Unbounded symmetric operators in *K*-homology and the Baum–Connes conjecture

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Abstract

Using the unbounded picture of analytical *K*-homology, we associate a well-defined *K*-homology class to an unbounded symmetric operator satisfying certain mild technical conditions. We also establish an "addition formula" for the Dirac operator on the circle and for the Dolbeault operator on closed surfaces. Two proofs are provided, one using topology and the other one, surprisingly involved, sticking to analysis, on the basis of the previous result. As a second application, we construct, in a purely analytical language, various homomorphisms linking the homology of a group in low degree, the *K*-homology of its classifying space and the analytic *K*-theory of its *C**-algebra, in close connection with the Baum–Connes assembly map. For groups classified by a 2-complex, this allows to reformulate the Baum–Connes conjecture.
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MSC: primary 19K33; 19K35; secondary 19K56; 19L64

Keywords: Unbounded symmetric operators; *K*-homology; *K K*-theory; Group homology; Baum–Connes conjecture

Contents

Part I. Introduction

1. Statement of the main results

The non-commutative geometry approach to K-homology rests on the concept of *unbounded Fredholm module*, due to Connes [12, Chapter I, Section 6]. Subsequently, this object was renamed *K-cycle* [13, Definition 11 in Section IV.2.γ] and then, quite conveniently, *spectral triple* (see [14]) to emphasize the connection with spectral geometry. Recall that, if $\mathcal{A}$ is an involutive algebra represented on the Hilbert space $\mathcal{H}$,

and D is af self-adjoint operator on $\mathcal{H}$ with compact resolvent, the triple $(\mathcal{A}, \mathcal{H}, D)$ is *spectral* if D almost commutes with any $a \in \mathcal{A}$, i.e. if $[D, a]$ is bounded for every $a \in \mathcal{A}$.

Given a separable C^*-algebra A, our goal is to define certain classes in the K-homology group $K^*(A) := KK_*(A, \mathbb{C})$, using unbounded symmetric operators (that are definitely not assumed to be self-adjoint). Not surprisingly, this will force the operators considered to fulfill some technical conditions. The following definition lists the properties we need (more details are provided in Section 4 below, in particular concerning deficiency indices and invertibility of $T^*T + 1$).

Definition 1.1. We call a triple $(\mathcal{H}, \pi, T)$ a *symmetric unbounded Fredholm module of degree i* over the separable C^*-algebra A if it consists of the following data:

(a) an integer $i \in \{0, 1\}$;
(b) a Hilbert space $\mathcal{H}$;
(c) a $*$-representation $\pi: A \longrightarrow B(\mathcal{H})$ of A;
(d) a densely defined closed symmetric operator T on $\mathcal{H}$ with domain $\mathrm{dom}(T)$.

These data are required to fulfill the following conditions:

(i) the deficiency indices of T coincide and are finite;
(ii) the operator $\pi(a)(T^*T + 1)^{-1}$ is compact for every $a \in A$;
(iii) the operators $[\pi(b), T]$ and $[\pi(b), T^*]$ are densely defined and $[\pi(b), T^*]$ is bounded for every b in some dense *subspace* $\mathcal{B}$ of A.
(The subspace $\mathcal{B}$ is not required to be a subalgebra.) If $i = 0$, we moreover require $\mathcal{H}$ to be $\mathbb{Z}/2$-graded, π to preserve the grading, and T to reverse it.

Of course, this would define an unbounded Fredholm module $[\mathcal{H}, \pi, T] \in KK_i(A, \mathbb{C})$ in the sense of Connes, if T would moreover be self-adjoint. In fact, our two main results read as follows.

Theorem 1.2. *Let $(\mathcal{H}, \pi, T)$ be a symmetric unbounded Fredholm module of degree i over a separable C^*-algebra A, as defined above. Then, there exists, in the unbounded picture of analytical K-homology, a well-defined K-homology class*

$$[\mathcal{H}, \pi, T] \in K^i(A),$$

that is canonical, and coincides with the usual class in case T is self-adjoint. More precisely, given an arbitrary self-adjoint extension $\widetilde{T}$ of T—and at least one such extension exists—, one has

$$[\mathcal{H}, \pi, T] = [\mathcal{H}, \pi, \widetilde{T}] \in K^i(A),$$

independently of the choice of $\widetilde{T}$.

As a consequence of this first result (more precisely of a generalization of it), we will derive the following theorem.

Theorem 1.3. *Let* $(\mathcal{H}, \pi, T_1)$ *and* $(\mathcal{H}, \pi, T_2)$ *be two unbounded Fredholm modules of degree i for a separable C*-algebra A* (*in the usual sense of Connes*), *with the same Hilbert space* $\mathcal{H}$ *and the same* ∗-*representation* π. *Suppose that* T_1 *and* T_2 *admit*, *as a common restriction*, *an operator T satisfying the two conditions*

(a) *T is densely defined*;
(b) $[\pi(b), T^{**}]$ *is densely defined and bounded for every b in some dense* ∗-*closed subspace* $\mathcal{B}$ *of A.*

Then, *one has the following equality of K-homology classes*:

$$[\mathcal{H}, \pi, T_1] = [\mathcal{H}, \pi, T_2] \in K^i(A).$$

Furthermore, *since* T_1 *enters in an unbounded Fredholm module*, *there exists a* ∗-*closed dense subspace* $\mathcal{B}'$ *of A such that* $[\pi(b'), T_1]$ *is densely defined and bounded for every* $b' \in \mathcal{B}'$, *and condition* (b) *above can be replaced by the next one while keeping the same conclusion*:

(b′) $[\pi(b'), T^{**}]$ *is densely defined for every* b' *in* $\mathcal{B}'$.

In both theorems, the separability assumption is needed to apply Baaj–Julg's results [2] (see the proof of Proposition 2.3 therein). Recall, for later applications, that for X a compact Hausdorff space, the C^*-algebra $C(X)$ is separable if and only if X is metrizable (or equivalently, second-countable). For example, as is well-known, a CW-complex is metrizable if and only if it is locally finite [19, Proposition 1.5.17]. At this point, let us mention that *throughout the paper*, *we assume that all spaces and maps between them are pointed.*

Here is a description of the content of the paper.

One of our goals is to apply our results to establish an "addition formula" concerning certain differential operators on closed manifolds of dimension one and two. More precisely, we would like to study the behaviour under connected sum of the K-homology class given by the Dirac operator in dimension one and by the Dolbeault operator in dimension two. In fact, in dimension one, the situation is well-behaved for the usual connected sum, but in dimension two, this leads to the introduction of a variation of the connected sum. For both considered dimensions, we will present two proofs of each "addition formula", one using standard and well-established tools from algebraic topology (in particular, "topological index theory"), and the other one in a purely analytical framework on the basis of Theorem 1.3. One of the interest of this latter approach is that the analytical proof is astonishingly involved. The context will be explained in detail in Section 2, and the proofs are presented later, in Section 7 for the topological proof, in Section 8 for the analytical proof in dimension one, and in Section 9 for the analytical proof in dimension two.

In Section 3, we explain in detail the framework of our application of these results in connection with the Baum–Connes conjecture, which aims at computing the K-group $K_*(C_r^*\Gamma)$ for a group Γ (see there for the notation and definitions). The K-homology group $K_j(B\Gamma)$ and the homology group $H_j(\Gamma; \mathbb{Z})$ will be involved for $j = 1, 2$. More precisely, for both values of j, we will construct two maps

$$\beta_j^{(t)} : H_j(\Gamma; \mathbb{Z}) \longrightarrow K_j(B\Gamma) \qquad \text{and} \qquad \beta_j^{(a)} : H_j(\Gamma; \mathbb{Z}) \longrightarrow K_j(C_r^*\Gamma).$$

Our main concern will be to define these maps in a purely analytical language, i.e. using the unbounded picture of analytical K-homology, and, as a major difficulty, to prove that they are well-defined group homomorphisms, while sticking to this analytical language. It turns out that the proof of this property will precisely amount to the "addition formulae" for suitable differential operators as in Section 2, hence the close connection with Theorem 1.3.

In Section 4, we state a general theorem, namely Theorem 4.1, that allows to associate to an unbounded symmetric Fredholm module $(\mathcal{H}, \pi, T)$ a "usual" KK-theory class in some group $KK_i(A, C(\mathcal{U}))$, where $\mathcal{U}$ is a suitable non-empty compact Hausdorff space depending on T. Evaluation at an arbitrary point u of $\mathcal{U}$ will provide the K-homology class $[\mathcal{H}, \pi, T] \in K^*(A)$ we are looking for. The punch-line is that this will *not* depend on the choice of u (the point being path-connectedness of $\mathcal{U}$). As a consequence, Theorem 1.2 follows from this.

The proof of Theorem 4.1, and hence of Theorem 1.2, is presented in Section 5.

In Section 6, we address a generalization of Theorem 1.2, where we reduce the assumptions on the triple $(\mathcal{H}, \pi, T)$ to the strict minimum (according to our proof); as the main relaxation of assumptions, finiteness of deficiency indices will be dropped. Using this generalization, we then establish Theorem 1.3. After this section, we move, for the rest of the paper, to the applications, namely on the "addition formulae" and around the Baum–Connes conjecture.

As we have said, we will present the proofs of the "addition formulae" for the Dirac and the Dolbeault operators in Section 7 (topological in both dimensions), in Section 8 (analytical in dimension one) and in Section 9 (analytical in dimension two).

We treat the case $j = 1$ of our application to the Baum–Connes conjecture in Section 10. In this case, $H_1(\Gamma; \mathbb{Z})$ is Γ^{ab}, the abelianization of Γ. We will see that $\beta_1^{(a)}$ is exactly the map $\Gamma^{ab} \longrightarrow K_1(C_r^*\Gamma)$ induced by the canonical inclusion of Γ in the group of invertibles of $C_r^*\Gamma$. It was proved by Elliott and Natsume [17] (and reproved in [8]) that $\beta_1^{(a)}$ is rationally injective.

In Section 11, for $j = 2$, Zimmermann's description of $H_2(\Gamma; \mathbb{Z})$ in [45] allows us to define $\beta_2^{(t)}$ and $\beta_2^{(a)}$. We were not able to prove rational injectivity of $\beta_2^{(a)}$.

In Section 12, we draw consequences of our constructions for groups which admit a 2-dimensional classifying space; we call these groups *two-dimensional*. We use our maps $\beta_j^{(a)}$ to propose, for these groups, an equivalent formulation of the Baum–Connes conjecture with the left-hand side replaced by integral group homology.

We point out that [30,31] contain closely related results; the relation of the maps $\beta_j^{(a)}$ with algebraic K-theory (and Steinberg symbols for $j = 2$) is studied in [32].

2. Description of the application to analysis on manifolds

We explain here two applications of Theorem 1.3 in the context of differential operators on manifolds. One of the applications is for the circle, i.e. in dimension one, and the other is in dimension two, more precisely for Riemann surfaces. Explicitly, we will state two "addition formulae" for suitable differential operators. The topological proof is presented in Section 7, and the analytical proof in Section 8 for the one-dimensional case, and in Section 9 for the two-dimensional case.

We first recall that for a σ-compact Hausdorff topological space X, for instance a CW-complex, one has a canonical and natural isomorphism

$$K_*(X) \cong RKK_*(X, \mathbb{C}),$$

where K_* is K-homology with compact supports, and RKK_* is Kasparov's KK-theory with compact supports, that we will see in the unbounded picture of K-homology (more on this is contained in Sections 4, 8 and 9). If X is compact Hausdorff, one further has $RKK_*(X, \mathbb{C}) = KK_*(C(X), \mathbb{C})$.

Now, we start with the one-dimensional situation. Consider the Dirac operator on the circle S^1 and the corresponding K-homology class, namely

$$D := \frac{1}{i} \cdot \frac{d}{d\theta} \qquad \text{and} \qquad [D] \in K_1(S^1) \cong KK_1(C(S^1), \mathbb{C});$$

details are provided in Section 8. Now, the "addition formula" reads as follows.

Theorem 2.1. *Let X be a pointed CW-complex, and let $f_1, f_2 : S^1 \longrightarrow X$ be two pointed continuous maps, that are constant in a small neighbourhood of the base-point of S^1. Consider the connected sum of these two maps (along a closed interval sitting inside the given neighbourhood for both copies of S^1)*

$$f_1 \# f_2 : S^1 \# S^1 \longrightarrow X,$$

and identify the closed oriented manifold $S^1 \# S^1$ with S^1 as usual. Then, in K-homology, one has

$$(f_1 \# f_2)_*[D] = (f_1)_*[D] + (f_2)_*[D] \ \in \ K_1(X) \cong RKK_1(X, \mathbb{C}).$$

We pass to the two-dimensional setting. Let Σ_g be a closed oriented surface of genus $g \geqslant 0$ (in particular, without boundary). We fix an auxiliary Kähler structure on Σ_g, i.e.
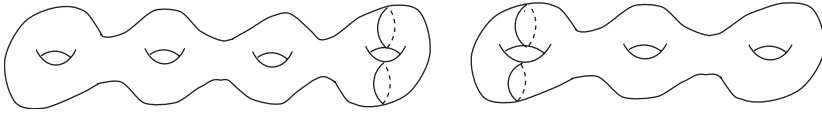
Fig. 1.
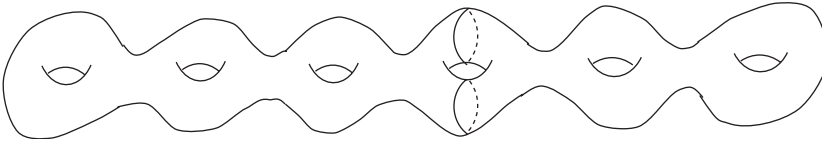
Fig. 2.

we view Σ_g as a complex curve equipped with a suitably compatible Riemannian metric. Consider the Dolbeault operator and its K-homology class

$$\bar{\partial}_{\Sigma_g} := \bar{\partial} \oplus \bar{\partial}^* \qquad \text{and} \qquad [\bar{\partial}_g] := [\bar{\partial}_{\Sigma_g}] \in K_0(\Sigma_g) \cong KK_0(C(\Sigma_g), \mathbb{C});$$

again, we will be more explicit in Section 9. As we will explain is that section, the connected sum for surfaces does *not* satisfy the "same addition formula" as in Theorem 2.1. We will explain that the exact source of the problem is the non-additivity of the Euler characteristic under the connected sum. We therefore introduce a modified version of the usual connected sum.

Thus, let Σ_{g_1} and Σ_{g_2} be surfaces of genus g_1 and g_2 respectively. By cutting out a handle in Σ_{g_1} (resp. in Σ_{g_2}), see Fig. 1, and gluing along the boundary circles in an orientation preserving way, we get a closed oriented surface $\Sigma_{g_1} \natural \Sigma_{g_2}$ of genus $g_1 + g_2 - 1$, as in Fig. 2.

We assume that the base-point of Σ_{g_1} is identified with the base-point of Σ_{g_2} in this operation (in particular, both base-points sit on two corresponding circles among the four boundary circles). We single out that the Euler characteristic is additive for this modified connected sum, i.e.

$$\chi(\Sigma_{g_1} \natural \Sigma_{g_2}) = \chi(\Sigma_{g_1}) + \chi(\Sigma_{g_2}).$$

In this situation, the "addition formula" reads as follows.

Theorem 2.2. *Let X be a pointed CW-complex. For $i = 1, 2$, let $f_i \colon \Sigma_{g_i} \longrightarrow X$ be a pointed continuous map, that is constant in a small neighbourhood of a handle of Σ_{g_i} ($g_i \geqslant 0$). Consider the modified connected sum* (*along the two given handles*)

$$f_1 \natural f_2 \colon \Sigma_{g_1} \natural \Sigma_{g_2} \longrightarrow X$$

of these two maps, and identify the closed oriented manifold $\Sigma_{g_1} \natural \Sigma_{g_2}$ with $\Sigma_{g_1+g_2-1}$ in the usual way. Then, in K-homology, one has

$$(f_1 \natural f_2)_*[\bar{\partial}_{g_1+g_2-1}] = (f_1)_*[\bar{\partial}_{g_2}] + (f_2)_*[\bar{\partial}_{g_2}] \in K_0(X) \cong RKK_0(X, \mathbb{C}).$$

For the analytical proofs of Theorems 2.1 and 2.2, we have to impose suitable boundary conditions on the glued parts of the considered manifolds; as a consequence, we must deal with symmetric non-self-adjoint operators. Theorems 1.2 and 1.3 then ensure the well definiteness of the corresponding "glued" K-homology classes.

