fig. 1 illustrates the fact that choosing an approximation order $N = 4$ is relevant by showing the obtained solution for different approximation orders ($N = 1, 2, 3, 4, 5$).

Figs. 2 and 3 show the state $x(t)$ and the control input $u(t)$ as functions of time t for different values of α. For $\alpha = 1$, the results match those of the analytical solution. Results are comparable to those obtained in [4,7].

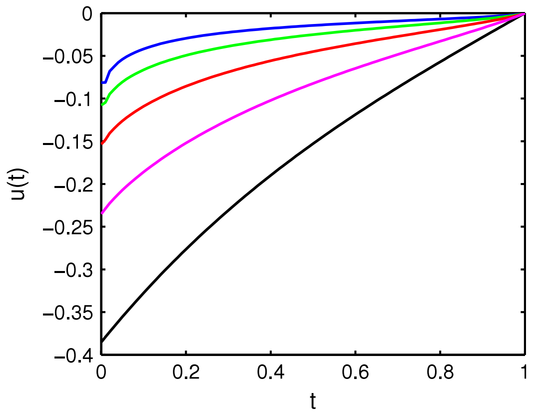

Fig. 3. Control $u(t)$ as a function of t for the LTI problem for different values of α (blue: $\alpha = 0.6$, green: $\alpha = 0.7$, red: $\alpha = 0.8$, magenta: $\alpha = 0.9$, black: $\alpha = 1$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

6.2. Linear time-variant problem

The second example studied here is also studied in [7]. It is a linear time-variant (LTV) problem stated as follows. Find the control $u(t)$, which minimizes the quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] \mathrm{d}t \tag{49}$$

subject to the following dynamics

$${}_0D_t^\alpha x = tx + u \tag{50}$$

with free terminal condition and the initial condition

$$x(0) = 1. \tag{51}$$

Using the proposed methodology, we reformulate the problem defined by Eqs. (49)–(51). Find the control $u(t)$, which minimizes the quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 (Cz(t) + Du(t))^2 + u^2(t)\mathrm{d}t \tag{52}$$

subjected to the following dynamics

$$\dot{z} = Az + B((Cz + Du)t + u) \tag{53}$$

and the initial condition

$$z(0) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}. \tag{54}$$

The bandwidth used for the ORA is [0.01, 100] rad/s and $N = 4$ as in the previous example. Fig. 4 illustrates the fact that choosing an approximation order $N = 4$ is relevant by showing the obtained solution for different approximation orders ($N = 1, 2, 3, 4, 5$).

Figs. 5 and 6 show the state $x(t)$ and the control $u(t)$ as functions of t for different values of α (0.6, 0.7, 0.8, 0.9, 1). For $\alpha = 1$, the optimal control problem has been analytically solved in [20]. In that paper, the author uses a scheme specific to integer order optimal control problems. The numerical solution obtained with the proposed methodology for $\alpha = 1$ is accurate and results for fractional orders of α matches those found in the literature.

6.3. Fractional order bang–bang optimal control

The third example studied here is called a free final time problem. It is a linear time-invariant problem stated as follows. Find the control $u(t)$ (satisfying $-2 \le u(t) \le 1$), which minimizes the quadratic performance index

$$J(u) = T \tag{55}$$

subject to the following dynamics

$${}_0D_t^\alpha x = u, \quad 1 < \alpha \le 2 \tag{56}$$

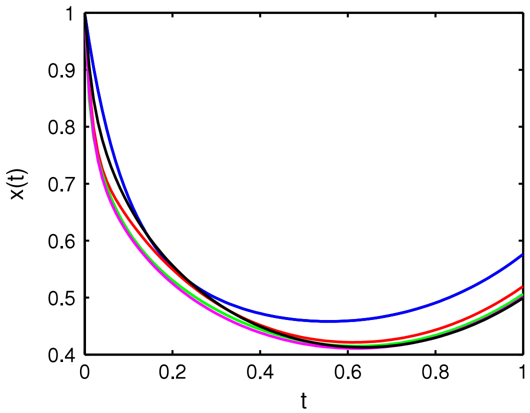

Fig. 4. State $x(t)$ as a function of t for the LTI problem for $\alpha = 0.9$ and different values of N (blue: 1, red: 2, green: 3, magenta: 4, black: 5). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

