View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

J. Math. Anal. Appl. 357 (2009) 447–467

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Steady compressible Navier–Stokes flow in a square

Tomasz Piasecki

Mathematical Institute, Polish Academy of Sciences, ul. Sniadeckich 8, 00-956 Warszawa, Poland

ARTICLE INFO

Article history:
Received 27 November 2008
Available online 22 April 2009
Submitted by J. Guermond

Keywords:
Navier–Stokes equations
Steady compressible flow
Inflow boundary condition
Slip boundary conditions
Strong solutions

ABSTRACT

We investigate a steady flow of compressible fluid with inflow boundary condition on the density and slip boundary conditions on the velocity in a square domain $Q \in \mathbf{R}^2$. We show existence if a solution $(v,\rho) \in W_p^2(Q) \times W_p^1(Q)$ that is a small perturbation of a constant flow ($\bar{v} \equiv [1,0]$, $\bar{\rho} \equiv 1$). We also show that this solution is unique in a class of small perturbations of the constant flow $(\bar{v},\bar{\rho})$. In order to show the existence of the solution we adapt the techniques known from the theory of weak solutions. We apply the method of elliptic regularization and a fixed point argument.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction and main results

The problems of steady compressible flows described by the Navier–Stokes equations are usually considered with the homogeneous Dirichlet boundary conditions on the velocity. It is worth from the mathematical point of view, as well as in the eye of applications, to investigate different types of boundary conditions. A significant feature of the compressible Navier–Stokes system is its mixed character: the continuity equation is elliptic in the velocity whereas the continuity equation is hyperbolic in the density. If we assume that the flow enters the domain, then the hyperbolicity of the continuity equation makes it necessary to prescribe the density on the inflow part of the boundary. A time-dependent compressible flow with inflow boundary condition has been considered by Valli and Zajaczkowski in [18]. The authors showed existence of a global in time solutions under some smallness assumptions on the data. They also obtained a stability result and existence of a stationary solution.

Plotnikov and Sokolowski investigated shape optimization problems with inflow boundary condition in 2D [16] and 3D [15], working with weak solutions. The analysis of domain dependence and other qualitative aspects of compressible flows in the framework of strong solutions encounters a barrier of lack of general existence results. Hence it is worth to cite two recent papers [13] and [14] by Plotnikov, Ruban and Sokolowski. In [13] an isentropic flow in a bounded domain past an obstacle is investigated. The authors show existence of a strong solution to the system with the right-hand side dependent on the obstacle. The result is subject to a certain condition on the geometry of the boundary and the boundary data. Next the convergence of appropriately defined finite differences with respect to the deformation of the obstacle is shown, that enables to define the shape derivative of the drag functional. In [14] Plotnikov, Ruban and Sokolowski investigated a complete heat-conducting system, showing existence of a strong solution for potential mass forces. A convergence to the solution of incompressible system when the viscosity tends to zero is also shown. The results are again subject to restrictions on the boundary and the boundary data similar as in [13].

Regular solutions to problems with inflow boundary conditions have been investigated by Kellogg and Kweon [6] and Kweon and Song [8]. The results obtained by these authors require some assumptions on the geometry of the boundary

E-mail address: T.Piasecki@impan.gov.pl.

0022-247X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2009.04.031

in the neighborhood of the points where the inflow and outflow parts of the boundary meet. In [7] Kweon and Kellogg investigated the case when the inflow and outflow parts of the boundary are separated, obtaining regular solutions.

What seems to be interesting is to investigate an inflow condition on the density combined with slip boundary conditions on the velocity, that allow to describe precisely the action between the fluid and the boundary. The slip boundary conditions have been investigated by Mucha [9] for incompressible flows, and also by Fujita [3] and Mucha and Pokorny [10] for compressible flows.

Here we investigate a steady flow of a viscous, barotropic, compressible fluid in a square domain in $\mathbf{R}^2$ satisfying inhomogeneous slip boundary conditions on the velocity combined with an inflow condition on the density. We impose that there is no flux across the bottom and the top of the square, so that it can be considered a finite, two-dimensional pipe. From the analytical point of view our domain prevents the singularity that appears in a general domain where the inflow and outflow parts of the boundary coincide.

We show existence of a solution that can be considered as a perturbation of a constant solution ($\bar{v} \equiv (1,0)$, $\bar{\rho} \equiv 0$). Under some smallness assumptions we can show an *a priori* estimate in a space $W^2_p(Q) \times W^1_p(Q)$ that is crucial in the proof of existence of the solution. Now let us formulate the problem under consideration more precisely.

The stationary compressible Navier–Stokes system describing the motion of the fluid, supplied with the slip boundary conditions, reads

$$\begin{aligned}
&\rho v \cdot \nabla v - \mu \Delta v - (\mu + \nu)\nabla \operatorname{div} v + \nabla p(\rho) = 0 \quad \text{in } Q,\\
&\operatorname{div}(\rho v) = 0 \quad \text{in } Q,\\
&n \cdot \mathbf{T}\big(v, \rho^\gamma\big) \cdot \tau + f v \cdot \tau = b \quad \text{on } \Gamma,\\
&n \cdot v = d \quad \text{on } \Gamma,\\
&\rho = \rho_{in} \quad \text{on } \Gamma_{in},
\end{aligned} \tag{1.1}$$

where $Q = [0,1] \times [0,1]$ is a square domain in $\mathbf{R}^2$ with the boundary Γ and $\Gamma_{in} = \{x \in \Gamma \colon \bar{v} \cdot n(x) < 0\}$. We will also denote $\Gamma_{out} = \{x \in \Gamma \colon \bar{v} \cdot n(x) > 0\}$ and $\Gamma_0 = \{x \in \Gamma \colon \bar{v} \cdot n(x) = 0\}$. Next, $b \in W^{1-1/p}_p(\Gamma)$, $d \in W^{2-1/p}_p(\Gamma)$ and $\rho_{in} \in W^{2-1/p}_p(\Gamma_{in})$ are given functions. $v = (v^{(1)}, v^{(2)})$ is the velocity field of the fluid and ρ is the density of the fluid. We assume that the pressure is a function of the density of the form $p(\rho) = \rho^\gamma$ for some $\gamma > 1$. The outward unit normal and tangent vectors are denoted respectively by n and τ. We assume $d = 0$ on Γ_0, what means that there is no flow across these parts of the boundary. Moreover,

$$\mathbf{T}(v, p) = 2\mu \mathbf{D}(v) + \nu \operatorname{div} v \mathbf{I} - p\mathbf{I}$$

is the stress tensor and

$$\mathbf{D}(v) = \frac{1}{2}\big\{v^i_{x_j} + v^j_{x_i}\big\}_{i,j=1,2}$$

is the deformation tensor. μ and ν are viscosity constants satisfying $\mu > 0$ and $\nu + 2\mu > 0$ and $f > 0$ is a friction coefficient. The slip boundary conditions $(1.1)_{3,4}$ are supplied with the condition $(1.1)_5$ prescribing the values of the density on the inflow part of the boundary. Under the assumptions on μ and ν the momentum equation $(1.1)_1$ is elliptic in u, whereas the continuity equation $(1.1)_2$ is hyperbolic in ρ.

Our method would also work with no modification if we considered a perturbation of the constant flow $(\bar{v}, \bar{\rho})$ satisfying $(1.1)_1$ with a term ρF on the r.h.s. provided that $\|F\|_{L_p}$ was small enough.

Since $\mathbf{T}(\bar{v}, \bar{\rho}^\gamma) = 0$, the constant flow $(\bar{v}, \bar{\rho})$ fulfills Eqs. (1.1) with boundary conditions $f\bar{v} \cdot \tau = f\tau^{(1)}$ and $n \cdot \bar{v} = n^{(1)}$.

Our main result is

Theorem 1. *Assume that* $\|b - f\tau^{(1)}\|_{W^{1-1/p}_p(\Gamma)}$, $\|d - n^{(1)}\|_{W^{2-1/p}_p(\Gamma)}$ *and* $\|\rho_{in} - 1\|_{W^{2-1/p}_p(\Gamma_{in})}$ *are small enough and* f *is large enough. Then there exists a solution* $(v, \rho) \in W^2_p(Q) \times W^1_p(Q)$ *to the system* (1.1) *and*

$$\|v - \bar{v}\|_{W^2_p} + \|\rho - \bar{\rho}\|_{W^1_p} \leqslant E, \tag{1.2}$$

where E *is a constant depending on the data, i.e. on* d, ρ_{in}, b, *the constants in the equation and the domain, that can be arbitrarily small provided that the data is small enough.*

Moreover, if (v_1, ρ_1) *and* (v_2, ρ_2) *are two solutions to* (1.1) *satisfying the estimate* (1.2) *then* $(v_1, \rho_1) = (v_2, \rho_2)$.

There are several difficulties in the proof of Theorem 1 that result, roughly speaking, from the mixed character of the problem. In a general domain a singularity appears in the points where the inflow and outflow parts of the boundary meet and we cannot apply the method used in this paper to obtain an *a priori* estimate. However, there is another difficulty in the analysis of the steady compressible Navier–Stokes system, independent on the domain. This difficulty lies in the term $u \cdot \nabla w$. Namely, if we want to apply some fixed point method then this term makes it impossible to show the compactness of the solution operator. We overcome this difficulty applying the method of elliptic regularization. We solve a sequence of approximate elliptic problems and show that this sequence converges to the solution of (1.1). This is a well-known method that has been usually applied to the issue of weak solutions [12,10], and differs from the approach of Kweon and Kellogg used to derive regular solutions in [6,7].

Let us now outline the strategy of the proof, and thus the structure of the paper. In Section 2 we start with removing inhomogeneity from the boundary conditions $(1.1)_{4,5}$. It leads to the system (2.3), and we can focus on this system instead of (1.1). In the same section we define an ϵ-elliptic regularization to the system (2.3) and introduce its linearization (2.4). In Section 3 we derive an ϵ-independent estimate on a solution of the linearized elliptic system (Theorem 2). Although linear, the system (2.4) has variable coefficients and thus its solution is not straightforward. In order to solve (2.4) we apply the Leray–Schauder fixed point theorem in Section 4, using a modification of the estimate from Theorem 2. In Section 5 we use the a priori estimate to apply the Schauder fixed point theorem to solve the approximate elliptic systems. In Section 6 we prove our main result, Theorem 1. The proof is divided into two steps. First we show that the sequence of approximate solutions converges to the solution of (2.3) and thus prove the existence of the solution to (1.1) satisfying the estimate (1.2). Next we show that this solution is unique in a class of small perturbations of the constant flow $(\bar{v},\bar{\rho})$. We see that the estimate from Theorem 3 is in fact used at three stages of the proof, therefore we show it in a detailed way in Section 3.

2. Preliminaries

In this section we remove the inhomogeneity from the boundary conditions $(2.3)_{4,5}$. Then we define an ϵ-elliptic regularization to the system (1.1). We also make some remarks concerning the notation. Let us construct $u_0 \in W^2_p(Q)$ and $w_0 \in W^2_p(Q)$ such that

$$n \cdot u_0|_\Gamma = d - n^{(1)} \quad \text{and} \quad w_0|_{\Gamma_{in}} = \rho_{in} - 1. \tag{2.1}$$

Due to the assumption of smallness of $d - n^{(1)}|_\Gamma$ and $\rho_{in} - 1|_{\Gamma_{in}}$ we can assume that

$$\|u_0\|_{W^2_p}, \|w_0\|_{W^2_p} \ll 1. \tag{2.2}$$

Now we consider

$$u = v - \bar{v} - u_0 \quad \text{and} \quad w = \rho - \bar{\rho} - w_0.$$

One can easily verify that (u, w) satisfies the following system:

$$\begin{aligned}
&\partial_{x_1} u - \mu\Delta u - (\nu+\mu)\nabla \operatorname{div} u + \gamma(w + w_0 + 1)^{\gamma-1}\nabla w = F(u, w) \quad \text{in } Q,\\
&(w + w_0 + 1)\operatorname{div} u + \partial_{x_1} w + (u + u_0)\cdot\nabla w = G(u, w) \quad \text{in } Q,\\
&n \cdot 2\mu\mathbf{D}(u)\cdot\tau + f u\cdot\tau = B \quad \text{on } \Gamma,\\
&n \cdot u = 0 \quad \text{on } \Gamma,\\
&w = 0 \quad \text{on } \Gamma_{in},
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
F(u, w) = {} & -(w + w_0 + 1)(u_0\cdot\nabla u + u\cdot\nabla u_0) - w(u_0\cdot\nabla u_0) - (w + w_0 + 1)u\cdot\nabla u - \gamma(w + w_0 + 1)^{\gamma-1}\nabla w_0\\
& + \mu\Delta u_0 + (\nu+\mu)\nabla \operatorname{div} u_0 - (w_0 + 1)u_0\cdot\nabla u_0,\\
G(u, w) = {} & -(w + w_0 + 1)\operatorname{div} u_0 - (u + u_0)\cdot\nabla w_0 - \partial_{x_1} w_0
\end{aligned}$$