Another occurrence of symmetric non-self-adjoint operators arises in analytic K-homology, in the discussion of excision in that framework, see [21, Section 10.8].

3. Description of the application to the Baum–Connes conjecture

We describe here our second application of Theorem 1.2, namely, in the framework of the celebrated Baum–Connes conjecture, that we also introduce with some explanations.

Let Γ be a countable discrete group. The *Baum–Connes conjecture* for Γ is the statement that the Baum–Connes assembly map, or analytical index map,

$$\mu_i^\Gamma : K_i^\Gamma(\underline{E\Gamma}) \longrightarrow K_i(C_r^*\Gamma) \qquad (i = 0, 1)$$

is an isomorphism. Here, $K_*^\Gamma(\underline{E\Gamma})$ is the Γ-equivariant *K*-homology with Γ-compact supports of $\underline{E\Gamma}$, the classifying space for proper Γ-actions, and $K_*(C_r^*\Gamma)$ is the analytical *K*-theory of $C_r^*\Gamma$, the reduced C^*-algebra of Γ. For precise definitions of the objects involved, various examples and the relevance of the conjecture to questions in topology, geometry, algebra and analysis, we refer to [4,33,38,42]; see also [23,40] for excellent surveys of progresses on the conjecture up to 1999. Recall also that both $K_*^\Gamma(\underline{E\Gamma})$ and $K_*(C_r^*\Gamma)$ are 2-periodic by virtue of Bott periodicity. For this reason, we will stick to the groups K_0 and K_1.

Denote by $F\Gamma$ the vector space of $\mathbb{C}$-valued functions on Γ, with finite support contained in the set of finite-order elements of Γ. The space $F\Gamma$ becomes a Γ-module by letting Γ act by conjugation; $H_*(\Gamma; F\Gamma)$ denotes the corresponding homology group. Baum and Connes defined in [3] a Chern character

$$\mathrm{ch}_*^\Gamma : K_i^\Gamma(\underline{E\Gamma}) \longrightarrow \bigoplus_{n=0}^\infty H_{2n+i}(\Gamma; F\Gamma) \qquad (i = 0, 1)$$

that becomes an isomorphism after tensoring with $\mathbb{C}$ (see also [31]).

Now, let $B\Gamma$ be the classifying space of Γ, and let $K_*(B\Gamma)$ denote its *K*-homology with compact supports, which is also 2-periodic by Bott periodicity. There is a canonical map $\varphi_*^\Gamma : K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$, which is an isomorphism for Γ torsion-free. Indeed, denote by $E\Gamma$ the universal cover of $B\Gamma$. This map φ_*^Γ is the composition of the canonical isomorphism $K_*(B\Gamma) \cong K_*^\Gamma(E\Gamma)$ with the forgetful map $K_*^\Gamma(E\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$

obtained by noticing that any free and proper Γ-action is proper. Of course, for Γ torsion-free, the spaces $E\Gamma$ and $\underline{E\Gamma}$ coincide (up to Γ-equivariant homotopy). Together with the Chern character ch_* in K-homology, these maps fit into the commutative diagram (see [4,31])

$$
\begin{array}{ccccc}
K_i(B\Gamma) & \xrightarrow{\varphi_i^\Gamma} & K_i^\Gamma(\underline{E\Gamma}) & \xrightarrow{\mu_i^\Gamma} & K_i(C_r^*\Gamma) \\
\text{ch}_* \downarrow & & \downarrow \text{ch}_*^\Gamma & & \\
\bigoplus_{n=0}^{\infty} H_{2n+i}(\Gamma;\mathbb{Z}) \rightarrow \bigoplus_{n=0}^{\infty} H_{2n+i}(\Gamma;\mathbb{Q}) \hookrightarrow \bigoplus_{n=0}^{\infty} H_{2n+i}(\Gamma;F\Gamma) & & & &
\end{array}
$$

for $i = 0$ and 1. (Throughout the paper, we identify the integral (resp. rational) homology of Γ with that of $B\Gamma$.) This shows in particular that φ_*^Γ is rationally injective. We let $\nu_*^\Gamma := \mu_*^\Gamma \circ \varphi_*^\Gamma : K_*(B\Gamma) \longrightarrow K_*(C_r^*\Gamma)$ be the Novikov assembly map. The reason for this terminology is that rational injectivity of ν_*^Γ implies the Novikov conjecture on higher signatures for the group Γ.

At the very beginning, this paper started out from a desire to exploit the bottom line of this diagram, in order to better understand the top line. Since, in favourable cases, geometry and topology provide explicit models for $B\Gamma$, from which group homology $H_*(\Gamma;\mathbb{Z})$ can be computed, or at least well-understood, it seems interesting to try to construct directly, out of integral homology classes, elements in $K_*(B\Gamma)$ and $K_*(C_r^*\Gamma)$. In other words, we are looking for maps

$$
\beta_j^{(\text{t})} : H_j(\Gamma;\mathbb{Z}) \longrightarrow K_i(B\Gamma) \qquad \text{and} \qquad \beta_j^{(\text{a})} : H_j(\Gamma;\mathbb{Z}) \longrightarrow K_i(C_r^*\Gamma),
$$

where $i \equiv j \pmod 2$, such that the diagram

$$
\begin{array}{ccc}
K_i(B\Gamma) & \xrightarrow{\ \nu_i^\Gamma\ } & K_i(C_r^*\Gamma) \\
& \nwarrow \beta_j^{(\text{t})} \quad \nearrow \beta_j^{(\text{a})} & \\
& H_j(\Gamma;\mathbb{Z}) &
\end{array}
$$

commutes. To ensure non-triviality, $\beta_j^{(\text{t})}$ should be rationally a right-inverse of the Chern character in degree j, i.e.

$$
(\text{ch}_j \otimes \text{id}_\mathbb{Q}) \circ (\beta_j^{(\text{t})} \otimes \text{id}_\mathbb{Q}) = \text{id}_{H_j(\Gamma;\mathbb{Q})}.
$$

Moreover, we do not want to define $\beta_j^{(\text{a})}$ merely as $\nu_j^\Gamma \circ \beta_j^{(\text{t})}$, but look instead for a direct and explicit construction. Indeed, it would follow from the Baum–Connes conjecture that $\beta_j^{(\text{a})}$ is rationally injective; one may then try to prove this directly.

To illustrate this program, let us consider the easy case where $j = 0$. Of course $H_0(\Gamma; \mathbb{Z}) \cong \mathbb{Z}$, and we define

$$\beta_0^{(t)}: \mathbb{Z} \longrightarrow K_0(B\Gamma), \quad n \longmapsto n \cdot \iota_*^{B\Gamma}[1],$$

where $\iota^{B\Gamma}: pt \longrightarrow B\Gamma$ is the inclusion of the base-point, and the class [1] is a pre-scribed generator of $K_0(pt) \cong \mathbb{Z}$. It is obvious that $\beta_0^{(t)}$ is a right-inverse of the map $\mathrm{ch}_0^{\mathbb{Z}}: K_0(B\Gamma) \longrightarrow H_0(\Gamma; \mathbb{Z})$, i.e. the integral Chern character in degree zero (compare with Lemma 12.1 below, and with [30]). On the other hand, we define

$$\beta_0^{(a)}: \mathbb{Z} \longrightarrow K_0(C_r^*\Gamma), \quad n \longmapsto n \cdot [1] = \mathrm{Sign}(n) \cdot \left[\mathrm{Diag}(\underbrace{1, \ldots, 1}_{|n| \text{ terms}}, 0, 0, \ldots) \right],$$

where [1] denotes, this time, the K-theory class of the unit in $C_r^*\Gamma$. It is an easy but instructive exercise (see e.g. [33, Example 2.11 in Part 2] or [42, Example 6.1.5]) to check that $v_0^\Gamma \iota_*^{B\Gamma}[1] = [1]$. Moreover, the canonical trace τ on $C_r^*\Gamma$ induces a map $\tau_*: K_0(C_r^*\Gamma) \longrightarrow \mathbb{R}$ such that $\tau_*[1] = 1$. This shows for free that $\beta_0^{(a)}$ is injective.

In this paper, we implement the program sketched above in the cases $j = 1$ and $j = 2$, exploiting especially simple descriptions of $H_j(\Gamma; \mathbb{Z})$ available in this range. The main feature is that the construction of $\beta_j^{(t)}$ is performed in the analytical and unbounded description—that is, "à la Kasparov and Connes"—of K-homology. In particular, the proof of the fact that $\beta_j^{(t)}$ is a group homomorphism is instructively subtle in the analytical framework and is presented in full details. Roughly speaking, this leads to the *construction of well-defined K-homology classes out of densely defined unbounded symmetric operators on Hilbert spaces*, *that are not necessarily self-adjoint*, precisely the subject of Theorem 1.2, hence the connection.

Part II. Symmetric unbounded Fredholm modules

4. Construction of the class $[\mathcal{H}, \pi, T]$

In this section, we provide some general information on unbounded symmetric operators and we construct the promised K-homology class $[\mathcal{H}, \pi, T]$. We also state a general result, Theorem 4.1, of which Theorem 1.2 is a direct corollary. The proof will be presented in Section 5.

Recall that for a densely defined closed operator T, the operator $T^*T + 1$ is densely defined, self-adjoint, injective on its domain and surjective, and its inverse satisfies $(T^*T + 1)^{-1} \in B(\mathcal{H})$ (see for example [41, Proposition A.8.4, p. 511]). Therefore, condition (ii) of Definition 1.1 makes sense.

It is well-known that a densely defined closed symmetric operator T is self-adjoint if and only if $\mathrm{Ker}(T^*-i)$ and $\mathrm{Ker}(T^*+i)$ are trivial. In general, there can exist none,

just one (in case T is already self-adjoint), or uncountably many self-adjoint extensions of T. In fact, they are canonically parameterized by the space

$$\mathcal{U} := \left\{ u \colon \mathrm{Ker}(T^* - i) \longrightarrow \mathrm{Ker}(T^* + i) \mid u \text{ is a unitary isomorphism} \right\},$$

equipped with the norm-topology inherited from $B\left(\mathrm{Ker}(T^* - i), \mathrm{Ker}(T^* + i)\right)$. This means in particular that T possesses self-adjoint extensions if and only if the *deficiency spaces* $\mathrm{Ker}(T^* - i)$ and $\mathrm{Ker}(T^* + i)$ of T have the same (possibly infinite) dimension; here dimension, is meant in the sense of the cardinal of a Hilbert basis. If the *deficiency indices* $\dim(\mathrm{Ker}(T^* - i))$ and $\dim(\mathrm{Ker}(T^* + i))$ of T are finite and equal, say equal to n, then $\mathcal{U}$ is a principal homogeneous space over the unitary group in dimension n, hence $\mathcal{U}$ is homeomorphic to $U(n)$ (non-canonically for $n > 0$); in particular, it is Hausdorff and compact. Explicitly, in the general case, the correspondence is given as follows (provided that $\mathcal{U}$ is non-empty):

$$\mathcal{U} \ni u \qquad \longleftrightarrow \qquad T_u \colon \mathrm{dom}(T_u) \longrightarrow \mathcal{H},$$

where T_u is the unbounded operator with domain

$$\mathrm{dom}(T_u) := \left\{ \xi + \eta + u(\eta) \,\middle|\, \xi \in \mathrm{dom}(T) \text{ and } \eta \in \mathrm{Ker}(T^* - i) \right\}$$

and given by the (well-defined) formula

$$T_u \left(\xi + \eta + u(\eta) \right) := T(\xi) + i\eta - i u(\eta).$$

Finally, letting "$\oplus$" stand for the algebraic direct sum (not necessarily orthogonal), we point out that $\mathrm{dom}(T^*) = \mathrm{dom}(T) \oplus \mathrm{Ker}(T^* - i) \oplus \mathrm{Ker}(T^* + i)$ and that every T_u is a restriction of T^*. For the details and proofs, we refer, for instance, to Reed-Simon [36, Section X.1, pp. 135–143].

For the sequel, we suppose that the deficiency indices of T coincide and are equal to $n < \infty$ (in particular, $\mathcal{U}$ is compact Hausdorff and $C(\mathcal{U})$ is a unital separable C^*-algebra). We consider the unbounded operator $T \hat{\otimes} 1$ on the Banach space $\mathcal{H} \hat{\otimes} C(\mathcal{U})$, viewed as a Hilbert C^*-module over $C(\mathcal{U})$ in the obvious way or, in other words, as a (constant) continuous field of Hilbert spaces over $\mathcal{U}$. The operator $T \hat{\otimes} 1$ has, as domain, the image of the algebraic tensor product $\mathrm{dom}(D) \otimes C(\mathcal{U})$ in $\mathcal{H} \hat{\otimes} C(\mathcal{U})$. It admits a canonical extension $\mathcal{T}$, which is the unbounded operator equal to T_u in the fibre over each $u \in \mathcal{U}$. Let us provide an explicit description of $\mathcal{T}$. First, we use the canonical isomorphism of Hilbert $C(\mathcal{U})$-modules $\mathcal{H} \hat{\otimes} C(\mathcal{U}) \cong C(\mathcal{U}, \mathcal{H})$ (see [28, p. 27]) to identify both Hilbert C^*-modules. As usual, $C(\mathcal{U}, \mathcal{H})$ is endowed with the $C(\mathcal{U})$-valued scalar product $\langle f \mid g \rangle := \left(u \mapsto \langle f(u) \mid g(u) \rangle_{\mathcal{H}} \right)$ for $f, g \in C(\mathcal{U}, \mathcal{H})$, therefore with the topology of uniform convergence. Then, $T \hat{\otimes} 1$ has, as domain, the dense subspace $\{ f \in C(\mathcal{U}, \mathcal{H}) \mid \mathrm{Im}(f) \subseteq \mathrm{dom}(T) \}$ of $C(\mathcal{U}, \mathcal{H})$,

and $\mathcal{T}$ is defined as the $C(\mathcal{U})$-linear operator with domain

$$\mathrm{dom}(\mathcal{T}) := \left\{ f \in C(\mathcal{U}, \mathcal{H}) \;\middle|\; f(u) \in \mathrm{dom}(T_u), \; \forall u \in \mathcal{U} \right\}$$

and given by

$$\mathcal{T} : \mathrm{dom}(\mathcal{T}) \longrightarrow C(\mathcal{U}, \mathcal{H}), \quad f \longmapsto \big(\mathcal{T}f : u \mapsto T_u\,(f(u))\big).$$

Observe that $\mathcal{T}$ is a restriction of the closed operator $T^* \hat{\otimes} 1$, consequently, it is well-defined, i.e. $\mathcal{T}f$ is continuous for every $f \in C(\mathcal{U}, \mathcal{H})$. Note also that $T^* \hat{\otimes} 1$ is symmetric if and only if T is self-adjoint, in which case $\mathcal{U}$ is a point and $\mathcal{T} = T$.

By definition, a densely defined closed operator $\mathcal{T}$ on a Hilbert C^*-module is called *regular* if $\mathcal{T}^*$ is densely defined and $\mathcal{T}^*\mathcal{T} + 1$ has dense range (of course, if $\mathcal{T}$ is symmetric, only the latter property is significant). For $\mathcal{T}$ regular, the operator $\mathcal{T}^*$ is regular, and $\mathcal{T}^*\mathcal{T}$ is densely defined, self-adjoint and regular (see [28, Lemma 9.1, Corollary 9.6 and Proposition 9.9]).