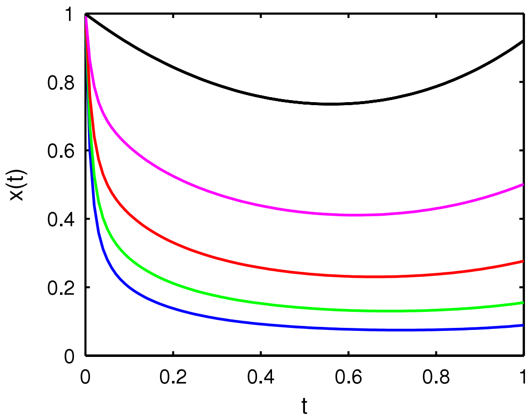

Fig. 5. State $x(t)$ as a function of t for the LTV problem for different values of α (blue: $\alpha = 0.6$, green: $\alpha = 0.7$, red: $\alpha = 0.8$, magenta: $\alpha = 0.9$, black: $\alpha = 1$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

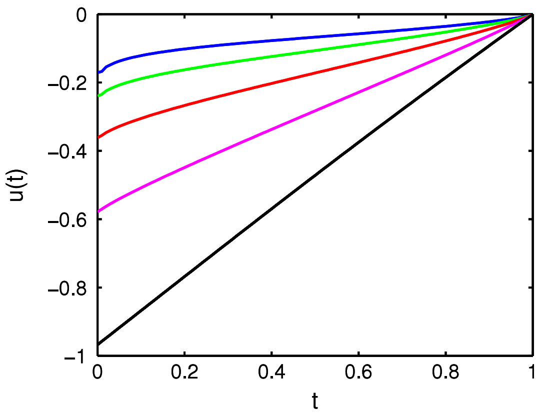

Fig. 6. Control $u(t)$ as a function of t for the LTV problem for different values of α (blue: $\alpha = 0.6$, green: $\alpha = 0.7$, red: $\alpha = 0.8$, magenta: $\alpha = 0.9$, black: $\alpha = 1$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

and the initial condition

$$\begin{aligned} x(0) &= 0 \\ \dot{x}(0) &= 0 \end{aligned} \tag{57}$$

final state constraints are

$$\begin{aligned} x(T) &= 300 \\ \dot{x}(T) &= 0. \end{aligned} \tag{58}$$

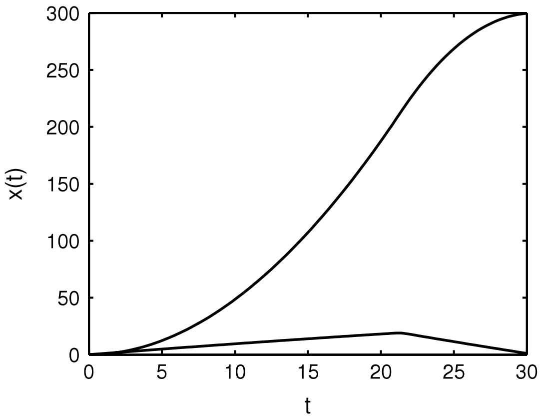

Fig. 7. States $x(t)$ and $\dot{x}(t)$ as functions of time t for the bang–bang control problem for $\alpha = 2$.

The analytical solution for this system for $\alpha = 2$, is given in [21] by $T^* = 30$ as

$$u(t) = \begin{cases} 1 & \text{for } 0 \le t < 20 \\ -2 & \text{for } 20 \le t \le 30 \end{cases} \tag{59}$$

$$x(t) = \begin{cases} t^2/2 & \text{for } 0 \le t < 20 \\ -t^2 + 60t - 600 & \text{for } 20 \le t \le 30 \end{cases} \tag{60}$$

$$\dot{x}(t) = \begin{cases} t & \text{for } 0 \le t < 20 \\ 60 - 2t & \text{for } 20 \le t \le 30. \end{cases} \tag{61}$$

Free final time problems can be transcribed into fixed final time problems by augmenting the system dynamics with additional states (one additional state for autonomous problems). The idea is to specify a nominal time interval, $[a\ b]$, for the problem and to use a scaling factor, adjustable by the optimization procedure, to scale the system dynamics and hence, in effect, scale the duration of the time interval. This scale factor, and the scaled time, are represented by the extra states. Then RIOTS_95 can minimize over the initial value of the extra states to adjust the scaling.