and

$$B = b - 2\mu n\cdot\mathbf{D}(u_0)\cdot\tau - f\tau^{(1)}.$$

In order to prove Theorem 1 it is enough to prove the existence of a solution (u, w) to the system (2.3) provided that $\|u_0\|_{W^2_p}, \|w_0\|_{W^2_p}$ and $\|B\|_{W^{1-1/p}_p(\Gamma)}$ are small enough. As we already mentioned, the presence of the term $u\cdot\nabla w$ in the continuity equation makes it impossible to show the compactness of a solution operator if we try to apply fixed point methods directly to the system (2.3). We overcome this difficulty applying the method of elliptic regularization. The method consists of adding an elliptic term $-\epsilon\Delta w$ to the r.h.s. of $(2.3)_2$ and introducing an additional Neumann boundary condition. Since the density is already prescribed on the inflow part of the boundary by $(2.3)_5$, we impose the Neumann condition only on the remaining part of the boundary. While we are passing to the limit with the density in W^1_p-norm, the Neumann condition will disappear. Similar approach has been applied to the issue of inviscid limit for the incompressible Euler system in [5]. Consider a following linear system with variable coefficients:

$$\begin{aligned}
&\partial_{x_1} u_\epsilon - \mu \Delta u_\epsilon - (\nu+\mu)\nabla \operatorname{div} u_\epsilon + \gamma(\bar{w}+w_0+1)^{\gamma-1}\nabla w_\epsilon = F_\epsilon(\bar{u},\bar{w}) \quad \text{in } Q,\\
&(\bar{w}+w_0+1)\operatorname{div} u_\epsilon + \partial_{x_1} w_\epsilon + (\bar{u}+u_0)\cdot\nabla w_\epsilon - \epsilon\Delta w_\epsilon = G_\epsilon(\bar{u},\bar{w}) \quad \text{in } Q,\\
&n\cdot 2\mu\mathbf{D}(u_\epsilon)\cdot\tau + f u_\epsilon\cdot\tau = B \quad \text{on } \Gamma,\\
&n\cdot u_\epsilon = 0 \quad \text{on } \Gamma,\\
&w_\epsilon = 0 \quad \text{on } \Gamma_{in},\\
&\frac{\partial w_\epsilon}{\partial n} = 0 \quad \text{on } \Gamma\setminus\Gamma_{in},
\end{aligned} \tag{2.4}$$

where $(\bar{u},\bar{w}) \in W^2_p(Q)\times W^1_p(Q)$ are given functions and $F_\epsilon(\bar{u},\bar{w})$ and $G_\epsilon(\bar{u},\bar{w})$ are regularizations to $F(\bar{u},\bar{w})$ and $G(\bar{u},\bar{w})$ obtained by replacing the functions u_0 and w_0 by their regular approximations u_0^ϵ and w_0^ϵ.

Let us define an operator $T_\epsilon : \mathcal{D}\subset W^2_p(Q)\times W^1_p(Q) \to W^2_p(Q)\times W^1_p(Q)$:

$$(u_\epsilon, w_\epsilon) = T_\epsilon(\bar{u},\bar{w}) \quad \Leftrightarrow \quad (u_\epsilon,w_\epsilon) \text{ is a solution to (2.4)}, \tag{2.5}$$

where $\mathcal{D}$ is a subset of $W^2_p(Q)\times W^1_p(Q)$ that we will define later. Using the operator T_ϵ we define an ϵ-elliptic regularization to the system (2.3).

Definition 1. By an ϵ-elliptic regularization to the system (2.3) we mean a system

$$(u_\epsilon, w_\epsilon) = T_\epsilon(u_\epsilon, w_\epsilon). \tag{2.6}$$

We want to show the existence of a solution to the ϵ-elliptic regularization to the system (2.3) applying the Schauder fixed point theorem. The strategy has been outlined in the introduction. In Section 4 we show that T_ϵ is well defined, which means that for given $(\bar{u},\bar{w})$ there exists a unique solution to (2.4) (Theorem 3). In fact we show that T_ϵ is well defined for ϵ small enough, but it suffices since we are interested in small values of ϵ.

In Section 5 we show that T_ϵ satisfies the assumptions of the Schauder fixed point theorem and thus we solve the system (2.6) for ϵ small enough.

As we already said, the key point is to derive an ϵ-independent estimate for the system (2.4), which is used at different stages of the proof. We derive such estimate in the next section. Before we proceed, we will finish this introductory part with a few remarks concerning notation.

For simplicity we will denote

$$\begin{aligned}
&a_0(\bar{w}) = \frac{\gamma(\bar{w}+w_0+1)^\gamma}{\nu+2\mu},\\
&a_1(\bar{w}) = \gamma(\bar{w}+w_0+1)^{\gamma-1},\\
&a_2(\bar{w}) = \gamma(\bar{w}+w_0+1)^{\gamma-2}.
\end{aligned} \tag{2.7}$$

By C we will denote a constant that depend on the data and thus can be controlled, not necessarily arbitrarily small. If the constant depend not only on the data, but also on ϵ, we will denote it by C_ϵ. Finally, by E we will denote a constant dependent on the data that can be arbitrarily small provided that the data is small enough.

Since we will usually use the spaces of functions defined on Q, we will omit Q in the notation of a space, for example we will denote the space $L_2(Q)$ by L_2. The spaces of functions defined on the boundary will be denoted by $L_2(\Gamma)$, etc.

We do not distinguish between the spaces of vector-valued and scalar-valued functions, for example we will write $u\in W^2_p$ instead of $u \in (W^2_p)^2$.

3. A priori estimate for the linearized elliptic system

In this section we show an ϵ-independent estimate on $\|u_\epsilon\|_{W^2_p} + \|w_\epsilon\|_{W^1_p}$, where (u_ϵ,w_ϵ) is a solution to (2.4). The first step is an estimate in $H^1\times L_2$. Next we eliminate the term $\operatorname{div} u$ from $(2.4)_2$ applying the Helmholtz decomposition and the properties of the slip boundary conditions. Then we derive the higher estimate using interpolation.

3.1. Estimate in $H^1\times L_2$

In order to prove a priori estimates on H^1-norm of the velocity and L^2-norm of the density for the system (2.4) let us define a space

$$V = \{v\in H^1(Q;\mathbf{R}^2):\ v\cdot n|_\Gamma = 0\}. \tag{3.1}$$

The estimate is stated in the following lemma.

Lemma 1. *Assume that ϵ, $\|\bar u\|_{W_p^2}$ and $\|\bar w\|_{W_p^1}$ are small enough and f is large enough. Then for sufficiently smooth solutions to system* (2.4) *the following estimate is valid*

$$\|u\|_{W_2^1}+\|w\|_{L_2}\leqslant C\big[\big\|F_\epsilon(\bar u,\bar w)\big\|_{V^*}+\big\|G_\epsilon(\bar u,\bar w)\big\|_{L_2}+\|B\|_{L^2(\Gamma)}+E\|w\|_{W_p^1}\big],\tag{3.2}$$

where V^* is the dual space of V.

Before we start the proof, we shall make a remark concerning the term $\|w\|_{W_p^1}$, that is rather unexpected in an energy estimate. Its presence is due to the functions $a_1(\bar w)$ and $(\bar w+w_0+1)$ on the r.h.s. of (2.4). However, this term does not cause any problems when we apply (3.2) to interpolate in the proof of Theorem 2, since it is multiplied by a small constant.

Proof of Lemma 1. The proof is divided into three steps. First we multiply $(2.4)_1$ by u and integrate over Q. We obtain an estimate on $\|u\|_{H^1}$ in terms of the data and $\|w\|_{L_2}$. Then we apply the second equation to estimate $\|w\|_{L_2}$ and finally combine these estimates to obtain (3.2). For simplicity we will write F and G instead of $F_\epsilon(\bar u,\bar w)$ and $G_\epsilon(\bar u,\bar w)$.

Step 1. We multiply $(2.4)_1$ by u and integrate over Q. Using the boundary conditions $(2.4)_{3,4}$ we get

$$\int_Q 2\mu\mathbf{D}^2(u)+\nu\operatorname{div}^2u\,dx+\int_\Gamma\left(f+\frac{n^{(1)}}{2}\right)|u|^2\,d\sigma+\int_Q\big[a_1(\bar w)\big]\nabla wu\,dx=\int_Q Fu\,dx+\int_\Gamma B(u\cdot\tau)\,d\sigma.\tag{3.3}$$

The boundary term on the l.h.s. will be positive provided that f is large enough. Next we integrate by parts the last term of the l.h.s. of (3.3). Using $(2.4)_2$ we obtain

$$\begin{aligned}\int_Q\big[a_1(\bar w)\big]\nabla wu\,dx&=-\int_Q\big[a_1(\bar w)\big]\operatorname{div}u w\,dx-\int_Q uw\nabla\big[a_1(\bar w)\big]\,dx\\&=\int_\Gamma\frac{[a_2(\bar w)]}{2}w^2n^{(1)}\,d\sigma-\frac12\int_Q w^2\big[\partial_{x_1}a_2(\bar w)+(\bar u+u_0)\nabla a_2(\bar w)\big]\,dx-\frac12\int_Q\big[a_2(\bar w)\big]\operatorname{div}(\bar u+u_0)w^2\,dx\\&\quad-\int_Q\big[a_2(\bar w)\big]G(\bar u,\bar w)w\,dx-\epsilon\int_Q\big[a_2(\bar w)\big]w\Delta w\,dx-\int_Q uw\nabla\big[a_1(\bar w)\big]\,dx.\end{aligned}$$

Since $n^{(1)}|_{\Gamma_{out}}\equiv1$, using (3.3) and the Korn inequality ((A.1), Appendix A) we get

$$\begin{aligned}&C_Q\|u\|^2_{W_2^1}+\int_{\Gamma_{out}}\big[a_2(\bar w)\big]w^2\,d\sigma\\&\quad\leqslant\underbrace{\int_Q a_2(\bar w)\operatorname{div}(\bar u+u_0)w^2\,dx}_{I_1}+\underbrace{\int_Q\big[a_2(\bar w)\big]Gw\,dx+\int_Q Fu\,dx+\int_\Gamma B(u\cdot\tau)\,d\sigma}_{I_2}\\&\qquad+\underbrace{\epsilon\int_Q\big[a_2(\bar w)\big]w\Delta w\,dx}_{I_3}+\underbrace{\int_Q uw\nabla\big[a_1(\bar w)\big]\,dx}_{I_4}+\underbrace{\int_Q w^2\big[\partial_{x_1}a_2(\bar w)+(\bar u+u_0)\nabla a_2(\bar w)\big]\,dx}_{I_5}.\end{aligned}\tag{3.4}$$

Obviously we have $I_1\leqslant E\|w\|_{L_2}$. Now we have to deal with the term with Δw. Due to the boundary conditions $(2.4)_{5,6}$ we have

$$I_3=\epsilon\int_Q\big[a_2(\bar w)\big]w\Delta w\,dx=-\epsilon\int_Q\big[a_2(\bar w)\big]|\nabla w|^2\,dx-\epsilon\int_Q w\nabla\big[a_2(\bar w)\big]\nabla w\,dx.\tag{3.5}$$

Using Hölder inequality we get

$$\left|\int_Q w\nabla\big[a_2(\bar w)\big]\nabla w\,dx\right|\leqslant\big\|\nabla\big[a_2(\bar w)\big]\big\|_{L_p}\|w\nabla w\|_{L_{p^*}}\leqslant\big\|\nabla\big[a_2(\bar w)\big]\big\|_{L_p}\|\nabla w\|_{L_2}\|w\|_{L_q}\leqslant C\|\nabla w\|^2_{L_2},$$

where $q=\frac{2p}{p-2}<+\infty$ and $p^*=\frac{p}{p-1}$. Thus the term with ϵ on the r.h.s. of (3.4) will be negative provided that $\|\bar w\|_{W_p^1}$ will be small enough. Next,

$$I_4\leqslant\left|\int_Q uw\nabla\big[a_1(\bar w)\big]\,dx\right|\leqslant C\big\|\nabla\big[a_1(\bar w)\big]\big\|_{L_p}\|u\|_{W_2^1}\|w\|_{L_2}\leqslant E\big(\|u\|^2_{W_2^1}+\|w\|^2_{L_2}\big).$$

The last term of the r.h.s. is the most inconvenient and it must be estimated by W^1_p-norm of w, and this is the reason why this term appears in (3.2). Fortunately it is multiplied by a small constant what will turn out very important in the proof of Theorem 2. We have

$$I_5 \leqslant C\left\|a_2(\bar{w})\right\|_{W^1_p}\|w\|^2_{W^1_p} \leqslant E\|w\|^2_{W^1_p}.$$

Provided that the data is small enough, using the trace theorem to estimate the boundary term and the Hölder inequality we get

$$\|u\|^2_{W^1_2} + C\int_{\Gamma_{out}} w^2\,d\sigma \leqslant C\left[\left\|F_\epsilon(\bar{u},\bar{w})\right\|_{V^*} + \left\|G_\epsilon(\bar{u},\bar{w})\right\|_{L_2} + \|B\|_{L_2(\Gamma)}\right]\left(\|u\|_{W^1_2} + \|w\|_{L_2}\right) + E\|w\|^2_{W^1_p}. \tag{3.6}$$

Step 2. In order to derive (3.2) from (3.6) we need to find a bound on $\|w\|_{L_2}$. From $(2.4)_2$ we have

$$\partial_{x_1} w = G - (\bar{u}+u_0)\cdot\nabla w - (\bar{w}+w_0+1)\operatorname{div} u + \epsilon\Delta w,$$

thus

$$\begin{aligned} w^2(x_1,x_2) &= w^2(0,x_2) + \int_0^{x_1} 2w w_s(s,x_2)\,ds \\ &= \underbrace{\int_0^{x_1} 2w\left[G-(\bar{w}+w_0+1)\operatorname{div}u\right](s,x_2)\,ds}_{S_1} \underbrace{-\int_0^{x_1} 2w(\bar{u}+u_0)\cdot\nabla w(s,x_2)\,ds + 2\epsilon\int_0^{x_1} w\Delta w(s,x_2)\,ds}_{S_2}. \end{aligned}$$