Theorem 4.1. *Let $(\mathcal{H}, \pi, T)$ be a symmetric unbounded Fredholm module of degree i over a separable C^*-algebra A. Let $\mathcal{U}$ and $\mathcal{T}$ be as constructed above. Then, the operator $\mathcal{T}$ has a dense domain, is self-adjoint, and is regular, in the sense that $\mathcal{T}^2 + 1$ has dense range. Moreover, the triple $\big(\mathcal{H}\hat{\otimes}C(\mathcal{U}), \pi\hat{\otimes}1, \mathcal{T}\big)$ determines, in the unbounded picture of Kasparov's KK-theory, a well-defined class*

$$\langle \mathcal{H}, \pi, T \rangle := \big[\mathcal{H}\hat{\otimes}C(\mathcal{U}), \pi\hat{\otimes}1, \mathcal{T}\big] \; \in \; KK_i\,(A, C(\mathcal{U})).$$

In particular, $\mathcal{U}$ being path-connected, for every choice of points $u, v \in \mathcal{U}$, the corresponding evaluation maps $(\mathrm{ev}_u)_*, (\mathrm{ev}_v)_* : KK_i(A, C(\mathcal{U})) \longrightarrow KK_i(A, \mathbb{C})$ yield the same K-homology class, i.e.

$$[\mathcal{H}, \pi, T_u] = [\mathcal{H}, \pi, T_v] \; \in \; KK_i(A, \mathbb{C}) = K^i(A).$$

We can therefore denote this class unambiguously by $[\mathcal{H}, \pi, T] \in K^i(A)$.

The proof is the subject of Section 5 below.

5. Proof of Theorems 1.2 and 4.1

In the present section, we establish Theorem 4.1. Clearly, Theorem 1.2 is merely a part of it, so, we will not need to say more about its proof.

Proof of Theorem 4.1. During the proof, we keep notation as in Section 4. The proof consists in two steps. First, we have to show that the triple $\big(\mathcal{H}\hat{\otimes}C(\mathcal{U}), \pi\hat{\otimes}1, \mathcal{T}\big)$

indeed is a Kasparov triple in Baaj–Julg's unbounded description of *KK*-theory. The second step is simply the observation that the compact Hausdorff space $\mathcal{U}$ being path-connected, the evaluation maps $(\mathrm{ev}_u)_*$ $(\mathrm{ev}_v)_*$ as in the statement yield, as is well-known, the same *K*-homology class. So, we can focus exclusively on the first step. According to Baaj–Julg [2], we have to show that

(1) the domain $\mathrm{dom}(\mathcal{T})$ is dense;
(2) the operator $\mathcal{T}$ is self-adjoint;
(3) the operator $\mathcal{T}$ is regular;
(4) the operator $[\pi(b)\hat{\otimes}1, \mathcal{T}]$ is densely defined and bounded, for every $b \in \mathcal{B}$;
(5) the operator $(\pi(a)\hat{\otimes}1)(\mathcal{T}^2 + 1)^{-1}$ is compact, for every $a \in A$.

In (5), compactness is meant in the sense of Hilbert *C**-modules. Let us now establish these properties. (Concerning (4), see also Remark 5.1 below.)

(1) The domain $\mathrm{dom}(\mathcal{T})$ contains $\mathrm{dom}(T\hat{\otimes}1)$, so, it is dense.
(2) By definition, we have

$$\mathrm{dom}(\mathcal{T}^*) = \left\{ g \in C(\mathcal{U}, \mathcal{H}) \,\middle|\, \exists h \in C(\mathcal{U}, \mathcal{H}) \text{ st. } \langle \mathcal{T}f \mid g \rangle = \langle f \mid h \rangle, \ \forall f \in \mathrm{dom}(\mathcal{T}) \right\}.$$

The condition $\langle \mathcal{T}f \mid g \rangle = \langle f \mid h \rangle$ amounts to having $\langle T_u f(u) \mid g(u) \rangle_{\mathcal{H}} = \langle f(u) \mid h(u) \rangle_{\mathcal{H}}$ for every $u \in \mathcal{U}$. If $g \in \mathrm{dom}(\mathcal{T})$, then $\langle T_u f(u) \mid g(u) \rangle_{\mathcal{H}} = \langle f(u) \mid T_u g(u) \rangle_{\mathcal{H}}$ for every $u \in \mathcal{U}$, which means that $\langle \mathcal{T}f \mid g \rangle = \langle f \mid \mathcal{T}g \rangle$. This shows that $\mathcal{T} \subseteq \mathcal{T}^*$. We pass to the reverse inclusion. Since for every $\xi \in \mathrm{dom}(T_u)$, there exists $f \in \mathrm{dom}(\mathcal{T})$ with $f(u) = \xi$ (take for f the constant map), the condition for g to be in $\mathrm{dom}(\mathcal{T}^*)$ implies that $\langle T_u \xi \mid g(u) \rangle_{\mathcal{H}} = \langle \xi \mid h(u) \rangle_{\mathcal{H}}$ for every $\xi \in \mathrm{dom}(T_u)$. Since T_u is self-adjoint, by definition, this means that $g(u) \in \mathrm{dom}(T_u)$ and $h(u) = T_u g(u)$. Since this has to hold for every $u \in \mathcal{U}$, we see that $g \in \mathrm{dom}(\mathcal{T})$, so that $\mathrm{dom}(\mathcal{T}^*) \subseteq \mathrm{dom}(\mathcal{T})$.

(3) By [28, Lemma 9.8], to show that $\mathcal{T}$ is regular, we just have to check that $\mathcal{T} + i$ and $\mathcal{T} - i$ are surjective. Let $u \in \mathcal{U}$; since T_u is self-adjoint, the operator $T_u \pm i$ is surjective, and $(T_u \pm i)^{-1}$ is a bounded operator with range equal to $\mathrm{dom}(T_u)$. So, we can consider the $C(\mathcal{U})$-linear operator

$$\mathcal{S}_\pm : C(\mathcal{U}, \mathcal{H}) \longrightarrow C(\mathcal{U}, \mathcal{H}), \quad f \longmapsto \left(u \mapsto (T_u \pm i)^{-1} f(u) \right).$$

In fact, we have to prove that $\mathcal{S}_\pm$ is really well-defined, namely that the function $\mathcal{S}_\pm f : u \longmapsto (T_u \pm i)^{-1} f(u)$ is continuous. We do this just below and assume it for a while. Since $\mathrm{Im}((T_u \pm i)^{-1}) = \mathrm{dom}(T_u)$ for every $u \in \mathcal{U}$, we see that

$$\mathrm{dom}(\mathcal{T}\mathcal{S}_\pm) = \left\{ f \in C(\mathcal{U}, \mathcal{H}) \,\middle|\, (T_u \pm i)^{-1} f(u) \in \mathrm{dom}(T_u), \ \forall u \in \mathcal{U} \right\} = C(\mathcal{U}, \mathcal{H}).$$

Since, obviously, $(\mathcal{T} \pm i)\mathcal{S}_\pm = 1$ on $\mathrm{dom}(\mathcal{T}\mathcal{S}_\pm)$, the operator $\mathcal{T} \pm i$ is surjective. So, let us establish the continuity of $\mathcal{S}_\pm f$. Since by assumption T admits at least one self-adjoint extension, the operator $T \pm i$ is injective on its domain $\mathrm{dom}(T)$, therefore, the

operator $(T\pm i)^{-1}\colon \mathrm{Im}(T\pm i) \longrightarrow \mathrm{dom}(T) \subseteq \mathcal{H}$ is well-defined on its domain $\mathrm{Im}(T\pm i)$. It will be crucial for us to observe that $\mathrm{Im}(T\pm i)$ is closed, as follows from [15, Proposition X.2.5 (c)]; as a side-remark, note also that $\mathrm{Im}(T\pm i)$ is the whole of $\mathcal{H}$ if and only if T is self-adjoint, see [35, Theorem VIII.2, pp. 256–257]. Fix a point $u \in \mathcal{U}$. Consider the closed subspace $\mathcal{H}_u := \{\eta + u(\eta) \mid \eta \in \mathrm{Ker}(T^*{-}i)\}$ of $\mathcal{H}$ and the operator

$$R_u\colon \mathcal{H}_u \longrightarrow \mathrm{Ker}(T^*{-}i) \oplus \mathrm{Ker}(T^*{+}i) \subseteq \mathcal{H}, \quad \eta + u(\eta) \longmapsto i\eta - iu(\eta),$$

where the direct sum is an algebraic one, that is, of mere vector spaces. For $\eta \in \mathrm{Ker}(T^*{-}i)$, we compute that $(R_u + i)(\eta + u(\eta)) = 2i\eta \in \mathrm{Ker}(T^*{-}i)$ and that $(R_u - i)(\eta + u(\eta)) = -2iu(\eta) \in \mathrm{Ker}(T^*{+}i)$. So, we can view $R_u\pm i$ as an operator with codomain $\mathrm{Ker}(T^*{\mp}i)$, i.e.

$$R_u\pm i\colon \mathcal{H}_u \longrightarrow \mathrm{Ker}(T^*{\mp}i).$$

As a consequence, $T_u\pm i$ decomposes as an algebraic direct sum of two operators, as follows:

$$T_u\pm i = (T\pm i) \oplus (R_u\pm i)\colon \underbrace{\mathrm{dom}(T) \oplus \mathcal{H}_u}_{=\mathrm{dom}(T_u)} \longrightarrow \underbrace{\mathrm{Im}(T\pm i) \oplus \mathrm{Ker}(T^*{\mp}i)}_{=\mathcal{H}}.$$

(The last direct sum is orthogonal, but we will not need this fact.) From the above explicit computation of $R_u\pm i$, we deduce that

$$(R_u\pm i)^{-1}\colon \mathrm{Ker}(T^*{\mp}i) \longrightarrow \mathcal{H}_u \subseteq \mathcal{H}, \quad \xi \longmapsto \pm\tfrac{1}{2i}\left(\xi + u^{\pm 1}(\xi)\right).$$

So, we can write the operator $(T_u\pm i)^{-1}\colon \mathcal{H} \longrightarrow \mathrm{dom}(T_u) \subseteq \mathcal{H}$ as the direct sum

$$(T\pm i)^{-1} \oplus (R_u\pm i)^{-1}\colon \mathrm{Im}(T\pm i) \oplus \mathrm{Ker}(T^*{\mp}i) \longrightarrow \mathrm{dom}(T) \oplus \mathcal{H}_u \subseteq \mathcal{H}.$$

Finally, letting $P_\pm\colon \mathcal{H} \twoheadrightarrow \mathrm{Im}(T\pm i)$ and $Q_\pm\colon \mathcal{H} \twoheadrightarrow \mathrm{Ker}(T^*{\mp}i)$ denote the orthogonal projections (recall that $\mathrm{Im}(T\pm i)$ is closed !), we see that

$$\mathcal{S}_\pm f\colon u \longmapsto (T\pm i)^{-1}P_\pm f(u) \pm \tfrac{1}{2i}\left(Q_\pm f(u) + u^{\pm 1}\left(Q_\pm f(u)\right)\right).$$

Observe that the function $u \longmapsto u^{\pm 1}\left(Q_\pm f(u)\right)$ is the composition

$$\begin{aligned}
\mathcal{U} &\longrightarrow \mathcal{U} \times \mathrm{Ker}(T^*{\mp}i) \longrightarrow \mathrm{Ker}(T^*{\pm}i) \subseteq \mathcal{H} \\
u &\longmapsto \quad (u, Q_\pm f(u)) \\
&\qquad\qquad (v, \xi) \qquad \longmapsto v^{\pm 1}(\xi),
\end{aligned}$$

where the two indicated maps are continuous (for the latter, recall that $\mathcal{U}$ is equipped with the norm-topology). It follows that $\mathcal{S}_{\pm} f$ is continuous, as was to be shown.

(4) Fix an element $b \in \mathcal{B}$. Since $[\pi(b)\hat{\otimes}1, T\hat{\otimes}1] \subseteq [\pi(b)\hat{\otimes}1, \mathcal{T}] \subseteq [\pi(b)\hat{\otimes}1, T^*\hat{\otimes}1]$, the result follows.

(5) Finally, we fix an element $a \in A$. By assumption, the operator $T^*T + 1$ has dense range and $\pi(a)(T^*T + 1)^{-1}$ is compact. Since $T_u^2 + 1$ is an extension of $T^*T + 1$, it follows that $\pi(a)(T_u^2 + 1)^{-1}$ coincides with the compact operator $\pi(a)(T^*T + 1)^{-1}$. Consequently, we have

$$(\pi(a)\hat{\otimes}1)(\mathcal{T}^2 + 1)^{-1} = \pi(a)(T^*T + 1)^{-1}\hat{\otimes}1 \ \in \ \mathcal{K}(\mathcal{H})\hat{\otimes}C(\mathcal{U}) = \mathcal{K}\left(\mathcal{H}\hat{\otimes}C(\mathcal{U})\right)$$

(see [28, p. 10] for the final equality), and the proof is complete. □

Remark 5.1. Following Blackadar's treatment of the Baaj–Julg results, we did not require $[\pi(b)\hat{\otimes}1, \mathcal{T}]$ to have domain containing $\mathrm{dom}(\mathcal{T})$ for every $b \in \mathcal{B}$, but merely to have dense domain, see [9, pp. 163–165].

6. Generalization of Theorem 1.2 and Proof of Theorem 1.3

We start this section with some observations from which we derive a generalization of Theorem 1.2.

Observations 6.1.

(1) The assumption that T is closed is not really essential, since otherwise one can simply replace it by its closure $\bar{T} = T^{**}$, and then check/require properties (i), (ii) and (iii) of Definition 1.1 for the closure.

(2) The condition, weaker than (i) of Definition 1.1, saying that the deficiency indices of T coincide, but are not necessarily finite is enough to define $[\mathcal{H}, \pi, T] \in K^i(A)$ unambiguously. Indeed, the space $\mathcal{U}$ is always path-connected, so, we replace it everywhere by a path $\mathcal{P}_{uv}$ connecting two arbitrary points u and v in $\mathcal{U}$. For this, note that $\mathcal{P}_{uv}$ is a non-empty, compact Hausdorff and path-connected space, and that the map $\mathcal{P}_{uv} \times \mathrm{Ker}(T^*\mp i) \longrightarrow \mathrm{Ker}(T^*\pm i)$ taking (v, ξ) to $v^{\pm 1}(\xi)$ is also continuous. The proof is 'less canonical' in this case (since we are constrained to make a choice for the path $\mathcal{P}_{uv}$).

(3) Let T be a densely defined closed symmetric operator on a Hilbert space $\mathcal{H}$, and let $\pi \colon A \longrightarrow B(\mathcal{H})$ be a $*$-homomorphism. Then the following property—which does *not* involve T^*—implies (iii) of Definition 1.1:

(iii′) $[\pi(b), T]$ and $[\pi(b^*), T]$ are densely defined and $[\pi(b), T]$ is bounded for every b in a dense subspace $\mathcal{B}$ of A.

Indeed, to show that (iii) of 1.1 follows from (iii′), we first note that $[\pi(b), T^*]$ is densely defined and also closable, since its adjoint satisfies

$$[\pi(b), T^*]^* = (\pi(b)T^* - T^*\pi(b))^* \supseteq T^{**}\pi(b)^* - \pi(b)^*T^{**} = -[\pi(b^*), T],$$

so, is densely defined [15, Proposition X.1.6 (b) and Example X.1.1, pp. 305, 308]. Using the inclusion $T \subseteq T^*$, we get $[\pi(b), T^*]^* \subseteq [\pi(b), T]^*$. Since by assumption $[\pi(b), T]$ is densely defined and bounded, $[\pi(b), T]^* \in B(\mathcal{H})$. Therefore, $[\pi(b), T^*]^*$ is densely defined and bounded, so that the closure of $[\pi(b), T^*]$ satisfies $[\pi(b), T^*]^{**} \in B(\mathcal{H})$, showing that $[\pi(b), T^*]$ is, indeed, densely defined and bounded.

(4) Let $[\mathcal{H}, \pi, D] \in KK_i(A, \mathbb{C})$ be an unbounded Fredholm module in the usual sense, i.e. with D self-adjoint. Let T be a densely defined closed symmetric restriction of D. Then, the deficiency indices of T are automatically equal, so that T satisfies (i) provided one of them is finite; this happens exactly when the quotient $\mathrm{dom}(T^*)/\mathrm{dom}(T)$ is finite dimensional. Furthermore, T necessarily verifies (ii), since then $(T^*T + 1)^{-1} = (D^2 + 1)^{-1}$, and $\pi(a)(D^2 + 1)^{-1}$ is compact for every $a \in A$ by assumption.