The problem defined by Eqs. (55)–(58) can accordingly be reformulated as: find the control $u(t)$ (satisfying $-2 \le u(t) \le 1$), which minimizes the quadratic performance index

$$J(u) = T \tag{62}$$

subject to the following dynamics

$$\begin{aligned} \dot{x}_1 &= Tx_2 \\ {}_0D_t^{\beta} x_2 &= Tu \\ (\dot{T}) &= 0 \end{aligned} \tag{63}$$

where $\beta = \alpha - 1$ and the initial conditions are

$$\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= 0 \\ T(0) &= 10 \end{aligned} \tag{64}$$

where $T(0)$ is the initial value chosen by the user. Final state constraints are

$$\begin{aligned} x_1(T) &= 300 \\ x_2(T) &= 0. \end{aligned} \tag{65}$$

To ensure the applicability of our method, we need to define a new state vector $y(t)$ such that

$$y(t) = \begin{bmatrix} x_1(t) \\ z(t) \\ T \end{bmatrix} \tag{66}$$

where $z(t)$ is the state vector of the ORA for the fractional order system described by ${}_0D_t^{\beta} x_2 = u$.

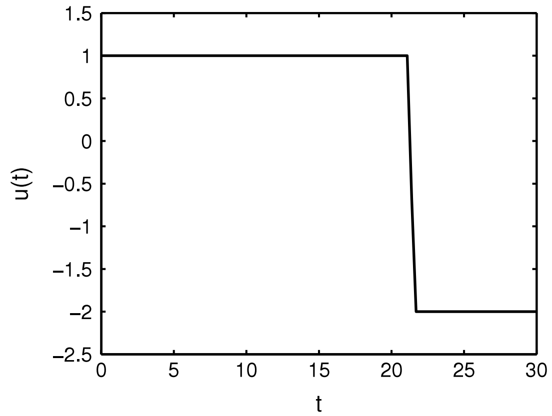

Fig. 8. Control signal $u(t)$ as a function of time t for the bang–bang control problem for $\alpha = 2$.

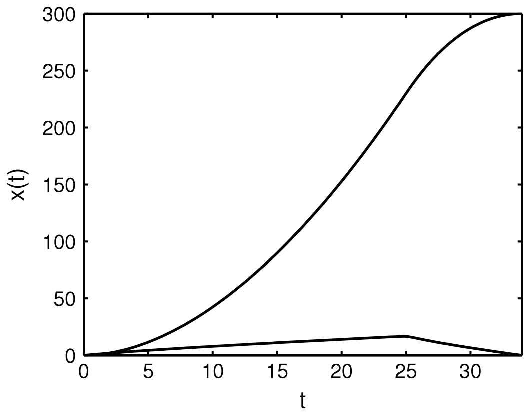

Fig. 9. States $x(t)$ and $\dot{x}(t)$ as functions of time t for the bang–bang control problem for $\alpha = 1.9$.

Table 1
Optimal durations of the solution of the bang–bang FOCP.

α	2	2	1.9	1.8	1.7	1.6
$T^*(s)$	30	30.56	35.71	46.28	61.15	82.16

Using the proposed methodology, we reformulate the problem defined by Eqs. (55)–(58). Find the control $u(t)$ (satisfying $-2 \le u(t) \le 1$), which minimizes the quadratic performance index

$$J(u) = T \tag{67}$$

subjected to the following dynamics

$$\dot{y} = \begin{bmatrix} C[y_2(t) \cdots y_{N+1}(t)]^{\mathrm{T}} + Du(t) \\ A[y_2(t) \cdots y_{N+1}(t)]^{\mathrm{T}} + Bu(t) \\ 0 \end{bmatrix} \tag{68}$$

and the initial condition

$$y(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 & T \end{bmatrix}^{\mathrm{T}} \tag{69}$$

and the final state constraints given by

$$\begin{aligned} y_1(T) &= 300 \\ C[y_2(T) \cdots y_{N+1}(T)]^{\mathrm{T}} + Du(T) &= 0. \end{aligned} \tag{70}$$

The bandwidth used for the ORA is [0.1, 10] rad/s and $N = 2$. Such values are chosen so as to reduce the computational burden and therefore the duration of the experiment.