S_1 can be estimated directly:

$$\int_Q S_1 \leqslant \left(\|G\|_{L_2} + C\|u\|_{H_1}\right)\|w\|_{L_2}. \tag{3.7}$$

It is a little more complicated to estimate S_2. We have

$$S_2 = -\int_0^{x_1}(\bar{u}+u_0)^{(1)}\partial_s w^2(s,x_2)\,ds - \int_0^{x_1}(\bar{u}+u_0)^{(2)}\partial_{x_2} w^2(s,x_2)\,ds + 2\epsilon\int_0^{x_1} w\Delta w(s,x_2)\,ds.$$

Now we integrate first and second component by parts. In the second component we use the fact that the integration interval does not depend on x_2. We get

$$\begin{aligned} S_2 &= -(\bar{u}+u_0)^{(1)}w^2(x_1,x_2) + \int_0^{x_1}(\bar{u}+u_0)^{(1)}_{x_1}w^2(s,x_2)\,ds \\ &\quad - \frac{\partial}{\partial x_2}\int_0^{x_1}(\bar{u}+u_0)^{(2)}w^2(s,x_2)\,ds + \int_0^{x_1}(\bar{u}+u_0)^{(2)}_{x_2}w^2(s,x_2)\,ds + 2\epsilon\int_0^{x_1} w\Delta w(s,x_2)\,ds \\ &= -(\bar{u}+u_0)^{(1)}w^2(x_1,x_2) + \int_0^{x_1} w^2\operatorname{div}(\bar{u}+u_0)(s,x_2)\,ds - \frac{\partial}{\partial x_2}\int_0^{x_1}(\bar{u}+u_0)^{(2)}w^2(s,x_2)\,ds + 2\epsilon\int_0^{x_1} w\Delta w(s,x_2)\,ds \\ &=: S_2^1 + S_2^2 + S_2^3 + S_2^4. \end{aligned}$$

The integrals of S_2^1 and S_2^2 can be estimated in a direct way:

$$\int_Q |S_2^1|, \int_Q |S_2^2| \leqslant E\|w\|^2_{L^2}. \tag{3.8}$$

Next,

$$\int_Q S_2^3 = \int_Q \frac{\partial}{\partial x_2}\left[\int_0^{x_1} u^{(2)}w^2(s,x_2)\,ds\right]dx = \int_\Gamma n^{(2)}\left[\int_0^{x_1}(\bar{u}+u_0)^{(2)}w^2(s,x_2)\,ds\right]d\sigma.$$

Now we remind that $w = 0$ on Γ_{in}. Moreover, the boundary conditions yield $(\bar{u} + u_0)^{(2)} = 0$ on Γ_0. Finally, on Γ_{out} we have $n^{(2)} = 0$. Thus

$$\int_Q S_2^3 = 0. \tag{3.9}$$

Finally,

$$\int_Q S_2^4\,dx = \int_0^1 \left[\int_0^1 \int_0^{x_1} w\Delta w(s, x_2)\,ds\,dx_2 \right] dx_1 = \int_0^1 \left[\int_{P_{x_1}} w\Delta w(x)\,dx \right] dx_1,$$

where $P_{x_1} := [0, x_1] \times [0, 1]$. We have

$$\int_{P_{x_1}} w\Delta w\,dx = -\int_{P_{x_1}} |\nabla w|^2\,dx + \int_{\partial P_{x_1}} w\nabla w \cdot n\,d\sigma \overset{(2.4)_{5,6}}{\leqslant} \int_0^1 w w_{x_1}(x_1, x_2)\,dx_2,$$

thus

$$\int_Q S_2^4\,dx \leqslant 2\epsilon \int_0^1 \int_0^1 w w_{x_1}(x_1, x_2)\,dx_2\,dx_1 = \epsilon \int_Q \partial_{x_1} w^2\,dx = \epsilon \int_{\Gamma_{out}} w^2 n^{(1)}\,d\sigma. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10) we get

$$\int_Q S_2 = \int_Q S_2^1 + S_2^2 + S_2^3 + S_2^4 \leqslant E\|w\|_{L^2}^2 + \epsilon \int_{\Gamma_{out}} w^2\,d\sigma.$$

Combining this estimate with (3.7) we get

$$\|w\|_{L^2}^2 \leqslant C\big(\big\|G_\epsilon(\bar{u}, \bar{w})\big\|_{L_2} + \|u\|_{W_2^1}\big)^2 + E\|w\|_{L^2} + \epsilon \int_{\Gamma_{out}} w^2\,d\sigma,$$

and thus

$$\|w\|_{L^2}^2 \leqslant C\big(\big\|G_\epsilon(\bar{u}, \bar{w})\big\|_{L_2} + \|u\|_{W_2^1}\big)^2 + C\epsilon \int_{\Gamma_{out}} w^2\,d\sigma. \tag{3.11}$$

Step 3. Substituting (3.11) to (3.6) we get

$$\|u\|_{W_2^1}^2 + \int_{\Gamma_{out}} w^2\,d\sigma \leqslant CD\big(\|u\|_{W_2^1} + \|w\|_{L_2}\big) + CD^2 + E\|w\|_{W_p^1}^2, \tag{3.12}$$

where $D = \|F_\epsilon(\bar{u}, \bar{w})\|_{V^*} + \|G_\epsilon(\bar{u}, \bar{w})\|_{L^2} + \|B\|_{L^2(\Gamma)}$. Combining this inequality with (3.11) we get

$$\big(\|u\|_{W_2^1} + \|w\|_{L_2}\big)^2 + (C - \epsilon) \int_{\Gamma_{out}} w^2\,d\sigma \leqslant CD\big(\|u\|_{W_2^1} + \|w\|_{L_2}\big) + D^2 + E\|w\|_{W_p^1}^2,$$

thus for ϵ small enough we obtain (3.2). $\square$

3.2. *Estimate for* $\|u\|_{W_p^2} + \|w\|_{W_p^1}$

The following theorem gives an ϵ-independent estimate on $\|u_\epsilon\|_{W_p^2} + \|w_\epsilon\|_{W_p^1}$ where (u_ϵ, w_ϵ) is a solution to (2.4).

Theorem 2. *Suppose that (u_ϵ, w_ϵ) is a solution to (2.4). Then the following estimate is valid provided that the data, $\|\bar{u}\|_{W_p^2}$ and $\|\bar{w}\|_{W_p^1}$ are small enough and f is large enough*

$$\|u_\epsilon\|_{W_p^2} + \|w_\epsilon\|_{W_p^1} \leqslant C\big[\big\|F_\epsilon(\bar{u}, \bar{w})\big\|_{L_p} + \big\|G_\epsilon(\bar{u}, \bar{w})\big\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)}\big], \tag{3.13}$$

where the constant C depends on the data but does not depend on ϵ.

The proof will be divided into three lemmas. In the first lemma we eliminate the term $\operatorname{div} u$ from $(2.4)_2$.

Lemma 2. *Let us define*

$$\bar{H} := -(\nu + 2\mu)\operatorname{div} u_\epsilon + \left[a_1(\bar{w})\right] w_\epsilon, \tag{3.14}$$

where (u_ϵ, w_ϵ) *is a solution to* (2.4) *and* $a_1(\bar{w})$ *is defined in* (2.7)*. Then*

$$\|\nabla \bar{H}\|_{L_p} \leqslant C\left[\left\|F_\epsilon(\bar{u}, \bar{w})\right\|_{L_p} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|u\|_{W_p^{1-1/p}(\Gamma)} + \|u\|_{W_p^1}\right] + E\|w\|_{W_p^1} \tag{3.15}$$

and w_ϵ *satisfies the following equation*

$$\left[a_0(\bar{w})\right] w + w_{x_1} + (\bar{u} + u_0) \cdot \nabla w - \epsilon \Delta w = \tilde{H}, \tag{3.16}$$

where

$$\tilde{H} = \frac{\bar{H}(\bar{w} + w_0 + 1)}{\nu + 2\mu} + G(\bar{u}, \bar{w}). \tag{3.17}$$

Proof. Let us rewrite $(2.4)_1$ as

$$\partial_{x_1} u_\epsilon - \mu \Delta u_\epsilon - (\nu + \mu)\nabla \operatorname{div} u_\epsilon + \gamma \nabla w_\epsilon = F_\epsilon(\bar{u}, \bar{w}) - \left[a_1(\bar{w}) - \gamma\right]\nabla w_\epsilon.$$

Taking the two-dimensional vorticity of $(2.4)_1$ we get

$$\begin{aligned} &\partial_{x_1} \alpha_\epsilon - \mu \Delta \alpha_\epsilon = \operatorname{rot}\left[F_\epsilon(\bar{u}, \bar{v}) - \left(a_1(\bar{w}) - \gamma\right)\nabla w_\epsilon\right] \quad \text{in } Q, \\ &\alpha_\epsilon = -\frac{f}{\mu}(u_\epsilon \cdot \tau) + \frac{B}{\mu} \quad \text{on } \Gamma, \end{aligned} \tag{3.18}$$

where $\alpha_\epsilon = \operatorname{rot} u_\epsilon = u^{(2)}_{\epsilon,x_1} - u^{(1)}_{\epsilon,x_2}$. The boundary condition $(3.18)_2$ has been shown in [11] in a more general case; a simplification of this proof yields $(3.18)_2$. Since our domain is a square, we can use the symmetry to deal with corner singularities and apply the standard L^p theory of elliptic equations [4] to obtain the estimate

$$\|\alpha_\epsilon\|_{W_p^1} \leqslant C\left[\left\|F_\epsilon(\bar{u}, \bar{w})\right\|_{L_p} + \left\|\left(a_1(\bar{w}) - \gamma\right)\nabla w_\epsilon\right\|_{L_p} + \left\|-\frac{f}{\mu}(u_\epsilon \cdot \tau) + \frac{B}{\mu}\right\|_{W_p^{1-1/p}(\Gamma)}\right]. \tag{3.19}$$

From the definition of $a_1(\bar{w})$ of (2.7) we see that $\|(a_1(\bar{w}) - \gamma)\|_{L_\infty}$ can be arbitrarily small provided that $\|\bar{w}\|_{W_p^1}$ is small enough. Moreover, from the boundary condition $(2.4)_4$ we have $u_\epsilon = \tau(u_\epsilon \cdot \tau)$ on Γ, thus (3.19) can be rewritten as

$$\|\alpha_\epsilon\|_{W_p^1} \leqslant C\left[\left\|F_\epsilon(\bar{u}, \bar{w})\right\|_{L_p} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|u_\epsilon\|_{W_p^{1-1/p}(\Gamma)} + E\|w_\epsilon\|_{W_p^1}\right]. \tag{3.20}$$

Now we apply the Helmholtz decomposition of u_ϵ (see Appendix A, (A.2)):

$$u_\epsilon = \nabla\phi + \nabla^\perp A. \tag{3.21}$$

For simplicity we omit the index ϵ in the notation of ϕ and A. We have $n \cdot \nabla^\perp A = \tau \cdot \nabla A = \frac{\partial}{\partial\tau} A$, thus the condition $n \cdot \nabla^\perp A|_\Gamma = 0$ yields $A|_\Gamma = \text{const}$. Moreover,

$$\operatorname{rot} u = \operatorname{rot}\left(\nabla\phi + \nabla^\perp A\right) = \operatorname{rot} \nabla^\perp A = \Delta A.$$

We see that A is a solution to the following boundary value problem:

$$\begin{cases} \Delta A = \alpha_\epsilon \in W_p^1(Q), \\ A|_\Gamma = \text{const}. \end{cases}$$

Applying again the elliptic theory we get

$$\|A\|_{W_p^3} \leqslant \|\alpha\|_{W_p^1} \leqslant C\left\{\left\|F_\epsilon(\bar{u}, \bar{w})\right\|_{L_p} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|u_\epsilon\|_{W_p^{1-1/p}(\Gamma)}\right\}. \tag{3.22}$$

Substituting the Helmholtz decomposition (A.2) to $(2.4)_1$ we get

$$\partial_{x_1}\left(\nabla\phi + \nabla^\perp A\right) - \mu\Delta\left(\nabla\phi + \nabla^\perp A\right) - (\nu + \mu)\nabla \operatorname{div}\left(\nabla\phi + \nabla^\perp A\right) + \left[a_1(\bar{w})\right]\nabla w = F_\epsilon(\bar{u}, \bar{w}),$$

but $\operatorname{div} \nabla\phi = \Delta\phi$ and thus

$$-(\nu+2\mu)\nabla\Delta\phi+\nabla\big(\big[a_1(\bar w)\big]w\big) = F(\bar u,\bar w)+\mu\Delta\nabla^{\perp}A+(\mu+\nu)\nabla\operatorname{div}\nabla^{\perp}A-\partial_{x_1}\nabla^{\perp}A+\partial_{x_1}\nabla\phi+w\nabla\big[a_1(\bar w)\big] =: \bar F, \tag{3.23}$$

what can be rewritten as

$$\nabla\big(-(\nu+2\mu)\Delta\phi+\gamma a_1(\bar w)w\big)=\bar F.$$

We have $\Delta\phi=\operatorname{div}u$, thus $\bar F=\nabla\bar H$ where $\bar H$ is defined in (3.14). From (3.23) we have

$$\|\bar F\|_{L_p}\leqslant C\big[\big\|F_\epsilon(\bar u,\bar w)\big\|_{L^p}+\|A\|_{W^3_p}+\big\|\nabla^2\phi\big\|_{L_p}\big]+\big\|\nabla\big[a_0(\bar w)\big]\big\|_{L_p}\|w_\epsilon\|_\infty$$
$$\leqslant C\big[\big\|F_\epsilon(\bar u,\bar w)\big\|_{L^p}+\|A\|_{W^3_p}+\|\phi\|_{W^2_p}\big]+E\|w_\epsilon\|_{W^1_p}$$

and from (3.22) and (A.3) we get (3.15). The proof is thus completed. □

In the next lemma we will use Eq. (3.16) to estimate $\|w\|_{W^1_p}$ in the terms of functions $\bar H$ and $G(\bar u,\bar w)$.