These observations combined with Theorem 1.2 lead us directly to the following statement.

Theorem 6.2. *Let A be a separable C^*-algebra. Suppose given the following data*:

(a) *an integer $i \in \{0, 1\}$*;
(b) *a Hilbert space $\mathcal{H}$*;
(c) *a $*$-representation $\pi: A \longrightarrow B(\mathcal{H})$ of A*;
(d) *a densely defined symmetric operator T on $\mathcal{H}$ with domain $\mathrm{dom}(T)$.*

These data are required to fulfill, firstly, the two conditions

 (i) *the deficiency indices of T^{**} coincide (as cardinals)*;
(ii) *the operator $\pi(a)(T^*T^{**} + 1)^{-1}$ is compact for every $a \in A$*;

and, secondly, one of the following two conditions:

 (iii) *the operator $[\pi(b), T^{**}]$ is densely defined, and $[\pi(b), T^*]$ is densely defined and bounded for every b in some norm-dense subspace $\mathcal{B}$ of A*;
(iii$'$) *$[\pi(b), T^{**}]$ and $[\pi(b^*), T^{**}]$ are densely defined and $[\pi(b), T^{**}]$ is bounded for every b in a dense subspace $\mathcal{B}$ of A.*

Thirdly, if $i = 0$, we moreover require $\mathcal{H}$ to be $\mathbb{Z}/2$-graded, π to preserve the grading, and T to reverse it. Then, there exists, in the unbounded picture of analytical K-homology, a well-defined K-homology class

$$[\mathcal{H}, \pi, T] \in K^i(A),$$

that is canonical, and coincides with the usual class in case T is self-adjoint. More precisely, given an arbitrary self-adjoint extension $\widetilde{T}$ of T—and at least one such extension exists—, one has

$$[\mathcal{H}, \pi, T] = [\mathcal{H}, \pi, \widetilde{T}] \in K^i(A),$$

independently of the choice of $\widetilde{T}$.

Definition 6.3. By extension, we call a triple $(\mathcal{H}, \pi, T)$ satisfying the hypotheses of Theorem 6.2 a *symmetric unbounded Fredholm module of degree i* over A.

We pass to the proof of Theorem 1.3.

Proof of Theorem 1.3. We proceed somehow as in 6.1 (4). Since T_1 is self-adjoint, its restriction T is symmetric with closure T^{**} satisfying $T \subseteq T^{**} \subseteq T_1$ and being symmetric. By hypothesis (a), T is densely defined, therefore, so is T^{**}. In particular, the densely defined closed symmetric operator T^{**} admits at least one self-adjoint extension, namely T_1, so that its deficiency indices coincide (see Section 4). This shows that T satisfies (i) of Theorem 6.2. Moreover, one has

$$(T^* T^{**} + 1)^{-1} = (T_1^2 + 1)^{-1}.$$

By assumption that $(\mathcal{H}, \pi, T_1)$ is a (usual) Fredholm module, $\pi(a)(T_1^2 + 1)^{-1}$ is compact for every $a \in A$. This implies (ii) of 6.2 for T. By hypothesis (b), we have that both the operators $[\pi(b), T^{**}]$ and $[\pi(b^*), T^{**}]$ are densely defined and bounded for every b in $\mathcal{B}$ (recall that $\mathcal{B}$ is $*$-closed). This implies (iii$'$) of 6.2 for T. All in all, we have a symmetric unbounded Fredholm module $(\mathcal{H}, \pi, T)$, and, applying Theorem 6.2 to it twice (T_2 is a self-adjoint extension of T as well), we get the equalities

$$[\mathcal{H}, \pi, T_1] = [\mathcal{H}, \pi, T] = [\mathcal{H}, \pi, T_2]$$

in $K^i(A)$, as desired. It remains to prove that condition (b$'$) implies condition (b). First, the operators $[\pi(b'), T^{**}]$ and $[\pi(b'^*), T^{**}]$ are densely defined and bounded for every b' in the subspace $\mathcal{B}'$. Secondly, for every $b' \in \mathcal{B}'$, the operator $[\pi(b'), T^{**}]$ is a restriction of $[\pi(b'), T_1]$, and is therefore also bounded, by choice of $\mathcal{B}'$. So, we get condition (b) with $\mathcal{B} := \mathcal{B}'$. The proof is now complete. $\square$

Part III. Proofs of the "addition formulae"

7. Topological proof of the "addition formulae"

This section is subdivided into three subsections. In the first one, we establish a useful general principle that will allow us to reduce the proofs (both topological and analytical) of Theorems 2.1 and 2.2 to the verification of one equality that embodies the pure substance of the "addition formulae", without any extraneous ornament. In the other two subsections, one for each treated dimension, we prove these theorems in the topological setting.

7.1. A general principle

We present here a general, but easy, principle on homology (and related) theories, that will be used on several occasions in the sequel, even for the analytical proofs. To

state it, we call a functor $F(-)$ from the category of CW-complexes to the category of abelian groups *additive* if, given two maps $f_1: X_1 \longrightarrow X$ and $f_2: X_2 \longrightarrow X$ of CW-complexes, one has a natural isomorphism

$$F(X_1 \amalg X_2) \cong F(X_1) \oplus F(X_2)$$

such that, using it as an identification,

$$(f_1 \amalg f_2)_*(x_1, x_2) = (f_1)_*(x_1) + (f_2)_*(x_2) \ \in \ F(X),$$

for every $x_1 \in F(X_1)$ and $x_2 \in F(X_2)$, where f_* stands for $F(f)$ whenever f is a map between CW-complexes. For example, an additive homology theory with compact supports, like integral homology or K-homology, is an additive functor. The point for us is that the assignment

$$X \longmapsto RKK_*(X, \mathbb{C})$$

is straightforwardly seen to be an additive functor in our sense, *without using the identification of* $RKK_*(X, \mathbb{C})$ *with* $K_*(X)$, so that, later, our analytical proofs will really be purely and strictly analytical.

Lemma 7.1. *Let* $F(-)$ *be an additive functor as defined above. Let* M_1 *and* M_2 *be connected oriented manifolds of the same dimension* $n > 0$. *Let* D_i $(i = 1, 2)$ *be a 'small' embedded open disk in* M_i, *whose boundary inside* M *contains the base-point, and form the oriented connected sum* $M_1 \# M_2$ *by gluing* $M_1 \setminus D_1$ *and* $M_2 \setminus D_2$ *along their boundaries. Consider the obvious maps*

$$j: M_1 \amalg M_2 \longrightarrow M_1 \vee M_2 \qquad and \qquad p: M_1 \# M_2 \longrightarrow M_1 \vee M_2,$$

given by identification of the base-points and pinching the boundary $\partial D_1 \approx \partial D_2$ *to a point, respectively. Let* X *be a pointed CW-complex and let* $f_i: M_i \longrightarrow X$ *be a continuous map, which, on* D_i, *is constant and equal to the base-point of* X. *Consider the connected sum* $f_1 \# f_2: M_1 \# M_2 \longrightarrow X$. *Finally, suppose given three elements*

$$x_1 \in F(M_1), \quad x_2 \in F(M_2) \quad and \quad x \in F(M_1 \# M_2)$$
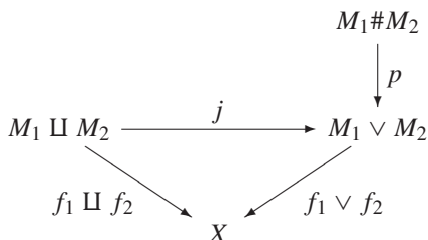
satisfying the compatibility condition

$$j_*(x_1, x_2) = p_*(x) \ \in \ F(M_1 \vee M_2).$$

Then, one has the equality

$$(f_1 \# f_2)_*(x) = (f_1)_*(x_1) + (f_2)_*(x_2) \ \in \ F(X).$$

Proof. We have the commutative diagram

$$
\begin{array}{ccc}
& & M_1 \# M_2 \\
& & \downarrow{\scriptstyle p} \\
M_1 \amalg M_2 \xrightarrow{\ \ j\ \ } & & M_1 \vee M_2 \\
\qquad\searrow{\scriptstyle f_1 \amalg f_2} & & \swarrow{\scriptstyle f_1 \vee f_2} \\
& X &
\end{array}
$$

Noticing that $f_1 \# f_2 = (f_1 \vee f_2) \circ p$, we compute

$$
\begin{aligned}
(f_1 \# f_2)_*(x) &= (f_1 \vee f_2)_*\, p_*(x) \\
&= (f_1 \vee f_2)_*\, j_*(x_1, x_2) \\
&= (f_1 \amalg f_2)_*(x_1, x_2) \\
&= (f_1)_*(x_1) + (f_2)_*(x_2),
\end{aligned}
$$

and this completes the proof.   □

As a prototypical illustration of Lemma 7.1, we deduce the following simple example on the homology of manifolds.

Example 7.2. Keep notation as in Lemma 7.1, but assume M_i $(i = 1, 2)$ to be closed and denote by $[M_i] \in H_n(M_i; \mathbb{Z})$ its fundamental class. Then, in the group $H_n(X; \mathbb{Z})$, one has

$$
(f_1 \# f_2)_*[M_1 \# M_2] = (f_1)_*[M_1] + (f_2)_*[M_2].
$$

Indeed, the map $j_*\colon H_n(M_1 \amalg M_2; \mathbb{Z}) \longrightarrow H_n(M_1 \vee M_2; \mathbb{Z})$ satisfies the compatibility condition

$$
j_*\,([M_1], [M_2]) = p_*[M_1 \# M_2]
$$

(as a computation using suitable triangulations shows), so, Lemma 7.1 applies to give the result. Now, suppose that $M_1 = \Sigma_{g_1}$ and $M_2 = \Sigma_{g_2}$ are closed oriented surfaces. As we have noticed, the Euler characteristic is *not* additive with respect to the connected sum. Note that this amounts to saying that the corresponding compatibility condition is *not* satisfied, as next indicated:

$$
j_*\left(\chi(\Sigma_{g_1})\cdot[1],\ \chi(\Sigma_{g_2})\cdot[1]\right) \neq p_*\left(\chi(\Sigma_{g_1}\#\Sigma_{g_2})\cdot[1]\right)\ \in\ H_0(\Sigma_{g_1}\vee\Sigma_{g_2};\mathbb{Z}),
$$

where [1] stands for the prescribed generator of the zeroth homology group of any connected CW-complex.

The next lemma is a slight variation of Lemma 7.1; for the modified connected sum "$\natural$", we refer to Figs. 1 and 2 in Section 2 and to the statement of Theorem 2.2.
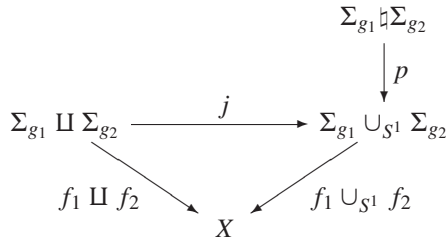
Lemma 7.3. *Keep the same notation and hypotheses as in Lemma 7.1, but with $M_1 = \Sigma_{g_1}$ and $M_2 = \Sigma_{g_2}$ being closed oriented surfaces, and D_i being a handle given as a small open tubular neighbourhood of a suitable non-retractable embedded circle C_i $(i = 1, 2)$. Let $\Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}$ denote the CW-complex obtained as the union of Σ_{g_1} and Σ_{g_2} with the circles C_1 and C_2 pointwise identified in an orientation-preserving way. Then, the equality*

$$(f_1 \natural f_2)_*(x) = (f_1)_*(x_1) + (f_2)_*(x_2) \ \in F(X)$$

holds for $x_1 \in F(\Sigma_{g_1})$, $x_2 \in F(\Sigma_{g_2})$ and $x \in F(\Sigma_{g_1} \natural \Sigma_{g_2})$ satisfying the compatibility condition $j_*(x_1, x_2) = p_*(x)$ in $F(\Sigma_{g_1} \cup_{S^1} \Sigma_{g_2})$, where j and p stand for the obvious identification and pinching maps

$$j: \Sigma_{g_1} \amalg \Sigma_{g_2} \longrightarrow \Sigma_{g_1} \cup_{S^1} \Sigma_{g_2} \qquad and \qquad p: \Sigma_{g_1} \natural \Sigma_{g_2} \longrightarrow \Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}.$$

Proof. This time, we have the commutative diagram

$$
\begin{array}{ccc}
 & & \Sigma_{g_1} \natural \Sigma_{g_2} \\
 & & \downarrow{\scriptstyle p} \\
\Sigma_{g_1} \amalg \Sigma_{g_2} \xrightarrow{\ j\ } & & \Sigma_{g_1} \cup_{S^1} \Sigma_{g_2} \\
\end{array}
$$

$$f_1 \amalg f_2 \searrow \qquad \swarrow f_1 \cup_{S^1} f_2$$

$$X$$

Noticing that $f_1 \natural f_2 = (f_1 \cup_{S^1} f_2) \circ p$, we can perform a similar computation as to establish Lemma 7.1. $\square$

The next result will turn useful on several occasions later on.

Proposition 7.4. *Let X be a pointed CW-complex. For $i = 1, 2$, let $f_i: \Sigma_{g_i} \longrightarrow X$ be a pointed continuous map, that is constant in a small neighbourhood of a handle of Σ_{g_i} $(g_i \geqslant 0)$. Consider the modified connected sum (along the two given handles) $f_1 \natural f_2: \Sigma_{g_1} \natural \Sigma_{g_2} \longrightarrow X$ of these two maps. Then, one has the equality*

$$\chi(\Sigma_{g_1} \natural \Sigma_{g_2}) = \chi(\Sigma_{g_1}) + \chi(\Sigma_{g_2})$$

and, in homology, one has

$$(f_1 \natural f_2)_*[\Sigma_{g_1} \natural \Sigma_{g_2}] = (f_1)_*[\Sigma_{g_1}] + (f_2)_*[\Sigma_{g_2}] \ \in \ H_2(X; \mathbb{Z}).$$

Proof. The first equality is obvious, since $\chi(\Sigma_g) = 2 - 2g$ for any $g \geqslant 0$. By the general principle 7.3, it suffices to check that

$$j_*\big([\Sigma_{g_1}], [\Sigma_{g_2}]\big) = p_*[\Sigma_{g_1+g_2-1}] \ \in \ H_2(\Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}; \mathbb{Z}).$$

One can either decree it to be visible and hence obvious, or, for instance, draw concrete triangulations on both surfaces, respecting the prescribed circles and handles, and then determine the corresponding triangulations for $\Sigma_{g_1} \natural \Sigma_{g_2}$ and $\Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}$, and finally deduce explicitly the maps j_* and p_* on the H_2-level to get the result.   □

7.2. Topological proof of Theorem 2.1

We begin by stating a result that will be needed for the topological proof of Theorem 2.2 as well (and also in our application to the Baum–Connes conjecture).