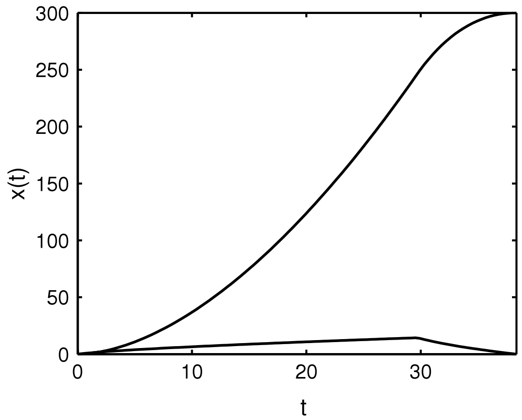

Fig. 10. States $x(t)$ and $\dot{x}(t)$ as functions of time t for the bang–bang control problem for $\alpha = 1.8$.

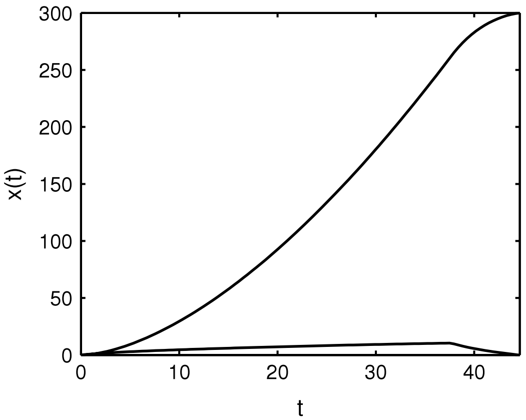

Fig. 11. States $x(t)$ and $\dot{x}(t)$ as functions of time t for the bang–bang control problem for $\alpha = 1.7$.

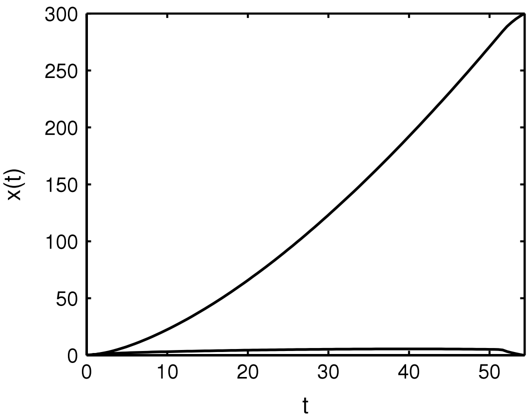

Fig. 12. States $x(t)$ and $\dot{x}(t)$ as functions of time t for the bang–bang control problem for $\alpha = 1.6$.

Figs. 7 and 8 show the state $x(t)$ and the control $u(t)$ as functions of t for $\alpha = 2$. Figs. 9–12 show the state $x(t)$ as a function of t for different values of α (1.9, 1.8, 1.7 and 1.6, respectively). As the order α approaches 2, the optimal duration nears its value for the double integrator case. Table 1 summarizes the optimal time durations of the different simulations under various α.

7. Conclusion

A new formulation towards solving a wide class of fractional optimal control problems has been introduced. The formulation made use of the Oustaloup recursive approximation to model the fractional dynamics of the system in terms of a state-space realization. This approximation created a bridge with classical optimal control problem and a readily-available optimal control solver was used to solve the fractional optimal control problem. The methodology allowed to reproduce results from the literature as well as solving a more complex problem of a fractional free final time problem. Numerical results show that the methodology, though simple, achieves good results. However, the choice of the bandwidth for the approximation is of utmost importance. For all examples, the solution for the integer order case of the problem is also obtained for the purpose of comparison.

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