Lemma 3. *Under the assumptions of Theorem* 2 *the following estimate is valid*:

$$\|w_\epsilon\|_{W^1_p}+\|w_{\epsilon,x_1}\|_{L_p(\Gamma_{in})}\leqslant C\|H\|_{W^1_p}, \tag{3.24}$$

where

$$H=\frac{\bar H}{\nu+2\mu}+G(\bar u,\bar w). \tag{3.25}$$

Proof. Throughout the proof we will omit the index ϵ denoting w_ϵ by w. The proof will be divided into four steps. First we estimate $\|w\|_{L_p}$, then $\|w_{x_1}\|_{L_p}$ and $\|w_{x_2}\|_{L_p}$ and finally combine these estimates.

Step 1. Multiplying (3.16) by $|w|^{p-2}w$ and integrating over Q we get

$$\underbrace{\int_Q a_0(\bar w)|w|^p}_{I_1^1}+\underbrace{\int_Q |w|^{p-2}ww_{x_1}}_{I_1^2}+\underbrace{\int_Q(\bar u+u_0)\cdot\nabla w|w|^{p-2}w}_{I_1^3}-\underbrace{\epsilon\int_Q\Delta w|w|^{p-2}w}_{I_1^4}=\underbrace{\int_Q\tilde H|w|^{p-2}w}_{I_1^5}. \tag{3.26}$$

We have

$$I_1^3=\frac1p\int_Q(u+u_0)\cdot\nabla|w|^p\,dx=-\frac1p\int_Q|w|^p\operatorname{div}(\bar u+u_0)\,dx+\frac1p\int_{\Gamma_{out}}u_0^{(1)}|w|^p\,d\sigma.$$

Next,

$$I_1^2=\frac1p\int_Q\partial_{x_1}|w|^p\,dx=\frac1p\int_\Gamma|w|^pn^{(1)}\,d\sigma=\frac1p\int_{\Gamma_{out}}|w|^p\,d\sigma.$$

Combining the last two equations we get

$$-\big(I_1^2+I_1^3\big)\leqslant E\|w\|^p_{L_p}-C\int_{\Gamma_{out}}\big(1+u_0^{(1)}\big)|w|^p\,d\sigma.$$

The boundary term is positive due to the assumption of smallness of u_0. The term with Δw:

$$I_1^4=\epsilon\int_Q\nabla w\cdot\nabla\big(|w|^{p-2}w\big)\,dx-\epsilon\int_\Gamma|w|^{p-2}w\frac{\partial w}{\partial n}\,d\sigma.$$

The boundary term vanishes due to the conditions $(2.4)_{5,6}$ and the first term of the r.h.s. is equal to

$$(p-1)\int_Q|w|^{p-2}|\nabla w|^2\,dx\geqslant 0.$$

The r.h.s. of (3.26) can be estimated directly:

$$I_1^5=\int_Q\tilde H|w|^{p-2}w\,dx\leqslant\|\tilde H\|_{L_p}\int_Q\big(|w|^{(p-1)p^*}\big)^{1/p^*}\,dx=\|\tilde H\|_{L_p}\|w\|^{p-1}_{L_p}.$$

The smallness of $\bar{w}$ and w_0 in W_p^1 implies that $a_0(\bar{w}) \geqslant C > 0$, thus combining the above estimates we get $C\|w\|_{L_p}^p \leqslant \|\tilde{H}\|_{L_p}\|w\|_{L_p}^{p-1} + E\|w\|_{L_p}^p$, thus

$$\|w\|_{L_p} \leqslant C\|\tilde{H}\|_{L_p}. \tag{3.27}$$

Step 2. In order to estimate w_{x_2} we differentiate (3.16) with respect to x_2, multiply it by $|w_{x_2}|^{p-2}w_{x_2}$ and integrate over Q. We get

$$\underbrace{\int_Q [a_0(\bar{w})]|w_{x_2}|^p}_{I_2^1} + \underbrace{\int_Q [a_0(\bar{w})]_{x_2} w|w_{x_2}|^p w_{x_2}}_{I_2^2} + \underbrace{\int_Q |w_{x_2}|^{p-2} w_{x_2} w_{x_2 x_1}}_{I_2^3} + \underbrace{\int_Q \big((\bar{u}+u_0)_{x_2}\cdot\nabla w\big)|w_{x_2}|^{p-2}w_{x_2}}_{I_2^4}$$
$$+ \underbrace{\int_Q \big((\bar{u}+u_0)\cdot\nabla w_{x_2}\big)|w_{x_2}|^{p-2}w_{x_2}}_{I_2^5} \underbrace{- \epsilon\int_Q \Delta w_{x_2}|w_{x_2}|^{p-2}w_{x_2}}_{I_2^6} = \underbrace{\int_Q \tilde{H}_{x_2}|w_{x_2}|^{p-2}w_{x_2}}_{I_2^7}.$$

We have

$$I_2^3 = \frac{1}{p}\int_Q \partial_{x_1}|w_{x_2}|^p\,dx = -\frac{1}{p}\int_{\Gamma_{in}} |w_{x_2}|^p\,d\sigma + \frac{1}{p}\int_{\Gamma_{out}} |w_{x_2}|^p\,d\sigma,$$

but the condition $w = 0$ on Γ_{in} implies $w_{x_2} = 0$ on Γ_{in}, thus

$$I_2^3 = \frac{1}{p}\int_{\Gamma_{out}} |w_{x_2}|^p\,d\sigma. \tag{3.28}$$

Obviously we have $I_2^4 \leqslant E\|\nabla w\|_{L_p}^p$. Next,

$$I_2^5 = -\frac{1}{p}\int_Q \operatorname{div}(\bar{u}+u_0)|w_{x_2}|^p\,dx + \frac{1}{p}\int_{\Gamma_{out}} u_0^{(1)} n^{(1)}|w_{x_2}|^p\,d\sigma.$$

Combining this equation with (3.28) we get

$$I_2^3 + I_2^5 = -\frac{1}{p}\int_Q \operatorname{div}(\bar{u}+u_0)|w_{x_2}|^p + \frac{1}{p}\int_{\Gamma_{out}} \big(1+u_0^{(1)}\big)|w_{x_2}|^p\,d\sigma.$$

The boundary term is nonnegative due to the smallness of u_0.

The last part of the l.h.s.:

$$I_2^6 = -\epsilon\int_Q \Delta w_{x_2}|w_{x_2}|^p w_{x_2}\,dx = \epsilon\int_Q \nabla w_{x_2}\cdot\nabla\big(|w_{x_2}|^{p-2}w_{x_2}\big)\,dx + \epsilon\int_\Gamma \frac{\partial w_{x_2}}{\partial n}|w_{x_2}|^{p-2}w_{x_2}\,d\sigma.$$

The first term equals $\int_Q (p-1)|w_{x_2}|^{p-2}|\nabla w_{x_2}|^2\,dx > 0$ and the boundary term vanishes due to the boundary condition $(2.4)_{4,5}$. Using the definition of $a_0(\bar{w})$ of (2.7) we get

$$\int_Q [a_0(\bar{w})]_{x_i} w|w_{x_i}|^{p-2}w_{x_i} \leqslant C\big\|(\bar{w}+w_0)_{x_i}\big\|_{L_p}\|w_{x_i}\|_{L_p}^{p-1}\|w\|_{W_p^1} \leqslant E\|w\|_{W_p^1}^p, \tag{3.29}$$

thus $I_2^2 \leqslant E\|w\|_{W_p^1}^p$. In order to estimate the r.h.s. we use the definition of $\tilde{H}$ and the Hölder inequality. We get

$$I_2^7 = \left|\int_Q \tilde{H}_{x_i}|w_{x_i}|^{p-2}w_{x_i}\,dx\right| \leqslant C\|H\|_{W_p^1}\|w_{x_i}\|_{L_p}^{p-1}. \tag{3.30}$$

The important fact that we could write H instead of $\tilde{H}$ on the r.h.s. easily results from the definition of $\tilde{H}$ of (3.17). Combining the above estimates we get

$$\|w_{x_2}\|_{L^p}^p \leqslant C\big[E\|\nabla w\|_{L_p}^p + C\|H\|_{W_p^1}\|w_{x_2}\|_{L_p}^{p-1}\big]. \tag{3.31}$$

Step 3. In order to estimate w_{x_1} we differentiate (2.4) with respect to x_1 and multiply by $|w_{x_1}|^{p-2}w_{x_1}$:

$$\underbrace{\int_Q a_0(\bar{w})|w_{x_1}|^p}_{I_3^1} + \int_Q \underbrace{\big[a_0(\bar{w})\big]_{x_1}|w_{x_1}|^{p-2}w_{x_1}}_{I_3^2} + \int_Q \underbrace{w_{x_1x_1}|w_{x_1}|^{p-2}w_{x_1}}_{I_3^3} + \underbrace{\int_Q (\bar{u}+u_0)\cdot\nabla w_{x_1}|w_{x_1}|^{p-2}w_{x_1}}_{I_3^4}$$
$$-\underbrace{\int_Q \epsilon\Delta w_{x_1}|w_{x_1}|^{p-2}w_{x_1}}_{I_3^5} = \underbrace{\int_Q \tilde{H}_{x_1}|w_{x_1}|^{p-2}w_{x_1} - (\bar{u}+u_0)_{x_1}\cdot\nabla w|w_{x_1}|^{p-2}w_{x_1}}_{I_3^6}.$$

We have

$$I_3^3 = \frac{1}{p}\int_Q \partial_{x_1}|w_{x_1}|^p\,dx = -\frac{1}{p}\int_{\Gamma_{in}}|w_{x_1}|^p\,d\sigma.$$

Next,

$$-I_3^5 = \epsilon\int_Q \nabla w_{x_1}\cdot\nabla\big(|w_{x_1}|^{p-2}w_{x_1}\big)\,dx - \epsilon\int_\Gamma \frac{\partial w_{x_1}}{\partial n}|w_{x_1}|^{p-2}w_{x_1}\,d\sigma.$$

The first term is nonnegative and the boundary term reduces to

$$\epsilon\int_{\Gamma_{in}} w_{x_1x_1}|w_{x_1}|^{p-2}w_{x_1}\,d\sigma. \tag{3.32}$$

Note that on Γ_{in} Eq. (3.16) takes the form:

$$\big(1+\bar{u}^1+u_0^1\big)w_{x_1} - \epsilon w_{x_1x_1} = \tilde{H}|_{\Gamma_{in}}.$$

Thus (3.32) can be rewritten as

$$\int_{\Gamma_{in}}\big[\big(1+\bar{u}^1+u_0^1\big)|w_{x_1}|^p - \tilde{H}|w_{x_1}|^{p-2}w_{x_1}\big]\,d\sigma.$$

Finally,

$$I_3^4 = -\frac{1}{p}\int_Q \operatorname{div}(\bar{u}+u_0)|w_{x_1}|^p\,dx - \frac{1}{p}\int_{\Gamma_{in}} u_0^1|w_{x_1}|^p\,d\sigma.$$

Combining the above results we get

$$C\int_Q |w_{x_1}|^p\,dx + \int_{\Gamma_{in}}\left(1-u^1-\frac{1}{p}\right)|w_{x_1}|^p\,d\sigma$$
$$\leqslant \frac{1}{p}\int_Q \operatorname{div}(\bar{u}+u_0)|w_{x_1}|^p\,dx + \int_Q \tilde{H}_{x_1}|w_{x_1}|^{p-2}w_{x_1}\,dx - \int_Q\big[a_0(\bar{w})\big]_{x_1}w|w_{x_1}|^{p-2}w_{x_1}\,dx$$
$$-\int_Q(\bar{u}+u_0)_{x_1}\cdot\nabla w|w_{x_1}|^{p-2}w_{x_1}\,dx + \int_{\Gamma_{in}}\tilde{H}|w_{x_1}|^{p-2}w_{x_1}\,d\sigma,$$

thus using (3.30) and (3.29) we obtain

$$(C-E)\|w_{x_1}\|^p_{L_p(Q)} + \left(1-\frac{1}{p}-E\right)\|w_{x_1}\|^p_{L_p(\Gamma_{in})}$$
$$\leqslant C\|H\|_{W_p^1}\|w_{x_1}\|^{p-1}_{L_p(Q)} + E\|\nabla w\|^p_{L_p(Q)} + \|\tilde{H}\|_{L_p(\Gamma_{in})}\|w_{x_1}\|^{p-1}_{L_p(\Gamma_{in})} + E\|w\|^p_{W_p^1}. \tag{3.33}$$

Step 4. Combining (3.33) and (3.31) we get

$$\|\nabla w\|^p_{L_p} + \|w_{x_1}\|^p_{L_p(\Gamma_{in})} \leqslant C\big[\big(\|H\|_{W_p^1} + \|w\|_{L_p}\big)\|\nabla w\|^{p-1}_{L_p} + \|\tilde{H}\|_{L_p(\Gamma_{in})}\|w_{x_1}\|^{p-1}_{L_p(\Gamma_{in})}\big].$$

Combining this estimate with (3.27) we get

$$\|w\|_{L_p} + \|\nabla w\|_{L_p} + \|w_{x_1}\|_{L_p(\Gamma_{in})} \leqslant C\big(\|\tilde{H}\|_{L_p} + \|H\|_{W^1_p} + \|\tilde{H}\|_{L_p(\Gamma_{in})}\big). \tag{3.34}$$

Due to (3.30) we have $\|H\|_{W^1_p}$ instead of $\|\tilde{H}\|_{W^1_p}$ on the r.h.s. and the proof of (3.24) is almost complete. Now it is enough to note that due to the smallness of $\bar{w}$ and w_0 in W^1_p we have

$$\|\tilde{H}\|_{L_p} \leqslant C\|H\|_{L_p} \quad \text{and} \quad \|\tilde{H}\|_{L_p(\Gamma_{in})} \leqslant C\|H\|_{L_p(\Gamma_{in})},$$

thus (3.34) can be rewritten as

$$\|w\|_{W^1_p} + \|w_{x_1}\|_{L_p(\Gamma_{in})} \leqslant C\big[\|H\|_{W^1_p} + \|H\|_{L_p(\Gamma_{in})}\big], \tag{3.35}$$

but we have $\|H\|_{L_p(\Gamma_{in})} \leqslant \|H\|_{L_p(\Gamma)} \leqslant C\|H\|_{W^1_p}$ by the trace theorem, thus (3.35) implies (3.24) □

In order to complete the proof of Theorem 2 we have to estimate H. We will make use of the interpolation inequality (Lemma 11 in Appendix A).