Lemma 7.5. *For a two-dimensional CW-complex X, there are canonical and natural isomorphisms (so-called integral Chern characters)*

$$\mathrm{ch}^{\mathbb{Z}}_{\mathrm{ev}} \colon K_0(X) \xrightarrow{\ \cong\ } H_0(X; \mathbb{Z}) \oplus H_2(X; \mathbb{Z}) \quad and \quad \mathrm{ch}^{\mathbb{Z}}_{\mathrm{odd}} \colon K_1(X) \xrightarrow{\ \cong\ } H_1(X; \mathbb{Z}).$$

They are compatible with the usual Chern character $\mathrm{ch}_* \colon K_*(X) \longrightarrow H_*(X; \mathbb{Q})$ *and the map in homology corresponding to the change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{Q}$.*

The compatibility in question is the obvious one; in case of doubt, see the diagrams in the proof of Lemma 12.1. See [6, Lemma 4] or [30, Proposition 2.1] for a proof of the lemma. The proof actually reveals more. Indeed, if $X^{[1]}$ denotes the 1-skeleton of X, consider the exact sequences both in K-homology and ordinary homology, associated with the cofibre sequence of spaces $X^{[1]} \longrightarrow X \longrightarrow X/X^{[1]}$. The isomorphisms in the lemma are the unique isomorphisms making the following diagram with exact rows commute:

$$
\begin{array}{ccccccccccc}
 & & \overbrace{\cong \mathbb{Z}}^{} & & & & & & & & \\
0 & \longrightarrow & K_0(X^{[1]}) & \longrightarrow & K_0(X) & \longrightarrow & \widetilde{K}_0(X/X^{[1]}) & \longrightarrow & K_1(X^{[1]}) & \longrightarrow & K_1(X) \longrightarrow 0 \\
 & & \| & & \vdots\downarrow & & \downarrow \cong & & \downarrow \cong & & \vdots\downarrow \\
0 & \longrightarrow & H_0(X) & \longrightarrow & H_{\mathrm{ev}}(X) & \longrightarrow & H_2(X/X^{[1]}) & \longrightarrow & H_1(X^{[1]}) & \longrightarrow & H_1(X) \longrightarrow 0 \\
 & & \underbrace{}_{\cong \mathbb{Z}} & & & & & & & &
\end{array}
$$

where H_* stands for *integral* homology. (Note that $X^{[1]}$ and $X/X^{[1]}$ are homotopy equivalent to a bouquet of circles and of 2-spheres, respectively). We present now an example, that we state as a lemma for later reference.

Lemma 7.6. *For the class $[D] \in K_1(S^1)$ of the Dirac operator over the circle, one has $\mathrm{ch}_{\mathrm{odd}}^{\mathbb{Z}}[D] = -[S^1]$ in $H_1(S^1; \mathbb{Z})$, where $[S^1]$ is the fundamental homology class of S^1 corresponding to the selected orientation; in particular, $K_1(S^1) \cong \mathbb{Z}$ is generated by $[D]$.*

Proof. This is well-known (up to the minus sign !), but since no proof seems to be available in the literature, we provide one here. Let $(e_n)_{n \in \mathbb{Z}}$ be the trigonometric basis of the Hilbert space $L^2(S^1)$. The phase F of D, i.e. the operator appearing in the polar decomposition $D = F \cdot |D|$, is given by $F(e_n) = e_n$ if $n > 0$, $F(e_0) = 0$ and $F(e_n) = -e_n$ if $n < 0$. The homotopy $t \mapsto F \cdot (D^*D)^{t/2}$ between F and D shows that $[F] = [D]$ in $K^1(C(S^1))$. Now, consider the rank one (hence compact) perturbation $F_\circ$ of F taking the same values on the e_n's, except that $F_\circ(e_0) = e_0$; of course, $[F_\circ] = [F]$ in $K^1(C(S^1))$. The operator $F_\circ$ is a self-adjoint involution and the corresponding projection $P := \frac{1+F_\circ}{2}$ is the Toeplitz projection on the Hardy space $\mathcal{H}^2(S^1)$ of S^1; indeed, $\mathcal{H}^2(S^1)$ is defined as the closed span of the set $\{e_n\}_{n \geqslant 0}$ in $L^2(S^1)$. By [21, 2.7.7, 2.7.9, 5.1.6 and pp. 213–214], this means that $[D]$, as an element of $\mathrm{Ext}(C(S^1)) \cong KK_1(C(S^1), \mathbb{C})$, corresponds to the Toeplitz extension of $C(S^1)$ by the compact operators on $\mathcal{H}^2(S^1)$ described in [21, (2.3.5)]). Now, given a unitary u in $C(S^1)$, consider the corresponding Toeplitz operator $T_u := PuP$ (see [21, Definition 2.7.7]). Then, for the canonical pairing (i.e. the Kasparov product)

$$\otimes : \underbrace{KK_1(\mathbb{C}, C(S^1))}_{\cong K^1(S^1)} \otimes_{\mathbb{Z}} \underbrace{KK_1(C(S^1), \mathbb{C})}_{\cong K_1(S^1)} \longrightarrow \underbrace{KK_0(\mathbb{C}, \mathbb{C})}_{\cong \mathbb{Z}}, \quad (x, y) \longmapsto x \otimes y,$$

writing $\mathrm{Wind}(u)$ for the winding number of u, one has

$$[u] \otimes [D] = \mathrm{Index}(T_u) = -\mathrm{Wind}(u),$$

where the first equality follows from [9, Theorem 18.10.2], and the second from [21, Theorem 2.3.2]. By Lemma 7.5 and by its well-known cohomological counterpart (see for instance [30, Lemma 5.1]), one has

$$K_1(S^1) \overset{\mathrm{ch}_{\mathrm{odd}}^{\mathbb{Z}}}{\cong} H_1(S^1; \mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad K^1(S^1) \overset{\mathrm{ch}_{\mathbb{Z}}^{\mathrm{odd}}}{\cong} H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}.$$

Consider the unitary $u_\circ = (z \mapsto z)$ in $C(S^1)$, whose class $[u_\circ]$ is the standard generator of $K^1(S^1)$, i.e. the one satisfying $\mathrm{ch}_{\mathbb{Z}}^{\mathrm{odd}}[u_\circ] = [S^1]$ in $H^1(S^1; \mathbb{Z})$. Then, one gets $[u_\circ] \otimes [D] = -1$. Altogether, this shows that $[D]$ indeed is a generator of $K_1(S^1)$ (another approach for this result is one based on the ideas of [5]). Now, since the

Chern character in K-homology and in K-theory of finite CW-complexes is induced by a map of spectra (in the sense of algebraic topology), for such a space X, there is a commutative diagram

$$
\begin{array}{ccc}
K^*(X) \otimes_{\mathbb{Z}} K_*(X) & \xrightarrow{\langle .,. \rangle_K} & \mathbb{Z} \\
{\scriptstyle \mathrm{ch}^* \otimes \mathrm{ch}_*} \downarrow & & \downarrow \\
H^*(X;\mathbb{Q}) \otimes_{\mathbb{Q}} H_*(X;\mathbb{Q}) & \xrightarrow{\langle .,. \rangle} & \mathbb{Q}
\end{array}
$$

where "$\langle .,. \rangle$" stands for the Kronecker product, and "$\langle .,. \rangle_K$" denotes the usual pairing between K-theory and K-homology (see [5]). As is folklore (see however [1] and [24, Section 6]), the diagram

$$
\begin{array}{ccc}
KK_*(\mathbb{C}, C(X)) \otimes_{\mathbb{Z}} KK_*(C(X), \mathbb{C}) & \xrightarrow{\otimes} & KK_*(\mathbb{C}, \mathbb{C}) \\
\cong \downarrow & & \downarrow \cong \\
K^*(X) \otimes_{\mathbb{Z}} K_*(X) & \xrightarrow{\langle .,. \rangle_K} & \mathbb{Z}
\end{array}
$$

does also commute. Since the integral (co)homology of the circle injects inside its rational (co)homology and since $\langle [S^1], [S^1] \rangle = \langle 1, [S^1] \cap [S^1] \rangle = \langle 1, 1 \rangle = 1$ by very choice of both orientation classes (see [16, VII.12.8] for the first equality), it follows that $\mathrm{ch}_{\mathrm{odd}}^{\mathbb{Z}}[D] = -[S^1]$. $\quad \square$

Now, we state a result that might be of independent interest, and to which the proof of Theorem 2.2 reduces.

Proposition 7.7. *Consider the obvious identification and pinching maps*

$$
j \colon S^1 \amalg S^1 \longrightarrow S^1 \vee S^1 \qquad and \qquad p \colon S^1 \cong S^1 \# S^1 \longrightarrow S^1 \vee S^1.
$$

Then, in K-homology, on has the equality

$$
j_*\left([D], [D]\right) = p_*[D] \ \in \ K_1(S^1 \vee S^1).
$$

Proof. By Lemma 7.5, we can identify K_1 with H_1 for all the (one-dimensional) spaces in sight in the statement, and therefore, applying Lemma 7.6, we are reduced to proving that $j_*\left(-[S^1], -[S^1]\right) = p_*\left(-[S^1]\right)$ in $H_1(S^1 \vee S^1)$, or equivalently that $j_*\left([S^1], [S^1]\right) = p_*[S^1]$, an equality that is obvious (compare with Example 7.2). $\quad \square$

Finally, we can pass to our first proof of Theorem 2.1.

Topological proof of Theorem 2.1. Our general principle embodied by Lemma 7.1 implies that the result follows from Proposition 7.7, that we have proven with purely topological methods.   □

7.3. Topological proof of Theorem 2.2

As in the preceding subsection, we present a result of independent interest to which Theorem 2.2 boils down.

Proposition 7.8. *Let Σ_{g_1} and Σ_{g_2} be two closed oriented surfaces of genus g_1 and g_2 respectively. Consider the obvious identification and pinching maps*

$$j: \Sigma_{g_1} \sqcup \Sigma_{g_2} \longrightarrow \Sigma_{g_1} \cup_{S^1} \Sigma_{g_2} \quad and \quad p: \Sigma_{g_1+g_2-1} \cong \Sigma_{g_1} \natural \Sigma_{g_2} \longrightarrow \Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}.$$

Then, in K-homology, on has the equality

$$j_* \left([\bar{\partial}_{g_1}], [\bar{\partial}_{g_2}] \right) = p_* [\bar{\partial}_{g_1+g_2-1}] \ \in \ K_0(\Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}).$$

Before we present the proof, we recall the following fundamental and classical result.

Lemma 7.9. *For a closed oriented surface Σ_g of genus g, denote by $\imath^{\Sigma_g}$ the inclusion of the base-point. Then, letting $[1]$ denote the canonical generator of the group $K_0(pt) \cong \mathbb{Z}$, one has*

$$K_0(\Sigma_g) \cong \mathbb{Z}^2 \qquad and \qquad K_1(\Sigma_g) \cong \mathbb{Z}^{2g}$$

with $\imath_*^{\Sigma_g}[1]$ and $[\bar{\partial}_g]$ as generators of $K_0(\Sigma_g)$; furthermore, one has

$$\mathrm{ch}_{\mathrm{ev}}^{\mathbb{Z}}[\bar{\partial}_g] = (1 - g) \cdot [1] + [\Sigma_g] \ \in \ H_0(\Sigma_g; \mathbb{Z}) \oplus H_2(\Sigma_g; \mathbb{Z}),$$

where $[1] \in H_0(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}$ is the canonical generator, and $[\Sigma_g] \in H_2(\Sigma_g; \mathbb{Z})$ is the fundamental class.

Proof. Consider a compact Kähler manifold M. Let $\mathrm{ch}_{\mathrm{ev}}: K_0(M) \longrightarrow H_{\mathrm{ev}}(M; \mathbb{Q})$ be the usual ("rational-valued") Chern character. Let $\bar{\partial}_M$ be the Dolbeault operator on M, and $[\bar{\partial}_M]$ its class in $K_0(M)$. The Riemann–Roch–Hirzebruch Formula for M (see [39, p. 29]) says precisely that $\mathrm{ch}_{\mathrm{ev}}[\bar{\partial}_M]$ is Poincaré-dual to the Todd class $\mathrm{Td}(TM) \in H^*(M; \mathbb{Q})$. Specializing to $M = \Sigma_g$, we see that $\mathrm{ch}_{\mathrm{ev}}[\bar{\partial}_g]$ is Poincaré-dual to the rational cohomology class (see [39, p. 3])

$$\mathrm{Td}(T\Sigma_g) = 1 + \tfrac{1}{2} c_1(T\Sigma_g) \ \in \ H^{\mathrm{ev}}(\Sigma_g; \mathbb{Q});$$

here, $T\Sigma_g$ is of course viewed as a complex line bundle over Σ_g. Since

$$\tfrac{1}{2}\langle c_1(T\Sigma_g), [\Sigma_g]\rangle = \mathrm{Index}(\bar{\partial}_{\Sigma_g}) = 1 - g$$

(cf. [39, p. 27]), the desired result concerning $\mathrm{ch}^{\mathbb{Z}}_{\mathrm{ev}}[\bar{\partial}_g]$ follows from Poincaré duality, the fact that $\mathrm{ch}_{\mathrm{ev}}$ and $\mathrm{ch}^{\mathbb{Z}}_{\mathrm{ev}}$ are compatible (see Lemma 7.5) for two-dimensional spaces, and the fact that the integral homology of Σ_g is torsion-free, so that the canonical map from integral homology to rational homology is injective in this case. The rest follows readily from Lemma 7.5 and the well-known integral homology of Σ_g. □

Proof of Proposition 7.8. By means of Lemmas 7.5 and 7.9, it suffices to check that

$$j_*\left(\underbrace{(1-g_1)}_{=\frac{1}{2}\chi(\Sigma_{g_1})}\cdot[1], \underbrace{(1-g_2)}_{=\frac{1}{2}\chi(\Sigma_{g_2})}\cdot[1]\right) = p_*\left(\underbrace{(1-(g_1+g_2-1))}_{=\frac{1}{2}\chi(\Sigma_{g_1}\natural\Sigma_{g_2})}\cdot[1]\right)$$

in the group $H_0(\Sigma_{g_1}\cup_{S^1}\Sigma_{g_2};\mathbb{Z})$, and that $j_*\left([\Sigma_{g_1}], [\Sigma_{g_2}]\right) = p_*[\Sigma_{g_1+g_2-1}]$ in the group $H_2(\Sigma_{g_1}\cup_{S^1}\Sigma_{g_2};\mathbb{Z})$. The first equality follows from Proposition 7.4, and the second from the proof of the latter. □

Next, we present our first proof of Theorem 2.2.

Topological proof of Theorem 2.2. The modified version of our general principle, stated as Lemma 7.3, implies that the result is a direct consequence of Proposition 7.7, whose proof given above was performed in the topological setting. □

Remark 7.10. Both these topological proofs, though their relative simplicity, leave an unsatisfactory feeling for the following reasons. It is not quite explicit *here* what K-homology is, and in particular how the classes $[D]$ and $[\bar{\partial}_g]$ are defined (except possibly in the proof of Lemma 7.6). Moreover, the Chern character plays a rather mysterious rôle. Reasoning the opposite way, it is rather pleasant that we did not have to define K-homology and these classes explicitly, and the question of the Chern character is something well-understood and absolutely central in connection with the Atiyah–Singer Index Theorem and of the Riemann–Roch–Hirzebruch Formula.

Remark 7.11. Keeping notation as in Theorem 2.2, we generally have

$$(f_1\#f_2)_*[\bar{\partial}_{g_1+g_2-1}] \neq (f_1)_*[\bar{\partial}_{g_2}] + (f_2)_*[\bar{\partial}_{g_2}] \in K_0(X),$$

so that there is no "addition formula" for Dolbeault operators with respect to the usual connected sum. Indeed, as the proof of the theorem presented above shows, the reason

for this is precisely that

$$j_*\left(\underbrace{(1-g_1)}_{=\frac{1}{2}\chi(\Sigma_{g_1})}\cdot[1], \underbrace{(1-g_2)}_{=\frac{1}{2}\chi(\Sigma_{g_2})}\cdot[1]\right) \neq p_*\left(\underbrace{1-(g_1+g_2)}_{=\frac{1}{2}\chi(\Sigma_{g_1}\#\Sigma_{g_2})}\cdot[1]\right) \in H_0(\Sigma_{g_1}\vee\Sigma_{g_2};\mathbb{Z}),$$

compare with Example 7.2 and Proposition 7.4. In other words, the obstruction at the source of the problem is the non-additivity of the Euler characteristic with respect to the usual connected sum.

8. Analytical proof of the Dirac-type "addition formula"

This section is partitioned into two subsections. In the first, we describe, with the necessary details, the class defined by the Dirac operator on the circle in *K*-homology, viewed using the unbounded picture of analytical *K*-homology. In the second, one of the cores of the paper, we prove Theorem 2.1 in this setting.