Lemma 4. *Under the assumptions of Theorem* 2, $\forall\delta > 0$ *we have*

$$\|H\|_{W^1_p} \leqslant \delta\|u\|_{W^2_p} + C(\delta)\big[\big\|F_\epsilon(\bar{u},\bar{w})\big\|_{L_p} + \big\|G_\epsilon(\bar{u},\bar{w})\big\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)} + E\|w\|_{W^1_p}\big], \tag{3.36}$$

where H *is defined in* (3.25)*.*

Proof. For simplicity let us denote $F := F_\epsilon(\bar{u},\bar{w})$ and $G := G_\epsilon(\bar{u},\bar{w})$. Applying the interpolation inequality (A.4) to the term $\|u\|_{W^1_p}$ in (3.15) we get

$$\|\nabla H\|_{L_p} \leqslant C\big[\|F\|_{L_p} + \|G\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)} + \|u\|_{W^{1-1/p}_p(\Gamma)} + \delta_1\|u\|_{W^2_p} + C(\delta_1)\|u\|_{H^1}\big] + E\|w\|_{W^1_p}.$$

In order to estimate $\|H\|_{L_p}$ we need to apply the interpolation inequality (A.4) and then the energy estimate (3.2). We get

$$\|H\|_{L_p} \leqslant \delta_2\|\nabla H\|_{L_p} + C(\delta_2)\big(\|F\|_{L_2} + \|G\|_{L_2} + \|B\|_{L_2(\Gamma)} + E\|w_\epsilon\|_{W^1_p}\big).$$

Combining the above estimates we get

$$\begin{aligned}\|\nabla H\|_{L_p} + \|H\|_{L_p} &\leqslant (1+\delta_2)\|\nabla H\|_{L_p} + C(\delta_2)\big[\|F\|_{L^2} + \|G\|_{L^2} + \|B\|_{L^2(\Gamma)} + E\|w_\epsilon\|_{W^1_p}\big] \\ &\leqslant \delta_3\|u_\epsilon\|_{W^2_p} + C(\delta_3)\big[\|F\|_{L_p} + \|G\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)} + \|u_\epsilon\|_{W^{1-1/p}_p(\Gamma)} + E\|w_\epsilon\|_{W^1_p}\big].\end{aligned} \tag{3.37}$$

Using the trace theorem, (A.4) and (3.2) we estimate the boundary term $\|u\|_{W^{1-1/p}_p(\Gamma)}$:

$$\|u_\epsilon\|_{W^{1-1/p}_p(\Gamma)} \leqslant \delta_4\|u_\epsilon\|_{W^2_p} + C(\delta_4)\big[\|F\|_{L_2} + \|G\|_{L_2} + \|B\|_{L^2(\Gamma)} + E\|w_\epsilon\|_{W^1_p}\big]. \tag{3.38}$$

Substituting (3.38) to (3.37) we get (3.36) with δ arbitrarily small since $\delta_1 \ldots \delta_4$ can be arbitrarily small. □

We are now ready to complete

Proof of Theorem 2. Let us fix $\eta > 0$. Provided that $\bar{w}$ and w_0 are small enough, combining (3.24) and (3.36) we get

$$\|w_\epsilon\|_{W^1_p(Q)} \leqslant \eta\|u\|_{W^2_p} + C(\eta)\big[\big\|F_\epsilon(\bar{u},\bar{w})\big\|_{L_p} + \big\|G_\epsilon(\bar{u},\bar{w})\big\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)}\big]. \tag{3.39}$$

The theory of elliptic equations applied to $(2.4)_1$ yields

$$\|u_\epsilon\|_{W^2_p} \leqslant \big\|F_\epsilon(\bar{u},\bar{w})\big\|_{L^p} + \|w_\epsilon\|_{W^1_p}. \tag{3.40}$$

Combining this estimate with (3.39) we get

$$\|u_\epsilon\|_{W^2_p} + \|w_\epsilon\|_{W^1_p} \leqslant \eta\|u_\epsilon\|_{W^2_p} + C_\eta\big[\big\|F_\epsilon(\bar{u},\bar{w})\big\|_{L_p} + \big\|G_\epsilon(\bar{u},\bar{w})\big\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)}\big].$$

Choosing for example $\eta = \frac{1}{2}$ we get (3.13). □

4. Solution of the linear system

In this section we will show that the operator T_ϵ is well defined. We have to show that the system (2.4) has a unique solution $(u, w) \in W^2_p \times W^1_p$ for $(\bar{u}, \bar{w}) \in W^2_p \times W^1_p$ small enough. The necessary result is stated in the following

Theorem 3. *Assume that $\|\bar{u}\|_{W^2_p} + \|\bar{w}\|_{W^1_p}$ is small enough. Then the system* (2.4) *has a unique solution $(u_\epsilon, w_\epsilon) \in W^2_p \times W^2_p$ and the estimate* (3.13) *holds.*

We shall make here one remark concerning the above theorem. The fact that $(u_\epsilon, w_\epsilon) \in W^2_p \times W^2_p$ is a consequence of the ellipticity of the system (2.4), but the estimate on $\|w\|_{W^2_p}$ depends on ϵ. What will be crucial for us is that (3.13) does not depend on ϵ.

The system (2.4) has variable coefficients thus its solution is not straightforward. In order to prove Theorem 3 we will apply the Leray–Schauder fixed-point theorem. Given $(\bar{u}, \bar{w}) \in W^2_p \times W^1_p$ we define an operator $S^\epsilon_{(\bar{u},\bar{w})} : W^2_p \times W^2_p \to W^2_p \times W^2_p$: $(u, w) = S^\epsilon_{(\bar{u},\bar{w})}(\tilde{u}, \tilde{w}) \Leftrightarrow (u, w)$ is a solution to

$$\begin{aligned}
&\partial_{x_1} u - \mu\Delta u - (\nu+\mu)\nabla \operatorname{div} u + \gamma\nabla w = F^\epsilon_{(\bar{u},\bar{w})}(\tilde{u}, \tilde{w}) \quad \text{in } Q,\\
&\operatorname{div} u + \partial_{x_1} w - \epsilon\Delta w = G^\epsilon_{(\bar{u},\bar{w})}(\tilde{u}, \tilde{w}) \quad \text{in } Q,\\
&n\cdot 2\mu\mathbf{D}(u)\cdot\tau + f u\cdot\tau = B \quad \text{on } \Gamma,\\
&n\cdot u = 0 \quad \text{on } \Gamma,\\
&w = 0 \quad \text{on } \Gamma_{in},\\
&\frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma\setminus\Gamma_{in},
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
&F^\epsilon_{(\bar{u},\bar{w})}(\tilde{u}, \tilde{w}) = -\big(a_1(\bar{w}) - \gamma\big)\nabla\tilde{w} + F_\epsilon(\bar{u}, \bar{w}),\\
&G^\epsilon_{(\bar{u},\bar{w})}(\tilde{u}, \tilde{w}) = -(\bar{w} + w_0)\operatorname{div}\tilde{u} - (\bar{u} + u_0)\cdot\nabla\tilde{w} + G_\epsilon(\bar{u}, \bar{w}).
\end{aligned} \tag{4.2}$$

We have to show that $S^\epsilon_{(\bar{u},\bar{w})}$ is well defined and verify that it satisfies the assumptions of the Leray–Schauder theorem. The reason to consider $S^\epsilon_{(\bar{u},\bar{w})}$ on $W^2_p \times W^2_p$ instead of $W^2_p \times W^1_p$ is that it is straightforward to show its complete continuity.

4.1. Solution of the system with constant coefficients

In this section we show that the operator $S^\epsilon_{(\bar{u},\bar{w})}$ is well defined. Thus we have to show that the system

$$\begin{aligned}
&\partial_{x_1} u - \mu\Delta u - (\nu+\mu)\nabla \operatorname{div} u + \gamma\nabla w = F \quad \text{in } Q,\\
&\operatorname{div} u + \partial_{x_1} w - \epsilon\Delta w = G \quad \text{in } Q,\\
&n\cdot 2\mu\mathbf{D}(u)\cdot\tau + f u\cdot\tau = B \quad \text{on } \Gamma,\\
&n\cdot u = 0 \quad \text{on } \Gamma,\\
&w = 0 \quad \text{on } \Gamma_{in},\\
&\frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma\setminus\Gamma_{in},
\end{aligned} \tag{4.3}$$

where $F, G \in W^1_p(Q)$ are given functions, has a unique solution $(u, w) \in W^2_p \times W^2_p$. We start with showing existence of a weak solution to the system (4.3). Let us recall the definition of space V of (3.1) and introduce another functional space $W = \{w \in H^1(Q):\ w|_{\Gamma_{in}} = 0\}$. Consider a bilinear form on $(V \times W)^2$:

$$\begin{aligned}
\mathbf{B}\big[(u, w), (v, \eta)\big] = &\int_Q \big\{v\partial_{x_1} u + 2\mu\mathbf{D}(u):\ \nabla v + \nu \operatorname{div} u \operatorname{div} v\big\}\,dx + \int_\Gamma f(u\cdot\tau)(v\cdot\tau)\,d\sigma\\
&- \gamma\int_Q w \operatorname{div} v\,dx + \gamma\int_Q \eta \operatorname{div} u\,dx + \gamma\int_Q \eta\partial_{x_1} w\,dx + \gamma\epsilon\int_Q \nabla w\cdot\nabla\eta\,dx
\end{aligned}$$

and a linear form on $(V \times W)$:

$$\mathbf{F}(v, \eta) = \int_Q F\cdot v\,dx + \int_\Gamma B(v\cdot\tau)\,dx + \int_Q G\eta\,dx.$$

By a weak solution to the system (4.3) we mean a couple $(u, w) \in V \times W$ satisfying

$$\mathbf{B}\big[(u, w), (v, \eta)\big] = \mathbf{F}(v, \eta) \quad \forall (v, \eta) \in V \times W. \tag{4.4}$$

Using the definition of V and W we can easily verify that

$$\mathbf{B}\big[(u, w), (u, w)\big] \geqslant \int_Q 2\mu \mathbf{D}^2(u) + \nu \operatorname{div}^2 u \, dx + \epsilon \int_Q |\nabla w|^2 \, dx \geqslant C_\epsilon \big[\|u\|_{H^1(Q)} + \|w\|_{H^1(Q)}\big],$$

thus existence of the weak solution to (4.3) easily follows from the Lax–Milgram lemma. Using standard techniques we show that the weak solution belongs to $W^2_p(Q) \times W^2_p(Q)$ and

$$\|u\|_{W^2_p} + \|w\|_{W^2_p} \leqslant C_\epsilon \big[\|F\|_{L_p} + \|G\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)}\big].$$

4.2. Complete continuity of $S^\epsilon_{(\bar{u},\bar{w})}$

In this section we show that $S^\epsilon_{(\bar{u},\bar{w})}$ is continuous and compact. Since it is a linear operator, it is enough to show its compactness, and this is quite obvious due to elliptic regularity of the system (4.3). Namely, if we take a sequence $(\tilde{u}^n, \tilde{w}^n)$ bounded in $W^2_p \times W^2_p$, then the sequence

$$\big(F_{(\bar{u},\bar{w})}\big(\tilde{u}^n, \tilde{w}^n\big), G_{(\bar{u},\bar{w})}\big(\tilde{u}^n, \tilde{w}^n\big)\big)$$

is bounded in $W^1_p \times W^1_p$. Thus the sequence $(u^n, w^n) = S^\epsilon_{(\bar{u},\bar{w})}(\tilde{u}^n, \tilde{w}^n)$ is bounded in $W^3_p \times W^3_p$ (the bound on $\|w\|_{W^3_p}$ depends on ϵ, but at this stage ϵ is fixed, so it does not matter). The compact imbedding theorem implies that (u^n, w^n) has a subsequence that converges in $W^2_p(Q) \times W^2_p(Q)$. Thus $S^\epsilon_{(\bar{u},\bar{w})}$ is compact.

4.3. Leray–Schauder a priori bounds

Next we have to show a λ-independent *a priori* estimate on solutions to the equations $(u_\lambda, w_\lambda) = \lambda S^\epsilon_{(\bar{u},\bar{w})}(u_\lambda, w_\lambda)$, that read

$$\begin{aligned}
&\partial_{x_1} u_\lambda - \mu \Delta u_\lambda - (\nu + \mu) \nabla \operatorname{div} u_\lambda + \gamma \nabla w_\lambda = \lambda F^\epsilon_{(\bar{u},\bar{w})}(u_\lambda, w_\lambda) \quad \text{in } Q,\\
&\operatorname{div} u_\lambda + \partial_{x_1} w_\lambda - \epsilon \Delta w_\lambda = \lambda G^\epsilon_{(\bar{u},\bar{w})}(u_\lambda, w_\lambda) \quad \text{in } Q,\\
&n \cdot 2\mu \mathbf{D}(u_\lambda) \cdot \tau + f u_\lambda \cdot \tau = B \quad \text{on } \Gamma,\\
&n \cdot u_\lambda = 0 \quad \text{on } \Gamma,\\
&w_\lambda = 0 \quad \text{on } \Gamma_{in},\\
&\frac{\partial w_\lambda}{\partial n} = 0 \quad \text{on } \Gamma \setminus \Gamma_{in},
\end{aligned} \tag{4.5}$$

for $\lambda \in [0, 1]$. Actually we should write $(u^\epsilon_\lambda, w^\epsilon_\lambda)$, but we will omit ϵ as it should not lead to any misunderstanding. The result is stated in the following

Lemma 5. *Let* $(u_\lambda, w_\lambda) = \lambda S^\epsilon_{(\tilde{u},\tilde{w})}(u_\lambda, w_\lambda)$*, then*

$$\|u_\lambda\|_{W^p_2} + \|w_\lambda\|_{W^p_2} \leqslant C_\epsilon \big[\big\|F(\bar{u}, \bar{w})\big\|_{L_p} + \big\|G(\bar{u}, \bar{w})\big\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)}\big]. \tag{4.6}$$

Proof. The proof is very similar to the proof of Theorem 2. First we repeat the proof of Lemma 1 obtaining the λ-independent energy estimate

$$\|u_\lambda\|_{H^1} + \|w_\lambda\|_{L_2} \leqslant C\big[\big\|F^\epsilon_{(\bar{u},\bar{w})}(u_\lambda, w_\lambda)\big\|_{L_2} + \big\|G^\epsilon_{(\bar{u},\bar{w})}(u_\lambda, w_\lambda)\big\|_{L_2} + \|B\|_{L^2(\Gamma)}\big] + E\|w\|_{W^1_p}. \tag{4.7}$$

Next we take the vorticity of (4.5):

$$\begin{aligned}
&\partial_{x_1} \alpha_\lambda - \mu \Delta \alpha_\lambda = \operatorname{rot}\big(\lambda F^\epsilon_{(\bar{u},\bar{v})}(u_\lambda, w_\lambda)\big) \quad \text{in } Q,\\
&\alpha_\lambda = -\frac{f}{\mu}(u_\lambda \cdot \tau) + \frac{B}{\mu} \quad \text{on } \Gamma,
\end{aligned}$$

where $\alpha_\lambda = \operatorname{rot} u_\lambda$. Thus

$$\|\alpha_\lambda\|_{W^p_1} \leqslant C\big\{\big\|F^\epsilon_{(\bar{u},\bar{v})}(u_\lambda, w_\lambda)\big\|_{L_p(Q)} + \|B\|_{W^{1-1/p}_p(\Gamma)} + \|u_\lambda\|_{W^{1-1/p}_p(\Gamma)}\big\}.$$

Now let $u_\lambda = \nabla\phi_\lambda + A_\lambda^\perp$. Substituting this decomposition to (4.5) we get

$$-(\nu+2\mu)\nabla\Delta\phi_\lambda + \nabla(\gamma w_\lambda) = \lambda F^\epsilon_{(\bar u,\bar w)}(u_\lambda, w_\lambda) + \mu\Delta A_\lambda^\perp - \partial_{x_1}A_\lambda^\perp + \partial_{x_1}\nabla\phi_\lambda - =: \bar F_\lambda,$$

what can be rewritten as $\nabla(-(\nu+2\mu)\Delta\phi_\lambda + \gamma w_\lambda) = \bar F_\lambda$. We denote as previously

$$\left(-(\nu+2\mu)\operatorname{div} u_\lambda + \left[a_1(\bar w)\right]w_\lambda\right) = \bar H_\lambda.$$

Combining this identity with $(4.5)_2$ we get an analog of (3.16):

$$\zeta_\lambda(\bar w)w_\lambda + w_{\lambda,x_1} + \lambda(\bar u + u_0)\cdot\nabla w_\lambda - \epsilon\Delta w_\lambda = \tilde H_\lambda, \tag{4.8}$$

where $\zeta_\lambda(\bar w) = \frac{\gamma}{\nu+2\mu}[1+\lambda(\bar w + w_0)]$ and $\tilde H_\lambda = \frac{1+\lambda(\bar w + w_0)}{\nu+2\mu}\bar H_\lambda + \lambda G$. Now we can repeat step by step the proof of Theorem 2 obtaining the estimate

$$\|w_\lambda\|_{W^1_p} \leqslant \eta\|u_\lambda\|_{W^2_p} + C_\eta\left[\left\|F^\epsilon_{(\bar u,\bar w)}(u_\lambda, w_\lambda)\right\|_{L_p} + \left\|G^\epsilon_{(\bar u,\bar v)}(u_\lambda, w_\lambda)\right\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)}\right] \tag{4.9}$$

for each $\eta > 0$. The estimates for $\|u_\lambda\|_{W^2_p}$ and $\|w_\lambda\|_{W^2_p}$ now easily result from the system (4.5). Namely, applying the standard elliptic theory to $(4.5)_1$ we obtain an estimate

$$\|u_\lambda\|_{W^2_p} \leqslant C\left[\|w_\lambda\|_{W^1_p} + \left\|F^\epsilon_{(\bar u,\bar w)}(u_\lambda, w_\lambda)\right\|_{L_p}\right] \tag{4.10}$$

that does not depend on λ. Next, from $(4.5)_2$ we get an elliptic estimate

$$\|w_\lambda\|_{W^2_p} \leqslant C_\epsilon\left(\|w_\lambda\|_{W^1_p} + \|u_\lambda\|_{W^1_p} + \left\|G^\epsilon_{(\bar u,\bar w)}(u_\lambda, w_\lambda)\right\|_{L_p}\right). \tag{4.11}$$

Combining (4.9), (4.10) and (4.11) we get

$$\|u_\lambda\|_{W^p_2} + \|w_\lambda\|_{W^p_2} \leqslant C_\epsilon\left[\|B\|_{W^{1-1/p}_p(\Gamma)} + \left\|F^\epsilon_{(\bar u,\bar w)}(u_\lambda, w_\lambda)\right\|_{L_p} + \left\|G^\epsilon_{(\bar u,\bar w)}(u_\lambda, w_\lambda)\right\|_{W^1_p}\right], \tag{4.12}$$

but from the definition of $F^\epsilon_{(\bar u,\bar w)}$ and $G^\epsilon_{(\bar u,\bar w)}$ we have

$$\left\|F^\epsilon_{(\bar u,\bar w)}(u_\lambda, w_\lambda)\right\|_{L_p} + \left\|G^\epsilon_{(\bar u,\bar w)}(u_\lambda, w_\lambda)\right\|_{W^1_p} \leqslant E\left(\|u_\lambda\|_{W^2_p} + \|w_\lambda\|_{W^2_p}\right) + \left\|F_\epsilon(\bar u,\bar w)\right\|_{L_p} + \left\|G_\epsilon(\bar u,\bar w)\right\|_{W^1_p}$$

and thus (4.12) yields (4.6). □

Now we are ready to complete

Proof of Theorem 3. We have shown that the operator $S^\epsilon_{(\bar u,\bar w)}$ satisfies the assumptions of the Leray–Schauder theorem. Thus there exists a fixed point $(u_\epsilon, w_\epsilon) = S^\epsilon_{(\bar u,\bar w)}(u_\epsilon, w_\epsilon)$. The fixed point is a solution to (2.4). Its uniqueness follows directly from the estimate (3.13). □

We have shown the existence of a unique solution to the system (2.4) under some smallness assumptions on $\bar u$ and $\bar w$. Thus we define the domain $\mathcal{D}$ of the operator T:

$$\mathcal{D} = \left\{(\bar u,\bar w) \in W^2_p(Q)\times W^1_p(Q)\colon \text{ Theorem 3 holds for } (\bar u,\bar w)\right\}. \tag{4.13}$$

5. Solution of the regularized system

In this section we show existence of a solution to an ϵ-elliptic regularization to the system (2.3). The result is stated in the following

Theorem 4. *Assume that the data and $\epsilon > 0$ are small enough and f is large enough. Then there exists a fixed point $(u^*_\epsilon, w^*_\epsilon) = T_\epsilon(u^*_\epsilon, w^*_\epsilon)$ and*

$$\left\|u^*_\epsilon\right\|_{W^2_p} + \left\|w^*_\epsilon\right\|_{W^1_p} \leqslant M, \tag{5.1}$$

where M depends on the data but does not depend on ϵ and can be arbitrarily small provided that the data is small enough.

In order to prove Theorem 4 we apply the Schauder fixed point theorem to the operator T_ϵ defined in (2.5). We start to verify the assumptions of the Schauder theorem with the following

Lemma 6. *Assume that u_0 and w_0 are small enough. Then $T_\epsilon(B) \subset B$ for some ball $B \subset W^2_p(Q)\times W^1_p(Q)$.*

Proof. From the definition of $F_\epsilon(\bar u,\bar w)$ and $G_\epsilon(\bar u,\bar w)$ we have

$$\left\|F_\epsilon(\bar u,\bar w)\right\|_{W^1_p}+\left\|G_\epsilon(\bar u,\bar w)\right\|_{W^1_p}\leqslant E+\left(\|\bar u\|_{W^2_p}+\|\bar w\|_{W^1_p}\right)^2. \tag{5.2}$$

Thus we can rewrite the estimate (3.13) as

$$\|u_\epsilon\|_{W^2_p}+\|w_\epsilon\|_{W^1_p}\leqslant C\left[D+\left(\|\bar u\|_{W^2_p}+\|\bar w\|_{W^1_p}\right)^2\right], \tag{5.3}$$

where D can be arbitrarily small provided that $\|u_0\|_{W^2_p}$ and $\|w_0\|_{W^1_p}$ are small enough. In (3.13) we only need an estimate on $\|F_\epsilon(\bar u,\bar w)\|_{L_p}$ that holds also for $F(\bar u,\bar w)$, but we will need the estimate in W^1_p to show the compactness of T_ϵ and this is the reason why we introduce the regularization F_ϵ. Let us assume that the data is small enough to ensure $D\leqslant\frac{1}{4C^2}$, where C and D are the constants from (5.3). Assume further that $\|\bar u\|_{W^2_p}+\|\bar w\|_{W^1_p}\leqslant\sqrt{D}$. Then from (5.3) we get

$$\|u_\epsilon\|_{W^2_p}+\|w_\epsilon\|_{W^1_p}\leqslant 2CD\leqslant\sqrt{D}.$$

Thus $T_\epsilon(B)\subset B$ where $B=B(0,\sqrt{D})\subset W^2_p(Q)\times W^1_p(Q)$. □

In the next lemma we show that T_ϵ is a continuous operator on $\mathcal{D}$, where $\mathcal{D}$ is defined in (4.13). The proof applies the estimate (3.13) which requires some smallness assumption, but this assumption is also included in the definition of $\mathcal{D}$ and therefore we can prove the continuity on the whole $\mathcal{D}$.