8.1. Class of the Dirac operator for S^1 in analytic K-homology

Consider the Dirac operator on the circle S^1 (equipped with the standard orientation, more precisely the standard Spin^c-structure), namely

$$D := \frac{1}{i}\cdot\frac{d}{d\theta},$$

where $\frac{d}{d\theta}$ stands for the distributional derivative with domain

$$\operatorname{dom}(D) := \left\{\xi \in L^2(S^1) \mid \frac{d\xi}{d\theta} \in L^2(S^1) \text{ and } \xi(0) = \xi(1)\right\}.$$

To be extremely precise, and for later use, let us give some explanations and recall some basic and well-known facts. First, $[a, b]$ will denote an arbitrary compact interval (with $a < b$), and θ (or θ_0) a variable in it. We consider S^1 as the unit interval $[0, 1]$ with 0 and 1 identified, and we view $L^2(S^1)$ as $L^2[0, 1]$ in the obvious way, namely considering a function $\xi(e^{2\pi i\theta})$ as a function of the variable $\theta \in [0, 1]$, denoted by $\xi(\theta)$ for simplicity. Every class $\xi \in L^2[a, b]$ defines a distribution on the interval $[a, b]$ given by

$$T_\xi\varrho(\theta) := \int_a^b \xi(\theta)\varrho(\theta)\,d\theta \qquad (\varrho \in C^\infty[a, b]).$$

The *distributional derivative* of ξ is the distribution

$$T'_\xi\varrho(\theta) := -\int_a^b \xi(\theta)\frac{d\varrho}{d\theta}(\theta)\,d\theta \qquad (\varrho \in C^\infty[a, b]).$$

One writes $\frac{d\xi}{d\theta} \in L^2[a,b]$ if there exists a class $\eta \in L^2[a,b]$ such that $T_\eta = T'_\xi$ (and then, this class is unique). In this case, we will always consider ξ as being the unique *continuous* function on $[a,b]$ representing its class in $L^2[a,b]$; explicitly, it is given by

$$\xi(\theta) = \int_a^\theta \frac{d\xi}{d\theta}(\theta_0)\, d\theta_0.$$

(This expression is meaningful by the Cauchy–Schwarz inequality.) We now define two kinds of Sobolev spaces. Firstly, we set

$$W^1[a,b] := \left\{ \xi \in L^2[a,b] \,\middle|\, \frac{d\xi}{d\theta} \in L^2[a,b] \right\}.$$

Secondly, for the circle, we define

$$W^1(S^1) := \left\{ \xi \in W^1[0,1] \,\middle|\, \xi(0) = \xi(1) \right\}.$$

Note that now the domain of D has a transparent meaning, since by definition it is the latter Sobolev space, i.e. $\mathrm{dom}(D) = W^1(S^1)$. Observe that $W^1[0,1]$ identifies with $AC[0,1]$, the space of absolutely continuous functions on $[0,1]$, cf. [35, pp. 258, 305]. We keep this notation in the sequel. The operator D is self-adjoint on its domain $W^1(S^1)$ (see e.g. [36, Example 1 in X.1, p. 141]), as required to be part of the following unbounded Fredholm module:

$$[D] := [L^2(S^1), \mathcal{M}, D] \in KK_1(C(S^1), \mathbb{C}),$$

where $\mathcal{M}$ is the $*$-representation of $C(S^1)$ on $L^2(S^1)$ by pointwise multiplication.

8.2. Analytical proof of Theorem 2.1

Recalling the notation introduced in the previous subsection, we can directly move to the announced proof.

Analytical of proof of Theorem 2.1. By our general principle 7.1, and keeping the same notation, it boils down to proving that

$$j_* \left([D], [D]\right) = p_*[D] \in KK_1(C(S^1 \vee S^1), \mathbb{C}).$$

Of course, this is precisely the content of Proposition 7.7, that we have already proved, but using topology. So, here, as promised, we will reprove this in the realm of analytical K-homology. Let us now make the unbounded Fredholm modules $p_*[D]$

and $j_*([D], [D])$ explicit. First, the pinching map $p: S^1 \longrightarrow S^1 \vee S^1$ is given by

$$p(\theta) = \begin{cases} (2\theta)_1 & \text{if } 0 \leqslant \theta \leqslant \frac{1}{2}, \\ (2\theta - 1)_2 & \text{if } \frac{1}{2} \leqslant \theta \leqslant 1. \end{cases}$$

Here, $(2\theta)_1$ means that we view 2θ as living in the first copy of S^1 in $S^1 \vee S^1$, and analogously for $(2\theta - 1)_2$. Then, $p_*[D] \in KK_1(C(S^1 \vee S^1), \mathbb{C})$ is described as

$$p_*[D] = \left[L^2(S^1), \mathcal{M}', D \right],$$

where, for $f \in C(S^1 \vee S^1)$, $\mathcal{M}'(f) = \mathcal{M}(f \circ p)$, the multiplication by $f \circ p$ on $L^2(S^1)$, and, as before, D is $\frac{1}{i} \cdot \frac{d}{d\theta}$ with domain $W^1(S^1)$. On the other hand,

$$j_*([D], [D]) = \left[L^2(S^1) \oplus L^2(S^1), \mathcal{M}_1 \oplus \mathcal{M}_2, D \oplus D \right],$$

where the direct sum is an orthogonal direct sum, and, for $f \in C(S^1 \vee S^1)$ and $i = 1, 2$, we have $\mathcal{M}_i(f) = \mathcal{M}(f_i)$ with f_i standing for the restriction of f to the i-th copy of S^1 in $S^1 \vee S^1$. Here and below, we make the obvious identifications

$$L^2(S^1) \oplus L^2(S^1) = L^2(S^1 \vee S^1) = L^2[0, 1] \oplus L^2[1, 2] = L^2[0, 2].$$

Since the domains of the operators play a crucial rôle, let us give the domain of $D \oplus D$ very explicitly:

$$\mathrm{dom}(D \oplus D) = W^1(S^1) \oplus W^1(S^1)$$
$$= \left\{ (\xi_1, \xi_2) \in L^2[0, 2] \,\middle|\, \begin{array}{l} \xi_1 \in W^1[0, 1] \text{ and } \xi_1(0) = \xi_1(1) \\ \xi_2 \in W^1[1, 2] \text{ and } \xi_2(1) = \xi_2(2) \end{array} \right\}.$$

To compare the unbounded Fredholm modules $p_*[D]$ and $j_*([D], [D])$, we first have to compare the corresponding Hilbert spaces. Consider the "doubling" unitary

$$U: \begin{cases} L^2(S^1) \oplus L^2(S^1) \xrightarrow{\cong} L^2(S^1) \\ (\xi_1, \xi_2) \longmapsto \left(\theta \mapsto \begin{cases} \sqrt{2} \cdot \xi_1(2\theta), & \text{if } 0 \leqslant \theta \leqslant \frac{1}{2} \\ \sqrt{2} \cdot \xi_2(2\theta - 1), & \text{if } \frac{1}{2} < \theta \leqslant 1 \end{cases}, \text{ a.e.} \right) \end{cases}$$

The inverse U^* of U is given by the formula

$$(U^*\xi)(\theta) = \left(\frac{1}{\sqrt{2}} \cdot \xi\left(\frac{\theta}{2}\right), \frac{1}{\sqrt{2}} \cdot \xi\left(\frac{\theta+1}{2}\right) \right),$$

for $\xi \in L^2(S^1)$ and $\theta \in [0, 1]$. Clearly, $U(\mathcal{M}_1 \oplus \mathcal{M}_2)U^* = \mathcal{M}'$ holds, so that we get

$$j_* ([D], [D]) = \left[L^2(S^1), \mathcal{M}', U(D \oplus D)U^* \right].$$

It remains to discuss the relationship between the operators D and $U(D \oplus D)U^*$. The domain of the latter will be denoted by $\mathcal{E}$ and is given by

$$\mathcal{E} := \mathrm{dom} \left(U(D \oplus D)U^* \right) = U \left(\mathrm{dom}(D \oplus D) \right) = U \left(W^1(S^1) \oplus W^1(S^1) \right)$$

$$= \left\{ (\xi_1, \xi_2) \in L^2[0, 1] \, \middle| \, \begin{array}{l} \xi_1 \in W^1[0, \frac{1}{2}] \text{ and } \xi_1(0) = \xi_1(\frac{1}{2}) \\ \xi_2 \in W^1[\frac{1}{2}, 1] \text{ and } \xi_2(\frac{1}{2}) = \xi_2(1) \end{array} \right\},$$

where we identify $L^2[0, 1]$ with $L^2[0, \frac{1}{2}] \oplus L^2[\frac{1}{2}, 1]$ in the obvious way. It is readily checked that $U(D \oplus D)U^*$ equals $\frac{1}{2i} \cdot \frac{d}{d\theta}$ on this domain. It is well-known that an unbounded Fredholm module $[\mathcal{H}, \pi, F]$ is equal to $[\mathcal{H}, \pi, \lambda \cdot F]$ for every positive real number $\lambda > 0$ (the triples are operator-homotopic). So, we have to show that

$$\left[L^2(S^1), \mathcal{M}', \underbrace{\frac{1}{i} \cdot \frac{d}{d\theta} \text{ on } W^1(S^1)}_{=D} \right] = \left[L^2(S^1), \mathcal{M}', \underbrace{\frac{1}{i} \cdot \frac{d}{d\theta} \text{ on } \mathcal{E}}_{=2U(D \oplus D)U^*} \right]$$

in $KK_1(C(S^1 \vee S^1), \mathbb{C})$. The difficulty we alluded to on several occasions is that the unitary U does *not* map the domain $W^1(S^1) \oplus W^1(S^1)$ to $W^1(S^1)$, i.e. $\mathcal{E}$ and $W^1(S^1)$ are different. For this reason, we define two new (dense) domains $\mathcal{D}_\circ$ and $\mathcal{D}$, both contained in $W^1(S^1)$ and with $\mathcal{D}$ also contained in $\mathcal{E}$, which are more adapted to the situation, namely

$$\{ \mathcal{D}_\circ := \left\{ \xi \in W^1(S^1) \, \middle| \, \xi(0) = \xi(1) = 0 \right\}$$

and

$$\mathcal{D} := \left\{ \xi \in W^1(S^1) \, \middle| \, \xi(0) = \xi\left(\tfrac{1}{2}\right) = \xi(1) = 0 \right\}.$$

Then U maps $\mathcal{D}_\circ \oplus \mathcal{D}_\circ$ isometrically onto $\mathcal{D}$, and we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}_\circ \oplus \mathcal{D}_\circ & \xrightarrow[\cong]{\; U \, |_{\mathcal{D}_\circ \oplus \mathcal{D}_\circ} \;} & \mathcal{D} \\
{\scriptstyle 2(D \oplus D)|_{\mathcal{D}_\circ \oplus \mathcal{D}_\circ}} \downarrow & & \downarrow {\scriptstyle D|_{\mathcal{D}}} \\
L^2(S^1) \oplus L^2(S^1) & \xrightarrow[\cong]{\; U \;} & L^2(S^1)
\end{array}
$$

As already singled out, the operator D is self-adjoint on its domain $W^1(S^1)$, therefore, $2U(D \oplus D)U^*$ is also self-adjoint. The operator $T := D|_{\mathcal{D}}$ is closed, symmetric, but *not* self-adjoint. In fact, its adjoint T^* is determined as in [35, Example in VIII.2, pp. 257–259]: it is the operator $T^* = \frac{1}{i} \cdot \frac{d}{d\theta}$ on the domain $\operatorname{dom}(T^*) = W^1[0, 1]$. So, we are faced with two genuinely distinct self-adjoint extensions of T, namely, one is D with domain $W^1(S^1)$, the other one is $2U(D \oplus D)U^*$ with domain $\mathcal{E}$. Therefore, we are urged to try to apply Theorem 1.3 to show that these operators define the same class in analytic K-homology. Before we proceed, as a side-remark, we mention that the deficiency indices of T are both equal to 2, and we refer to Example 8.2 below for more details on this.

Firstly, T is a closed symmetric operator and its domain, $\mathcal{D}$, is dense, so that condition (a) of Theorem 1.3 is fulfilled. Secondly, to get condition (b′), let us determine a ∗-closed dense subspace $\mathcal{B}'$ of $C(S^1 \vee S^1)$ such that the operator $[\mathcal{M}'(f), T]$ is densely defined and bounded for every $f \in \mathcal{B}'$. To do so, we identify $C(S^1 \vee S^1)$ with the C^*-algebra $\left\{ f \in C(S^1) \,\middle|\, f(0) = f(\frac{1}{2}) \right\}$ in the obvious way, namely, still viewing S^1 as $[0, 1]/\partial[0, 1]$. We correspondingly take for $\mathcal{B}'$ the ∗-closed dense sub-algebra $\left\{ f \in C^\infty(S^1) \,\middle|\, f(0) = f(\frac{1}{2}) \right\}$, in other words,

$$\mathcal{B}' := \left\{ f \in C(S^1 \vee S^1) \,\middle|\, f \circ p \in C^\infty(S^1) \right\}.$$

Observing that the subspace $\mathcal{M}'(\mathcal{B}')(\mathcal{D})$ is equal to $\mathcal{D}$, the domain of T, we deduce that $\operatorname{dom}\left([\mathcal{M}'(f), T]\right) \supseteq \operatorname{dom}(T)$. For ξ in $\mathcal{D}$ and $f \in \mathcal{B}'$, we have

$$[\mathcal{M}'(f), T]\xi = \frac{1}{i} \cdot \left((f \circ p) \cdot \frac{d\xi}{d\theta} - \frac{d((f \circ p) \cdot \xi)}{d\theta} \right) = \frac{1}{i} \cdot \mathcal{M}\left(\frac{d(f \circ p)}{d\theta} \right) \xi.$$

As hoped, Theorem 1.3 applies and yields the desired equality of analytic K-homology classes defined by the two given self-adjoint extensions of T. This completes the proof. □

We thank G. Skandalis for pointing out to us the rôle of distinct self-adjoint extensions, while we were trying to prove Theorem 10.1 below in an analytical way, which in fact amounts to the present proof as we will see.

Remark 8.1. The above proof shows that $(L^2(S^1), \mathcal{M}', T)$ is a symmetric unbounded Fredholm module, with T *non*-self-adjoint, and defining a non-trivial analytic K-homology class $[L^2(S^1), \mathcal{M}', T] \in KK_1(C(S^1 \vee S^1), \mathbb{C})$.

As a matter of illustration, we would now like to give some more information on the self-adjoint extensions of the operator T of the preceding proof.

Example 8.2. We keep notation as above. Obviously, one has

$$\mathrm{Ker}(T^*-i) = \left\{ (\xi_1, \xi_2) \in W^1[0, \tfrac{1}{2}] \oplus W^1[\tfrac{1}{2}, 1] \,\middle|\, \begin{matrix} \xi_1(\theta) = \lambda_1 \cdot e^{-\theta} \\ \xi_2(\theta) = \lambda_2 \cdot e^{-\theta} \end{matrix}, \ \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

and

$$\mathrm{Ker}(T^*+i) = \left\{ (\xi_1, \xi_2) \in W^1[0, \tfrac{1}{2}] \oplus W^1[\tfrac{1}{2}, 1] \,\middle|\, \begin{matrix} \xi_1(\theta) = \lambda_1 \cdot e^{\theta} \\ \xi_2(\theta) = \lambda_2 \cdot e^{\theta} \end{matrix}, \ \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

So, in our situation, we have a 'canonical' orthonormal basis for each deficiency space, namely $\{(e'_1, e'_2), (e''_1, e''_2)\}$ for the former and $\{(\varepsilon'_1, \varepsilon'_2), (\varepsilon''_1, \varepsilon''_2)\}$ for the latter, where e'_2, e''_1, ε'_2 and ε''_1 are zero functions, and, letting $\omega_0 := \sqrt{\frac{2}{e-1}}$,

$$e'_1(\theta) := \omega_0 \sqrt{e} \cdot e^{-\theta}, \ e''_2(\theta) := \omega_0 e \cdot e^{-\theta}, \ \varepsilon'_1(\theta) := \omega_0 \cdot e^{\theta} \text{ and } \varepsilon''_2(\theta) := \frac{\omega_0}{\sqrt{e}} \cdot e^{\theta}.$$

This gives an explicit homeomorphism between $\mathcal{U}$ and $U(2)$. By direct computation, one checks that for every $u \in \mathcal{U}$, the self-adjoint extension T_u of T is equal to $\frac{1}{i} \cdot \frac{d}{d\theta}$ on its domain $\mathrm{dom}(T_u)$. This is no surprise, since every T_u is a restriction of T^*, which is also $\frac{1}{i} \cdot \frac{d}{d\theta}$ on its domain $W^1[0, 1]$, as we have seen. One can wonder to which matrix in $U(2)$ does the operator D (resp. $2U^*(D \oplus D)U$) correspond to. One obtains

$$D \ \longleftrightarrow \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in U(2) \qquad \text{and} \qquad 2U^*(D \oplus D)U \ \longleftrightarrow \ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U(2).$$

In general, to a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2)$ with determinant Δ, corresponds the operator $\frac{1}{i} \cdot \frac{d}{d\theta}$ on the domain consisting of the functions $(\xi_1, \xi_2) \in W^1[0, \tfrac{1}{2}] \oplus W^1[\tfrac{1}{2}, 1]$ satisfying the boundary conditions

$$\begin{pmatrix} \xi_1(0) \\ \xi_2(\tfrac{1}{2}) \end{pmatrix} = \frac{1}{1 + (\alpha+\delta)\sqrt{e} + \Delta e} \begin{pmatrix} \alpha + (1+\Delta)\sqrt{e} + \delta e & \beta(1-e) \\ \gamma(1-e) & \delta + (1+\Delta)\sqrt{e} + \alpha e \end{pmatrix} \cdot \begin{pmatrix} \xi_1(\tfrac{1}{2}) \\ \xi_2(1) \end{pmatrix}.$$

9. Analytical proof of the Dolbeault-type "addition formula"

As in the case of the Dirac-type "addition formula", this section contains two subsections. In the first one, we depict the analytic K-homology class determined by the Dolbeault operator for Σ_g. In the second, we present the analytical proof of Theorem 2.2; again, this is one of the central parts of the paper.