Lemma 7. *T_ϵ is a continuous operator on $\mathcal{D}$.*

Proof. Let us have $(u_1,w_1)=T(\bar u_1,\bar w_1)$ and $(u_2,w_2)=T(\bar u_2,\bar w_2)$, then the functions u_1-u_2 and w_1-w_2 satisfy the equations

$$\begin{aligned}&\partial_{x_1}(u_1-u_2)-\mu\Delta(u_1-u_2)-(\nu+\mu)\nabla\operatorname{div}(u_1-u_2)+\gamma(\bar w_1+w_0+1)^{\gamma-1}\nabla(w_1-w_2)\\&\quad=F_\epsilon(\bar u_1,\bar w_1)-F_\epsilon(\bar u_2,\bar w_2)-\gamma\left[(\bar w_1+w_0+1)^{\gamma-1}-(\bar w_2+w_0+1)^{\gamma-1}\right]\nabla w_2\end{aligned}$$

and

$$\begin{aligned}&(\bar w_1+w_0+1)\operatorname{div}(u_1-u_2)+\partial_{x_1}(w_1-w_2)+(\bar u_1+u_0)\cdot\nabla(w_1-w_2)-\epsilon\Delta(w_1-w_2)\\&\quad=G_\epsilon(\bar u_1,\bar w_1)-G_\epsilon(\bar u_2,\bar w_2)-(\bar w_1-\bar w_2)\operatorname{div}u_2-(\bar u_1-\bar u_2)\cdot\nabla w_2,\end{aligned}$$

supplied with boundary conditions

$$\begin{aligned}&n\cdot 2\mu\mathbf{D}(u_1-u_2)\cdot\tau+f(u_1-u_2)\cdot\tau=0\quad\text{on }\Gamma,\\&n\cdot(u_1-u_2)=0\quad\text{on }\Gamma,\\&w_1-w_2=0\quad\text{on }\Gamma_{in},\\&\frac{\partial(w_1-w_2)}{\partial n}=0\quad\text{on }\Gamma\setminus\Gamma_{in}.\end{aligned} \tag{5.4}$$

If $(\bar u_1,\bar w_1),(\bar u_2,\bar w_2)\in\mathcal{D}$ then the system on (u_1-u_2,w_1-w_2) satisfies the assumptions of Theorem 2 and thus (3.13) yields

$$\begin{aligned}&\|u_1-u_2\|_{W^2_p}+\|w_1-w_2\|_{W^1_p}\\&\quad\leqslant\left\|F_\epsilon(\bar u_1,\bar w_1)-F_\epsilon(\bar u_2,\bar w_2)\right\|_{L_p}+\left\|G_\epsilon(\bar u_1,\bar w_1)-G_\epsilon(\bar u_2,\bar w_2)\right\|_{W^1_p}\\&\qquad+\left\|\left[(\bar w_1+w_0+1)^{\gamma-1}-(\bar w_1+w_0+1)^{\gamma-1}\right]\nabla w_2\right\|_{L_p}+\left\|(\bar w_1-\bar w_2)\operatorname{div}u_2\right\|_{W^1_p}+\left\|(\bar u_1-\bar u_2)\cdot\nabla w_2\right\|_{W^1_p}.\end{aligned} \tag{5.5}$$

From the definition of $F_\epsilon(\bar u,\bar w)$ and $G_\epsilon(\bar u,\bar w)$ we directly get

$$\begin{aligned}&\left\|F_\epsilon(\bar u_1,\bar w_1)-F_\epsilon(\bar u_2,\bar w_2)\right\|_{L_p}+\left\|G_\epsilon(\bar u_1,\bar w_1)-G_\epsilon(\bar u_2,\bar w_2)\right\|_{W^1_p}\\&\qquad+\left\|\left[(\bar w_1+w_0+1)^{\gamma-1}-(\bar w_1+w_0+1)^{\gamma-1}\right]\nabla w_2\right\|_{L_p}+\left\|(\bar w_1-\bar w_2)\operatorname{div}u_2\right\|_{W^1_p}\\&\quad\leqslant C\left(\|\bar u_1\|_{W^1_p},\|\bar w_1\|_{W^1_p},\|\bar u_2\|_{W^1_p},\|\bar w_2\|_{W^1_p}\right)\left[\|\bar u_1-\bar u_2\|_{W^2_p}+\|\bar w_1-\bar w_2\|_{W^1_p}\right].\end{aligned}$$

In order to estimate the last term of the r.h.s. of (5.5) we have to use higher norm of w_2:

$$\left\|(\bar u_1-\bar u_2)\cdot\nabla w_2\right\|_{W^1_p}\leqslant C\left(\|\bar w_2\|_{W^2_p}\right)\|\bar u_1-\bar u_2\|_{W^2_p}.$$

Since on this level ϵ is fixed, we can use the elliptic regularity of the system (2.4) that yields

$$\|\bar{w}_2\|_{W^2_p} \leqslant C_\epsilon\big[\big\|F_\epsilon(\bar{u}_2,\bar{w}_2)\big\|_{L_p} + \big\|G_\epsilon(\bar{u}_2,\bar{w}_2)\big\|_{W^1_p} + \|B\|_{L_p(\Gamma)}\big].$$

Combining the above estimates we get from (5.5):

$$\|u_1-u_2\|_{W^2_p} + \|w_1-w_2\|_{W^1_p} \leqslant C_\epsilon\big[\|\bar{u}_1-\bar{u}_2\|_{W^2_p} + \|\bar{w}_1-\bar{w}_2\|_{W^1_p}\big], \tag{5.6}$$

what completes the proof of continuity of T_ϵ. □

Now we need to prove that T_ϵ is a compact operator. The key is in the following lemma:

Lemma 8. *Let us have $(u,w)=T_\epsilon(\bar{u},\bar{w})$. Then $(u,w)\in W^3_p(Q)\times W^2_p(Q)$ and*

$$\|u\|_{W^3_p} + \|w\|_{W^2_p} \leqslant C_\epsilon\big[\|\bar{u}\|_{W^2_p} + \|\bar{w}\|_{W^1_p} + E\big]. \tag{5.7}$$

Proof. If $(u,w)=T_\epsilon(\bar{u},\bar{w})$ then in particular w satisfies

$$-\epsilon\Delta w = G_\epsilon(\bar{u},\bar{w}) - \partial_{x_1}w - (\bar{u}+u_0)\cdot\nabla w - (\bar{w}+w_0+1)\operatorname{div}u.$$

Thus by (3.13) we have

$$\|w\|_{W^2_p} \leqslant C_\epsilon\big[\big\|G_\epsilon(\bar{u},\bar{w})\big\|_{L_p} + C\big(\|u\|_{W^2_p}+\|w\|_{W^1_p}\big)\big] \leqslant C_\epsilon\big[\big\|F_\epsilon(\bar{u},\bar{w})\big\|_{L_p} + \big\|G_\epsilon(\bar{u},\bar{w})\big\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)}\big]. \tag{5.8}$$

Next, u satisfies the equation

$$-\mu\Delta u - (\nu+\mu)\nabla\operatorname{div}u = F_\epsilon(\bar{u},\bar{w}) - \partial_{x_1}u - \gamma(\bar{w}+w_0+1)^{\gamma-1}\nabla w,$$

what yields

$$\|u\|_{W^3_p} \leqslant C\big[\big\|F_\epsilon(\bar{u},\bar{w})\big\|_{W^1_p} + \|w\|_{W^2_p}\big] \overset{(5.8)}{\leqslant} C_\epsilon\big[\big\|F_\epsilon(\bar{u},\bar{w})\big\|_{W^1_p} + \big\|G_\epsilon(\bar{u},\bar{w})\big\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)}\big]. \tag{5.9}$$

Now, from (5.2) we get (5.7). □

With Lemma 8 the compactness of T_ϵ is a straightforward consequence of the compact imbedding theorem. Namely, if we take a sequence $(\bar{u}^n,\bar{w}^n)$ that is bounded in $W^2_p(Q)\times W^1_p(Q)$ and consider $(u^n,w^n)=T_\epsilon(\bar{u}^n,\bar{w}^n)$, then from (5.7) the sequence (u^n,w^n) is bounded in $W^3_p(Q)\times W^2_p(Q)$. Thus the compact imbedding theorem implies the existence of a subsequence (u^{n_k},w^{n_k}) that converges in $W^2_p(Q)\times W^1_p(Q)$, what means that T_ϵ is compact.

Proof of Theorem 4. The theorem results directly from the Schauder fixed point theorem for the operator T_ϵ. □

6. Proof of Theorem 1

In this section we prove our main result, Theorem 1, passing to the limit with ϵ in (2.4). The proof will be divided into two steps: the proof of existence of the solution and the proof of its uniqueness. These steps are quite separated since in order to prove uniqueness we will go back to the original system (1.1) and modify the proof of the estimate (3.2).

Step 1: Existence. Consider a decreasing sequence $\epsilon_n\to 0$. If ϵ_1 is small enough that Theorem 4 holds (what we can assume without loss of generality), then for each $n\in\mathbf{N}$ Theorem 4 gives a solution $(u_{\epsilon_n},w_{\epsilon_n})$ to an ϵ_n-elliptic regularization to (2.3).

By (5.1) the sequence $(u_{\epsilon_n},w_{\epsilon_n})$ is uniformly bounded in $W^2_p\times W^1_p$. The compact imbedding theorem implies that there exists a couple $(u,w)\in W^2_p\times W^1_p$ such that (up to a subsequence)

$$u_{\epsilon_n} \overset{W^2_p}{\rightharpoonup} u \quad\text{and}\quad w_{\epsilon_n} \overset{W^1_p}{\rightharpoonup} w. \tag{6.1}$$

From the definition of F_ϵ and G_ϵ we easily get

$$F_\epsilon(u_\epsilon,w_\epsilon) \overset{L_p}{\to} F(u,w) \quad\text{and}\quad G_\epsilon(u_\epsilon,w_\epsilon) \overset{L_p}{\to} G(u,w). \tag{6.2}$$

We have to show that (u,w) satisfies the system (2.3). Clearly we have

$$\Delta u_{\epsilon_n} \overset{L_p}{\rightharpoonup} \Delta u, \qquad \nabla\operatorname{div}u_{\epsilon_n} \overset{L_p}{\rightharpoonup} \nabla\operatorname{div}u,$$
$$\partial_{x_1}w_{\epsilon_n} \overset{L_p}{\rightharpoonup} \partial_{x_1}w, \qquad \nabla w \overset{L_p}{\rightharpoonup} w. \tag{6.3}$$

Thus it remains to show convergence in nonlinear terms, but this is also straightforward. We have $\forall\phi \in L_q$:

$$\int_Q \phi (w_\epsilon + w_0 + 1)^{\gamma-1} \nabla w_\epsilon \, dx = \int_Q \phi\big[(w_\epsilon + w_0 + 1)^{\gamma-1} - (w + w_0 + 1)^{\gamma-1}\big]\nabla w_\epsilon + \int_Q \phi (w + w_0 + 1)^{\gamma-1} \nabla w_\epsilon \, dx.$$

Since $\phi(w + w_0 + 1)^{\gamma-1} \in L_q$, the second term converges to $\int_Q \phi(w + w_0 + 1)^{\gamma-1}\nabla w \, dx$. The first term

$$\left|\int_Q \phi\big[(w_\epsilon + w_0 + 1)^{\gamma-1} - (w + w_0 + 1)^{\gamma-1}\big]\nabla w_\epsilon \, dx\right| \leqslant \big\|\phi\big[(w_\epsilon + w_0 + 1)^{\gamma-1} - (w + w_0 + 1)^{\gamma-1}\big]\big\|_{L_q} \|w_\epsilon\|_{W^1_p} \overset{\epsilon\to 0}{\to} 0,$$

since by the compact imbedding theorem $w_\epsilon \overset{L_q}{\to} w \; \forall 1 \leqslant q < +\infty$. Thus

$$\int_Q \phi (w_\epsilon + w_0 + 1)^{\gamma-1} \nabla w_\epsilon \, dx \to \int_Q \phi (w + w_0 + 1)^{\gamma-1} \nabla w \, dx. \tag{6.4}$$

Similarly we can show that

$$(w_\epsilon + w_0 + 1) \operatorname{div} u_\epsilon + (u_\epsilon + u_0) \cdot \nabla w_\epsilon \overset{L_p}{\rightharpoonup} (w + w_0 + 1) \operatorname{div} u + (u + u_0) \cdot \nabla w. \tag{6.5}$$

From (6.2), (6.3), (6.4) and (6.5) we see that (u, w) satisfies $(2.3)_{1,2}$ a.e. in Q. The trace theorem implies that

$$w_\epsilon|_{\gamma_{in}} \overset{L_p(\Gamma_{in})}{\rightharpoonup} w|_{\Gamma_{in}}, \qquad u|_\Gamma \overset{L_p(\Gamma)}{\to} u|_\Gamma, \qquad \mathbf{D}(u) \overset{L_p(\Gamma_{in})}{\rightharpoonup} u_\Gamma. \tag{6.6}$$

Thus u satisfies $(2.3)_{3,4}$ a.e. on Γ and w satisfies $(2.3)_5$ a.e. on Γ_{in}. Now take $v = u + u_0 + \bar{v}$ and $\rho = w + w_0 + \bar{\rho}$, where u_0 and w_0 are extensions to the boundary data defined in (2.1) and $(\bar{v}, \bar{\rho}) \equiv ([1,0], 1)$ is the constant solution. Then (v, ρ) satisfies the system (1.1).

In order to show the estimate (1.2) we repeat the proof of Theorem 2 obtaining

$$\|u\|_{W^2_p} + \|w\|_{W^1_p} \leqslant C\big[\big\|F(u,w)\big\|_{L_p} + \big\|G(u,w)\big\|_{W^1_p} + \|B\|_{W^{1-1/p}_p(\Gamma)}\big]. \tag{6.7}$$

We have

$$\big\|F(u,w)\big\|_{L_p} + \big\|G(u,w)\big\|_{W^1_p} \leqslant D + \big(\|u\|_{W^2_p} + \|w\|_{W^1_p}\big)^2, \tag{6.8}$$

where D can be arbitrarily small provided that the data is small enough. From (6.7) and (6.8) we conclude (1.2).