9.1. Class of the Dolbeault operator for Σ_g in analytic K-homology

We fix an auxiliary Kähler structure on Σ_g, i.e. we view Σ_g as a complex curve equipped with a suitably compatible Riemannian metric. We let $\bar\partial_{\Sigma_g} := \bar\partial \oplus \bar\partial^*$ be the Dolbeault operator, i.e.

$$L^2(\Lambda^{0,0}T^*\Sigma_g) \oplus L^2(\Lambda^{0,2}T^*\Sigma_g) \supset \mathrm{dom}(\bar\partial_{\Sigma_g}) \xrightarrow{\ \bar\partial \oplus \bar\partial^*\ } L^2(\Lambda^{0,1}T^*\Sigma_g).$$

Here, $L^2(\Lambda^{0,j}T^*\Sigma_g)$ is the Hilbert space of L^2-forms of bidegree $(0, j)$ on $T^*\Sigma_g$ (see for instance [18, pp. 73–74]). In other words, we view Σ_g as a Kähler manifold equipped with the 'anti-canonical' Spin^c-structure, and $\bar\partial_{\Sigma_g}$ is $\frac{1}{\sqrt{2}}D_\nabla$, where D_∇ is the Dirac operator corresponding to the Levi–Civita connection ∇ (for the details, see [18, pp. 77–81]). The domain of $\bar\partial_{\Sigma_g}$ is $W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_g)$, a Sobolev space on which the operator $\bar\partial_{\Sigma_g}$ is self-adjoint (see [18, pp. 100–101] or [10, Chapter 20]). To simplify the notation, we let $[\bar\partial_g]$ denote the class of the operator $\bar\partial_{\Sigma_g}$ in $KK_0(C(\Sigma_g), \mathbb{C}) \cong K_0(\Sigma_g)$. Explicitly, $[\bar\partial_g]$ is given by the unbounded Fredholm module

$$[\bar\partial_g] := \left[L^2(\Lambda^{0,*}T^*\Sigma_g), \mathcal{M}, \bar\partial_{\Sigma_g} \right] \ \in \ KK_0(C(\Sigma_g), \mathbb{C}),$$

where $L^2(\Lambda^{0,*}T^*\Sigma_g) = \bigoplus_{j=0}^{2} L^2(\Lambda^{0,j}T^*\Sigma_g)$ is $\mathbb{Z}/2$-graded by even and odd degree forms, and $\mathcal{M}$ is the $*$-representation of $C(\Sigma_g)$ on $L^2(\Lambda^{0,*}T^*\Sigma_g)$ given by pointwise multiplication. By connectedness of the Teichmüller space, the class $[\bar\partial_g]$ is independent of the choice of the Kähler structure. For a later application, let $\mathrm{Lip}(\Sigma_g)$ be the $*$-closed dense sub-algebra of $C(\Sigma_g)$ consisting of the Lipschitz functions; by Rademacher's Theorem (see [43, Theorem 11A]), Lipschitz functions are differentiable almost everywhere on Σ_g and we single out that $[\mathcal{M}(\vartheta), \bar\partial_{\Sigma_g}]$ is densely defined and bounded for every function $\vartheta \in \mathrm{Lip}(\Sigma_g)$.

9.2. Analytical proof of Theorem 2.2

For the analytical proof, we will need the notions and notation introduced in Section 8.2, and we will apply Theorem 1.3.

Analytical proof of Theorem 2.2. Again, using (the slight variation of) our general principle 7.3, still with the same notation, it remains to prove that

$$j_* \left([\bar\partial_{g_1}], [\bar\partial_{g_2}] \right) = p_*[\bar\partial_{g_1+g_2-1}] \ \in \ KK_0(C(\Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}), \mathbb{C}).$$

As in the one-dimensional case, this is exactly Proposition 7.8, and, this time, we will establish it while sticking to analysis. We start by carefully describing the two Fredholm

modules under consideration. For the sake of readability, we set $g := g_1 + g_2 - 1$ and $X := \Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}$. First, in the group $KK_0(C(X), \mathbb{C})$, we have

$$p_* \left[\bar{\partial}_g \right] = \left[L^2(\Lambda^{0,*}T^*\Sigma_g), \mathcal{M}', \bar{\partial}_{\Sigma_g} \right],$$

where, for a function $\vartheta \in C(X)$, the operator $\mathcal{M}'(\vartheta)$ is fibre-wise multiplication by $\vartheta \circ p \in C(\Sigma_g)$ on the Hilbert space $L^2(\Lambda^{0,*}T^*\Sigma_g)$ of L^2-sections of the vector bundle $\Lambda^{0,*}T^*\Sigma_g$; as before, the domain of $\bar{\partial}_{\Sigma_g}$ is $W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_g)$. On the other hand, we get

$$j_* \left([\bar{\partial}_{g_1}], [\bar{\partial}_{g_2}] \right) = \left[L^2(\Lambda^{0,*}T^*\Sigma_{g_1}) \oplus L^2(\Lambda^{0,*}T^*\Sigma_{g_2}), \mathcal{M}_1 \oplus \mathcal{M}_2, \bar{\partial}_{\Sigma_{g_1}} \oplus \bar{\partial}_{\Sigma_{g_2}} \right],$$

where the direct sum is an orthogonal and graded one, and, for $\vartheta \in C(X)$ and $i = 1, 2$, we have $\mathcal{M}_i(\vartheta) = \mathcal{M}(\vartheta_i)$ with ϑ_i standing for the restriction $\vartheta|_{\Sigma_{g_i}}$; the domain of $\bar{\partial}_{\Sigma_{g_1}} \oplus \bar{\partial}_{\Sigma_{g_2}}$ is the orthogonal direct sum

$$\mathrm{dom}(\bar{\partial}_{\Sigma_{g_1}} \oplus \bar{\partial}_{\Sigma_{g_2}}) = W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_{g_1}) \oplus W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_{g_2}).$$

Now, we would like to determine a grading-preserving unitary isomorphism

$$U \colon L^2(\Lambda^{0,*}T^*\Sigma_{g_1}) \oplus L^2(\Lambda^{0,*}T^*\Sigma_{g_2}) \xrightarrow{\cong} L^2(\Lambda^{0,*}T^*\Sigma_g).$$

We can modify $\Sigma_{g_1} \natural \Sigma_{g_2}$ by an orientation-preserving analytic diffeomorphism, so, we may suppose that the modified connected sum $\Sigma_{g_1} \natural \Sigma_{g_2} \cong \Sigma_g$ is obtained from Σ_{g_1} and Σ_{g_2} by gluing the open manifolds $V_1 := \Sigma_{g_1} \setminus C_1$ and $V_2 := \Sigma_{g_2} \setminus C_2$ along the closed manifold $K := S^1 \sqcup S^1$, with C_1 and C_2 as in Lemma 7.3, i.e.

$$\Sigma_{g_1} \natural \Sigma_{g_2} = \underbrace{(\Sigma_{g_1} \setminus C_1)}_{=V_1} \sqcup \underbrace{(S^1 \sqcup S^1)}_{=K} \sqcup \underbrace{(\Sigma_{g_2} \setminus C_2)}_{=V_2}.$$

This way, we can consider V_i ($i = 1, 2$) as an analytic open sub-manifold of both Σ_{g_i} and $\Sigma_{g_1} \natural \Sigma_{g_2}$, and the complement in the latter of the union $V_1 \sqcup V_2$ is of measure zero. Moreover, p merely identifies the two copies of S^1 pointwise. Now, with this in mind, we define U almost everywhere by the formula

$$U(\omega_1, \omega_2) := \begin{cases} \sqrt{2} \cdot \omega_1|_{V_1} & \text{on } V_1 \subset \Sigma_g, \\ \sqrt{2} \cdot \omega_2|_{V_2} & \text{on } V_2 \subset \Sigma_g. \end{cases}$$

The inverse is simply given (almost everywhere) by

$$U^* \omega := \left(\frac{1}{\sqrt{2}} \cdot \omega|_{V_1}, \frac{1}{\sqrt{2}} \cdot \omega|_{V_2} \right).$$

Is it obvious that U intertwines $\mathcal{M}_1 \oplus \mathcal{M}_2$ and $\mathcal{M}'$, i.e. $U(\mathcal{M}_1 \oplus \mathcal{M}_2)U^* = \mathcal{M}'$. It follows that

$$j_* \left([\bar{\partial}_{g_1}], [\bar{\partial}_{g_2}] \right) = \left[L^2(\Lambda^{0,*}T^*\Sigma_g), \mathcal{M}', U(\bar{\partial}_{\Sigma_{g_1}} \oplus \bar{\partial}_{\Sigma_{g_2}})U^* \right],$$

where the operator appearing has domain

$$\operatorname{dom}\left(U(\bar{\partial}_{\Sigma_{g_1}} \oplus \bar{\partial}_{\Sigma_{g_2}})U^* \right) = U \left(W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_{g_1}) \oplus W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_{g_2}) \right).$$

To see what happens at the level of the domains of the unbounded operators involved, we first define a dense subspace $\widetilde{\mathcal{D}}$ of $L^2(\Lambda^{0,\mathrm{ev}}T^*\Sigma_g)$ by

$$\mathcal{D} := \left\{ \omega \in W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_g) \,\Big|\, \omega|_K = 0 \right\} = H_0^1(\Lambda^{0,\mathrm{ev}}T^*V_1) \oplus H_0^1(\Lambda^{0,\mathrm{ev}}T^*V_2).$$

For the definition of the Sobolev space $H_0^1(\Lambda^{0,\mathrm{ev}}T^*V_i)$, for the latter equality and for the sense to give to the equation $\omega|_K = 0$, we refer to [41, p. 290, Example 4.5.2 on p. 294 and to Proposition 4.4.5 on p. 287] respectively. Similarly, for $i = 1, 2$, we define a dense subspace $\mathcal{D}_i$ in $L^2(\Lambda^{0,\mathrm{ev}}T^*\Sigma_{g_i})$ by

$$\mathcal{D}_i := \left\{ \omega_i \in W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_{g_i}) \,\Big|\, \omega_i|_{C_i} = 0 \right\} = H_0^1(\Lambda^{0,\mathrm{ev}}T^*V_i).$$

The point is that U maps $\mathcal{D}_1 \oplus \mathcal{D}_2$ isometrically onto $\mathcal{D}$, and, since V_1 and V_2 are analytic open sub-manifolds of Σ_g and since Dolbeault operators are local (i.e. defined locally), there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}_1 \oplus \mathcal{D}_2 & \xrightarrow[\cong]{\;\;\;\;\;U\;\;\;\;\;} & \mathcal{D} \\[4pt]
{\scriptstyle \bar{\partial}_{\Sigma_{g_1}} \oplus \bar{\partial}_{\Sigma_{g_2}}|_{\mathcal{D}_1 \oplus \mathcal{D}_2}} \Big\downarrow & & \Big\downarrow {\scriptstyle \bar{\partial}_{\Sigma_g}|_{\mathcal{D}}} \\[4pt]
L^2(\Lambda^{0,1}T^*\Sigma_{g_1}) \oplus L^2(\Lambda^{0,1}T^*\Sigma_{g_2}) & \xrightarrow[\cong]{\;\;\;\;\;U\;\;\;\;\;} & L^2(\Lambda^{0,1}T^*\Sigma_g)
\end{array}
$$

So, letting $T := \bar{\partial}_{\Sigma_g}|_{\mathcal{D}}$, we are faced with two self-adjoint extensions of the densely defined symmetric operator T, namely $\bar{\partial}_{\Sigma_g}$ with domain $W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_g)$ and $U(\bar{\partial}_{\Sigma_{g_1}} \oplus \bar{\partial}_{\Sigma_{g_2}})U^*$ with domain $U \left(W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_{g_1}) \oplus W^1(\Lambda^{0,\mathrm{ev}}T^*\Sigma_{g_2}) \right)$, and we have to show that they define the same K-homology class. Again, the instructive difficulty is that these two domains are distinct. As in the one-dimensional case, we will now verify that Theorem 1.3 applies to establish the desired K-equality. Condition (a) of Theorem 1.3 being clearly fulfilled by T, to get condition (b′), let us determine a $*$-closed dense subspace $\mathcal{B}'$ of $C(X)$ such that the operator $[\mathcal{M}'(\vartheta), T]$ is densely defined and bounded

for every $\vartheta \in \mathcal{B}'$. Let us consider the $*$-closed dense subalgebra $\mathrm{Lip}(X)$ of $C(X)$ consisting of the Lipschitz functions on $X = \Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}$. Given a function $\vartheta \in \mathrm{Lip}(X)$, the map $p : \Sigma_g \longrightarrow X$ being Lipschitz, we see that the composite satisfies $\vartheta \circ p \in \mathrm{Lip}(\Sigma_g)$. By the final sentence in Section 9.1, the operator $[\mathcal{M}'(\vartheta), T]$ is indeed densely defined and bounded for every $\vartheta \in \mathrm{Lip}(X)$. Finally, Theorem 1.3 applies and gives the desired equality of analytic K-homology classes defined by the two self-adjoint extensions of T at hand. This completes the proof. □

Remark 9.1. This proof shows that the triple $\left(L^2(\Lambda^{0,*}T^*\Sigma_{g_1+g_2-1}), \mathcal{M}', T \right)$ is a symmetric unbounded Fredholm module, with T *non*-self-adjoint, and defining a non-trivial analytic K-homology class $\left[L^2(\Lambda^{0,*}T^*\Sigma_{g_1+g_2-1}), \mathcal{M}', T \right]$ in the group $KK_0 \left(C(\Sigma_{g_1} \cup_{S^1} \Sigma_{g_2}), \mathbb{C} \right)$.

Remark 9.2. The deficiency indices of the symmetric unbounded operator T in the above proof are equal and countably infinite.

Remark 9.3. Contrarily to the Dirac case (see the commutative diagram in the analytical proof of Theorem 2.1 in Section 8.2), in the commutative diagram with Dolbeault operators in the proof above, there is no constant popping up like the 2 in the Dirac case. The reason for this is the equality

$$\underbrace{\mathrm{Area}(\Sigma_{g_1} \cup_{S^1} \Sigma_{g_2})}_{=\mathrm{Area}(\Sigma_{g_1})+\mathrm{Area}(\Sigma_{g_2})} = \mathrm{Area}(\Sigma_g),$$

of areas, whereas, in the Dirac case, with our choice of parametrizations, we have

$$\underbrace{\mathrm{Length}(S^1 \vee S^1)}_{=\mathrm{Length}(S^1)+\mathrm{Length}(S^1)} = 2 \cdot \mathrm{Length}(S^1).$$

Part IV. Application to the Baum–Connes conjecture

10. The first homology of a group and the Baum–Connes map

In this section, subdivided into five subsections, we treat our program described in Section 3 for the case $j = 1$.