Step 2: Uniqueness. In order to prove the uniqueness of the solution in a class of small perturbations of the constant flow $(\bar{v}, \bar{\rho})$ consider (v_1, ρ_1) and (v_2, ρ_2) both being solutions to (1.1) satisfying the estimate (1.2). We will apply the ideas of the proof of the energy estimate (3.2) in order to show that

$$\|v_1 - v_2\|^2_{H^1} + \|\rho_1 - \rho_2\|^2_{L_2} = 0. \tag{6.9}$$

For simplicity let us denote the differences $u := v_1 - v_2$ and $w := \rho_1 - \rho_2$. We will follow the notation of constants introduced before, namely E shall denote a constant dependent on the data that can be arbitrarily small provided that the data is small enough, whereas C will denote a constant dependent on the data that is controlled, but not necessarily small. In order to show (6.9) it is enough to prove that

$$\|u\|_{H^1} \leqslant E\|w\|_{L_2} \tag{6.10}$$

and

$$\|w\|_{L_2} \leqslant C\|u\|_{H^1}. \tag{6.11}$$

If we subtract the equations on (v_1, ρ_1) and (v_2, ρ_2) there appears a term $\rho_1^\gamma - \rho_2^\gamma$. We will use the fact that $\rho_1, \rho_2 \sim 1 \Rightarrow \rho_1^\gamma - \rho_2^\gamma \sim \gamma(\rho_1 - \rho_2)$, more precisely, we can write

$$\rho_1^\gamma - \rho_2^\gamma = (\rho_1 - \rho_2)\underbrace{\int_0^1 \gamma\big[t\rho_1 + (1-t)\rho_2\big]^{\gamma-1} dt}_{I_\gamma}$$

and we have $I_\gamma \simeq \gamma$. Now we easily verify that the difference (u, w) satisfies the system

$$\begin{aligned}
&w v_2 \cdot \nabla v_2 + \rho_1 u \cdot \nabla v_2 + \rho_1 v_1 \cdot \nabla u - \mu \Delta u - (\mu + \nu) \nabla \operatorname{div} u + I_\gamma \nabla w = 0,\\
&\rho_1 \operatorname{div} u + w \operatorname{div} v_2 + u \cdot \nabla \rho_2 + v_1 \cdot \nabla w = 0,\\
&n \cdot 2\mu \mathbf{D}(u) \cdot \tau|_\Gamma = 0,\\
&n \cdot u|_\Gamma = 0,\\
&w|_{\Gamma_{in}} = 0.
\end{aligned} \tag{6.12}$$

We modify the proof of (3.2), multiplying $(6.12)_1$ by $\rho_1 u$ and integrating over Q (the reason why we take $\rho_1 u$ instead of u will be explained soon). We get

$$\int_Q \big(2\mu \mathbf{D}^2(u) + \nu \rho_1 \operatorname{div}^2 u\big)\,dx + \underbrace{\int_Q \big[(\rho_1 - 1)\mathbf{D}(u) : \nabla u + \mathbf{D}(u) : (u \otimes \nabla \rho_1)\big]\,dx}_{I_1} - I_\gamma \int_Q w \rho_1 \operatorname{div} u\,dx + \int_\Gamma \rho_1 f u^2\,d\sigma$$

$$- \underbrace{\int_Q w u \nabla \rho_1\,dx}_{I_2} + \underbrace{\int_Q \rho_1^2 u^2 \cdot \nabla v_2\,dx}_{I_3} + \underbrace{\int_Q u w \rho_1 v_2 \cdot \nabla v_2\,dx}_{I_4} + \underbrace{\int_Q \rho_1^2 (v_1 \cdot \nabla u) \cdot u\,dx}_{I_5} = 0.$$

We have $|I_1| + |I_2| + |I_3| + |I_4| \leqslant E(\|u\|^2_{H^1} + \|w\|^2_{L_2})$. Now let us split I_5 into two parts:

$$2I_5 = \underbrace{\int_Q \big(\rho_1^2 v_1^{(1)} - 1\big)\partial_{x_1}|u|^2 + \rho_1^2 v_1^{(2)} \partial_{x_2}|u|^2\,dx}_{I_5^1} + \underbrace{\int_Q \partial_{x_1}|u|^2\,dx}_{I_5^2}.$$

We have $|I_5^1| \leqslant E\|u\|^2_{H^1}$ and $I_5^2 = \int_\Gamma |u|^2 n^{(1)}\,d\sigma$. The last term can be integrated by parts and combined with the boundary term involving friction. Thus applying the Korn inequality (A.1) we get

$$C\|u\|^2_{H_1} + \int_\Gamma \big(\rho_1 f + n^{(1)}\big)|u|^2\,d\sigma - I_\gamma \int_Q w \operatorname{div} u\,dx \leqslant E\|u\|^2_{H^1}.$$

For the friction coefficient f large enough the boundary term will be positive and thus

$$\|u\|^2_{H^1} \leqslant C \int_Q w \rho_1 \operatorname{div} u\,dx. \tag{6.13}$$

The reason why we multiplied $(6.12)_1$ by $\rho_1 u$ is that now we have this function on the r.h.s. of (6.13) instead of $\operatorname{div} u$. In order to derive (6.10) from (6.13) we express $\rho_1 \operatorname{div} u$ in terms of w using Eq. $(6.12)_2$. Thus we can rewrite (6.13) as

$$\|u\|^2_{H^1} \leqslant - \underbrace{\int_Q w^2 \operatorname{div} v_2\,dx}_{I_6} - \underbrace{\int_Q w v_1 \cdot \nabla w\,dx}_{I_7} - \underbrace{\int_Q w u \cdot \nabla \rho_2\,dx}_{I_8}. \tag{6.14}$$

Obviously $|I_6| \leqslant E\|w\|^2_{L_2}$ and, since $p > 2$, we have $|I_8| \leqslant \|\nabla \rho_2\|_{L_p} \|w\|_{L_2} \|u\|_{L_q}$ for some $q < \infty$. Thus from the imbedding theorem we get $|I_8| \leqslant E(\|w\|^2_{L_2} + \|u\|^2_{H^1})$. Integrating by parts in I_7 and using the boundary conditions we get

$$-2I_7 = \int_Q w^2 \operatorname{div} v_1\,dx - \int_{\Gamma_{out}} v_1^{(1)}\,d\sigma.$$

The boundary term is positive since $v_1^{(1)} \sim 1$, thus $-I_7 \leqslant C\|\nabla v_1\|_\infty \|w\|^2_{L_2} = E\|w\|_{L_2^2}$. Combining the estimates for I_6, I_7 and I_8 we get (6.10).

Now in order to complete the proof we have to show (6.11). Note that it is useless to multiply $(6.12)_2$ by w since we would obtain a term $w^2 \operatorname{div} v_2$. Thus we adapt again the approach from the proof of (3.2) and write an expression on a pointwise value of w^2:

$$\begin{aligned} w^2(x_1,x_2) &= \int_0^{x_1} w w_s(s,x_2)\,ds \\ &= -\int_0^{x_1} \frac{\rho_1}{v_1^{(1)}} w \operatorname{div} u(s,x_2)\,ds - \int_0^{x_1} \frac{1}{v_1^{(1)}}\big(w^2 \operatorname{div} v_2 + wu\cdot\nabla\rho_2\big)(s,x_2)\,ds - \int_0^{x_1} \frac{v_1^{(2)}}{v_1^{(1)}} w\partial_{x_2} w(s,x_2)\,ds \\ &=: w_1^2 + w_2^2 + w_3^2. \end{aligned}$$

Note that we have $\rho_1, v_1^{(1)} \sim 1$ and thus $\forall\delta > 0$:

$$\int_Q w_1^2\,dx \leqslant C\big(\|w\|_{L_2}\|\operatorname{div} u\|_{L_2}\big) \leqslant \delta\|w\|_{L_2}^2 + C(\delta)\|u\|_{H^1}^2. \tag{6.15}$$

Next we easily get $\int_Q w_3^2\,dx \leqslant E(\|w\|_{L_2}^2 + \|u\|_{H^1}^2)$, and we only have to deal with w_3^2. We have $\int_Q w_3^2\,dx = \int_0^1 [\int_{P_{x_1}} w_3^2\,dx]\,dx_1$. Consider the inner integral

$$\int_{P_{x_1}} w_3^2\,dx = -\int_{P_{x_1}} \partial_{x_2}\frac{v_1^{(2)}}{v_1^{(1)}} w^2\,dx + \int_{\partial P_{x_1}} w^2 v_1^{(1)} v_1^{(2)} n^{(2)}\,d\sigma.$$

The boundary term vanishes and thus

$$\int_Q w_3^2 \leqslant C\left\|\partial_{x_2}\frac{v_1^{(2)}}{v_2^{(1)}}\right\|_\infty \|w\|_{L_2}^2 \leqslant E\|w\|_{L_2}^2. \tag{6.16}$$

Choosing for example $\delta = \frac{1}{2}$ in (6.15) we get (6.11), what completes the proof of (6.9). We have shown that the solution is unique, and thus completed the proof of Theorem 1.

Acknowledgments

The author would like to thank Piotr Mucha and Milan Pokorny for fruitful discussions and remarks concerning this paper. The author has been supported by the Polish Ministry of Science and Higher Education grant No. N N201 364736.

Appendix A

Lemma 9 *(Korn inequality). Let $V = \{v \in H^1(Q)\colon (n\cdot v)|_\Gamma = 0\}$. Then $\exists C = C(Q)$:*

$$\int_Q 2\mu\mathbf{D}^2(u) + \nu \operatorname{div}^2 u\,dx \geqslant C_Q\,\|u\|_{W_2^1}^2. \tag{A.1}$$

The proof can be found in [9, Lemma 2.1] or in [17, Lemma 4].

Lemma 10 *(Helmoltz decomposition). For $v \in W_p^1(Q)$ there exists a couple of functions $(\phi, A) \in (W_p^2)^2$ such that $n\cdot\nabla^\perp A|_\Gamma = 0$*

$$v = \nabla\phi + \nabla^\perp A. \tag{A.2}$$

Moreover,

$$\|\phi\|_{W_p^2} + \|A\|_{W_p^2} \leqslant C\|v\|_{W_p^1}. \tag{A.3}$$

The proof can be found in [2]. The last auxiliary result we need is the following interpolation inequality:

Lemma 11. $\forall\epsilon > 0$ $\exists C(\epsilon, p, Q)$ *such that* $\forall f \in W_p^1(Q)$:

$$\|f\|_{L_p} \leqslant \epsilon\|\nabla f\|_{L_p} + C(\epsilon, p, Q)\|f\|_{L_2}. \tag{A.4}$$

Proof. Inequality (A.4) results from the following inequality [1, Theorem 5.8]:

$$\|f\|_{L_p} \leqslant K\|f\|_{W_2^1}^{\theta}\|f\|_{L_2}^{1-\theta} \tag{A.5}$$

for each $2 \leqslant p < \infty$, where $\theta = \frac{n(p-2)}{2p}$ and $K = K(p, Q)$. Using Cauchy inequality with ϵ we get (A.4). □

References

[1] R. Adams, J. Fournier, Sobolev Spaces, 2nd ed., Elsevier, Amsterdam, 2003.
[2] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, vol. I, Springer-Verlag, New York, 1994.
[3] H. Fujita, Remarks on the Stokes flow under slip and leak boundary conditions of friction type, in: Topics in Mathematical Fluid Mechanics, in: Quad. Mater., vol. 10, 2002, pp. 73–94.
[4] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed., Springer-Verlag, Berlin, 1983.
[5] O. Glass, P.B. Mucha, Inviscid limit for the 2-D stationary Euler system with arbitrary force in simply connected domains, Appl. Math. 35 (1) (2008) 49–67.
[6] R.B. Kellogg, J.R. Kweon, Compressible Navier–Stokes equations in a bounded domain with inflow boundary condition, SIAM J. Math. Anal. 28 (1) (1997) 94–108.
[7] R.B. Kellogg, J.R. Kweon, Smooth solution of the compressible Navier–Stokes equations in an unbounded domain with inflow boundary condition, J. Math. Anal. Appl. 220 (1998) 657–675.
[8] J.R. Kweon, M. Song, Boundary geometry and regularity of solution to the compressible Navier–Stokes equations in bounded domains of $\mathbf{R}^n$, ZAMM Z. Angew. Math. Mech. 86 (6) (2006) 495–504.
[9] P.B. Mucha, On Navier–Stokes equations with slip boundary conditions in an infinite pipe, Acta Appl. Math. 76 (2003) 1–15.
[10] P.B. Mucha, M. Pokorny, On a new approach to the issue of existence and regularity for the steady compressible Navier–Stokes equations, Nonlinearity 19 (2006) 1747–1768.
[11] P.B. Mucha, R.B. Rautmann, Convergence of Rothe's scheme for the Navier–Stokes equations with slip boundary conditions in 2D domains, ZAMM Z. Angew. Math. Mech. 86 (9) (2006) 691–701.
[12] A. Novotny, I. Straskraba, Introduction to the Mathematical Theory of Compressible Flows, Oxford Science Publications, Oxford, 2004.
[13] P.I. Plotnikov, E.V. Ruban, J. Sokolowski, Inhomogeneous boundary value problems for compressible Navier–Stokes equations: Well-posedness and sensitivity analysis, SIAM J. Math. Anal. 40 (3) (2008) 1152–1200.
[14] P.I. Plotnikov, E.V. Ruban, J. Sokolowski, Inhomogeneous boundary value problems for compressible Navier–Stokes an transport equations, J. Math. Pures Appl. (2009), doi:10.1016/j.matpur.2009.02.001, in press.
[15] P.I. Plotnikov, J. Sokolowski, Domain dependence of solutions to compressible Navier–Stokes equations, SIAM J. Control Optim. 45 (4) (2006) 1165–1197.
[16] P.I. Plotnikov, J. Sokolowski, On compactness, domain dependence and existence of steady state solutions to compressible isothermal Navier–Stokes equations, J. Math. Fluid Mech. 7 (2005) 529–573.
[17] V.A. Solonnikov, V.E. Scadilov, On a boundary value problem for a stationary system of Navier–Stokes equations, Tr. Mat. Inst. Steklova 125 (1973) 186–199.
[18] A. Valli, W.M. Zajaczkowski, Navier–Stokes equations for compressible fluids: Global existence and qualitative properties of the solutions in the general case, Comm. Math. Phys. 103 (2) (1986) 259–296.