10.1. Topological and analytical definitions of $\beta_1^{(a)}$

We denote the abelianization of the group Γ by Γ^{ab} and we identify it with $H_1(\Gamma; \mathbb{Z})$ in the usual way. We write γ^{ab} for the class of the element $\gamma \in \Gamma$ in the quotient

group Γ^{ab}. We consider

$$\beta_1^{(\mathrm{a})} \colon \Gamma^{\mathrm{ab}} \longrightarrow K_1(C_r^*\Gamma), \quad \gamma^{\mathrm{ab}} \longmapsto [\gamma] = \left[\mathrm{Diag}(\gamma, 1, 1, \ldots) \right],$$

the canonical homomorphism induced by the homomorphism $\bar{\beta}_1^{(\mathrm{a})} \colon \Gamma \longrightarrow K_1(C_r^*\Gamma)$ coming from the inclusion of Γ into the group of invertible elements in $C_r^*\Gamma$. In the analytical description of KK-theory, the class $[\gamma] \in K_1(C_r^*\Gamma) \cong KK_1(\mathbb{C}, C_r^*\Gamma)$ is given via the equality

$$[\gamma] = \alpha_*^{\gamma} \left[\mathcal{E}, \tfrac{d}{dx} \right] \in KK_1(\mathbb{C}, C_r^*\Gamma),$$

where the notation is as follows. First, $[\mathcal{E}, \tfrac{d}{dx}] = [\mathcal{E}, \pi_{\circ}, \tfrac{d}{dx}] \in KK_1(\mathbb{C}, C_r^*\mathbb{Z}) \cong \mathbb{Z}$ is the 'standard' generator, with $\mathcal{E}$ denoting the separation-completion of the algebra $C_c^{\infty}(\mathbb{R})$ of compactly supported smooth complex-valued functions on the real line with respect to the $C_r^*\mathbb{Z}$-valued scalar product determined by

$$\langle \xi_1 | \xi_2 \rangle (n) := \langle \xi_1 | \varrho(n) \cdot \xi_2 \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \overline{\xi_1(x)} e^{-2\pi i n x} \xi_2(x) \, dx,$$

for $\xi_1, \xi_2 \in C_c^{\infty}(\mathbb{R})$ and $n \in \mathbb{Z}$, where ϱ is the action of $\mathbb{Z}$ on $C_c^{\infty}(\mathbb{R})$ by point-wise multiplication by integer powers of the function $e^{-2\pi i x}$; and $\pi_{\circ} \colon \mathbb{C} \longrightarrow \mathcal{L}_{C_r^*\mathbb{Z}}(\mathcal{E})$ is the unit, i.e. the $*$-homomorphism taking $\lambda \in \mathbb{C}$ to $\lambda \cdot \mathrm{id}_{\mathcal{E}}$; compare with [33, Section 4.2 in Part 2]. Second, α_*^{γ} stands for the map

$$\alpha_*^{\gamma} \colon KK_1(\mathbb{C}, C_r^*\mathbb{Z}) \longrightarrow KK_1(\mathbb{C}, C_r^*\Gamma)$$

induced by the composition of $*$-homomorphisms

$$C_r^*\mathbb{Z} = C^*\mathbb{Z} \xrightarrow{\widehat{\alpha^{\gamma}}} C^*\Gamma \xrightarrow{\lambda_{\Gamma}} C_r^*\Gamma$$

defined using *maximal* group-C^*-algebras, where the first indicated $*$-homomorphism is induced by the obvious group homomorphism determined by γ, namely

$$\alpha^{\gamma} \colon \mathbb{Z} \longrightarrow \Gamma, \quad n \longmapsto \gamma^n;$$

the map λ_{Γ} is the canonical epimorphism. It is also possible to describe $[\gamma]$ directly as an unbounded Kasparov element (in the sense of Baaj–Julg [2]), namely,

$$[\gamma] = [\mathcal{E}', \tfrac{d}{dx}] = [\mathcal{E}', \pi_{\circ}', \tfrac{d}{dx}] \in KK_1(\mathbb{C}, C_r^*\Gamma),$$

where $\mathcal{E}'$ is the separation-completion of $C_c^\infty(\mathbb{R})$ with respect to the $C_r^*\Gamma$-valued scalar product determined by

$$\langle \xi_1 | \xi_2 \rangle \, (\gamma') := \sum_{n \in (\alpha^\gamma)^{-1}(\gamma')} \langle \xi_1 | \varrho(n) \cdot \xi_2 \rangle_{L^2(\mathbb{R})} \,,$$

for $\xi_1, \xi_2 \in C_c^\infty(\mathbb{R})$ and $\gamma' \in \Gamma$, and where $\pi_o' : \mathbb{C} \longrightarrow \mathcal{L}_{C_r^*\Gamma}(\mathcal{E}')$ is the unit.

10.2. Topological definition of $\beta_1^{(t)}$

We begin by constructing a homomorphism $\beta_1^{(t)} : \Gamma^{\mathrm{ab}} \longrightarrow K_1(B\Gamma)$ in such a way that $v_1^\Gamma \circ \beta_1^{(t)} = \beta_1^{(a)}$. This was previously done by Natsume [34], under the extra assumption that Γ is a torsion-free group. Our definition of $\beta_1^{(t)}$ will be in two steps: first, we define a (set-theoretic !) map $\bar{\beta}_1^{(t)} : \Gamma \longrightarrow K_1(B\Gamma)$; next, we prove that $\bar{\beta}_1^{(t)}$ is a group homomorphism. Since the target group is abelian, this will imply that $\bar{\beta}_1^{(t)}$ factors through the desired homomorphism $\beta_1^{(t)}$.

To define $\bar{\beta}_1^{(t)}$, we notice that since $\pi_1(B\Gamma) = \Gamma$, every element $\gamma \in \Gamma$ defines (up to homotopy) a pointed continuous map $\gamma : S^1 \longrightarrow B\Gamma$. Keeping notation as in Lemma 7.5, we let

$$[S^1]_K := (\mathrm{ch}_{\mathrm{odd}}^{\mathbb{Z}})^{-1}[S^1] \; \in \; K_1(S^1)$$

be the (unique) K-homology class with integral Chern character given by the fundamental class (the usual orientation, and even Spin^c-structure, is fixed on S^1). Letting $D := \frac{1}{i} \cdot \frac{d}{d\theta}$ be the Dirac operator, see Section 8, by Lemma 7.6, we have

$$[S^1]_K = -[D] = [-D] = [i \cdot \tfrac{d}{d\theta}] \; \in \; K_1(S^1).$$

By functoriality, we get a homomorphism $\gamma_* : K_1(S^1) \longrightarrow K_1(B\Gamma)$ and we set $\bar{\beta}_1^{(t)}(\gamma) := \gamma_*[S^1]_K$, for $\gamma \in \Gamma$, so that

$$\beta_1^{(t)} : \Gamma^{\mathrm{ab}} \longrightarrow K_1(B\Gamma), \quad \gamma^{\mathrm{ab}} \longmapsto \gamma_*[S^1]_K = -\gamma_*[D].$$

10.3. Analytical definition of $\beta_1^{(t)}$

We describe the map $\beta_1^{(t)} : \Gamma^{\mathrm{ab}} \longrightarrow K_1(B\Gamma)$ analytically, using the unbounded picture for K-homology, see [2]. The element $[S^1]_K$ is then described as the unbounded Fredholm module (see Section 8.1)

$$[S^1]_K = [-D] = [L^2(S^1), \mathcal{M}, -D] \; \in \; KK_1(C(S^1), \mathbb{C}),$$

where $\mathcal{M}$ is the $*$-representation of $C(S^1)$ on $L^2(S^1)$ by pointwise multiplication.

If $\gamma \in \Gamma$ corresponds to a map $\gamma: S^1 \longrightarrow B\Gamma$, and if X is an arbitrary compact subspace of $B\Gamma$ containing $\gamma(S^1)$, as for example $\gamma(S^1)$ itself, then $\beta_1^{(t)}(\gamma^{ab})$ is described by the image of the Fredholm module

$$\gamma_*[L^2(S^1), \mathcal{M}, -D] \in KK_1(C(X), \mathbb{C})$$

(where γ is viewed as a map from S^1 to X) under the homomorphism

$$KK_1(C(X), \mathbb{C}) \longrightarrow RKK_1(B\Gamma, \mathbb{C})$$

induced by the inclusion (recall that $RKK_1(B\Gamma, \mathbb{C})$ is by definition the colimit, over the compact subspaces Y of $B\Gamma$, of the abelian groups $KK_1(C(Y), \mathbb{C})$). Assume moreover that the map $\gamma: S^1 \longrightarrow X$ is Lipschitz; up to homotopy, one can always make this assumption on the map γ (with $\gamma(S^1)$ as suitable X). Then, letting $\gamma^*: C(X) \longrightarrow C(S^1)$ take a function f to $f \circ \gamma$, we see that the $*$-closed subalgebra $\mathrm{Lip}(X)$ of $C(X)$ is dense and $\gamma^* \mathrm{Lip}(X)$ verifies $\gamma^* \mathrm{Lip}(X) \subseteq \mathrm{Lip}(S^1)$ and consists therefore of functions that are differentiable almost everywhere by Rademacher's Theorem (see [43, Theorem 11 A]), so that

$$\gamma_*[L^2(S^1), \mathcal{M}, -D] = [L^2(S^1), \mathcal{M} \circ \gamma^*, -D] \in KK_1(C(X), \mathbb{C}).$$

10.4. Properties of $\beta_1^{(t)}$

Theorem 10.1. *The map $\bar{\beta}_1^{(t)}: \Gamma \longrightarrow K_1(B\Gamma)$ is a group homomorphism. Consequently, the map*

$$\beta_1^{(t)}: \Gamma^{ab} \longrightarrow K_1(B\Gamma), \quad \gamma^{ab} \longmapsto \gamma_*[S^1]_K = -\gamma_*[D]$$

is a well-defined group homomorphism.

This will be proved in Section 10.5 below. Before the proof, assuming Theorem 10.1 for a while, we deduce some consequences.

Remark 10.2. We claim that $\varphi_1^\Gamma \circ \bar{\beta}_1^{(t)}$ *is zero on torsion elements of* Γ, where, recall, φ_1^Γ denotes the canonical map $K_1(B\Gamma) \longrightarrow K_1^\Gamma(\underline{E\Gamma})$. Indeed, if $\gamma \in \Gamma$ has order $n \geqslant 1$, the map $\gamma_*: K_1(S^1) \longrightarrow K_1(B\Gamma)$ factorizes as

$$K_1(S^1) \xrightarrow{\quad \gamma_* \quad} K_1(B\Gamma)$$

$$K_1(B\mathbb{Z}/n)$$

with arrows labeled $B\mathrm{incl}_*$

where incl: $\mathbb{Z}/n \hookrightarrow \Gamma$ takes 1 to γ. On the other hand, the diagram

$$
\begin{array}{ccc}
K_1(B\mathbb{Z}/n) & \xrightarrow{B\text{incl}_*} & K_1(B\Gamma) \\
\varphi_1^{\mathbb{Z}/n} \downarrow & & \downarrow \varphi_1^{\Gamma} \\
K_1^{\mathbb{Z}/n}(\underline{E}\mathbb{Z}/n) & \xrightarrow{E\text{incl}_*} & K_1^{\Gamma}(\underline{E}\Gamma)
\end{array}
$$

commutes. However, one can take $\underline{E}\mathbb{Z}/n = pt$, so that $K_1^{\mathbb{Z}/n}(\underline{E}\mathbb{Z}/n) = 0$. Our claim follows. This observation is elaborated on in [30].

Proposition 10.3. *Let* $\text{ch}_{\text{odd}}: K_1(B\Gamma) \longrightarrow \bigoplus_{n=1}^{\infty} H_{2n+1}(\Gamma; \mathbb{Q})$ *be the odd Chern character. Then* $(\text{ch}_{\text{odd}} \otimes \text{id}_{\mathbb{Q}}) \circ (\beta_1^{(t)} \otimes \text{id}_{\mathbb{Q}}) = \text{id}_{H_1(\Gamma; \mathbb{Q})}$ *holds, in particular,* $\beta_1^{(t)}$ *is rationally injective.*

Proof. Fix an element $\gamma \in \Gamma$, and denote by $\hat{\alpha}^{\gamma}: \mathbb{Z} \longrightarrow \Gamma$ the homomorphism taking 1 to γ. Note that the pointed continuous map $\gamma: B\mathbb{Z} = S^1 \longrightarrow B\Gamma$ considered earlier is merely $B\hat{\alpha}^{\gamma}$. Due to the naturality of the Chern character in K-homology, we have a commutative diagram

$$
\begin{array}{ccc}
K_1(S^1) \otimes \mathbb{Q} & \xrightarrow{\hat{\alpha}^{\gamma}_*} & K_1(B\Gamma) \otimes \mathbb{Q} \\
\text{ch}_{\text{odd}} \otimes \text{id}_{\mathbb{Q}} \downarrow \cong & & \cong \downarrow \text{ch}_{\text{odd}} \otimes \text{id}_{\mathbb{Q}} \\
H_1(S^1; \mathbb{Q}) & \xrightarrow{\hat{\alpha}^{\gamma}_*} & H_{\text{odd}}(B\Gamma; \mathbb{Q})
\end{array}
$$

Then, dropping "$\otimes \text{id}_{\mathbb{Q}}$" and "$\otimes 1$" from the notation, we compute

$$
\text{ch}_{\text{odd}}\, \beta_1^{(t)}(\gamma^{\text{ab}}) = \text{ch}_{\text{odd}}\, \hat{\alpha}^{\gamma}_*[S^1]_K = \hat{\alpha}^{\gamma}_*\, \text{ch}_{\text{odd}}[S^1]_K = \hat{\alpha}^{\gamma}_*[S^1] = \gamma^{\text{ab}},
$$

where we have used the fact that for S^1, the usual Chern character takes $[S^1]_K$ to the fundamental class $[S^1]$ in rational homology (see Lemma 7.6). $\square$

Theorem 10.4. *The equality* $\beta_1^{(a)} = v_1^{\Gamma} \circ \beta_1^{(t)}$ *holds.*

Proof. Clearly it is enough to prove that $\bar{\beta}_1^{(a)} = v_1^{\Gamma} \bar{\beta}_1^{(t)}$. As in the proof of Proposition 10.3, we fix $\gamma \in \Gamma$ and write $\hat{\alpha}^{\gamma}: \mathbb{Z} \longrightarrow \Gamma$ for the corresponding homomorphism.

Consider the diagram

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\;\;\hat{\alpha}^{\gamma}\;\;} & \Gamma
\end{array}
$$
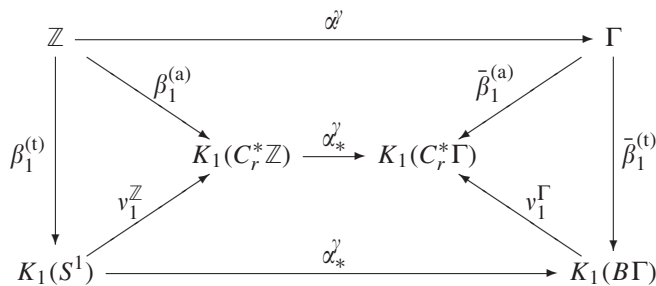
We have $\alpha_*^{\gamma} \beta_1^{(a)} = \bar{\beta}_1^{(a)} \hat{\alpha}^{\gamma}$ by obvious reasons, $\bar{\beta}_1^{(t)} \hat{\alpha}^{\gamma} = \alpha_*^{\gamma} \beta_1^{(t)}$ by definition of $\bar{\beta}_1^{(t)}$, and $\alpha_*^{\gamma} v_1^{\mathbb{Z}} = v_1^{\Gamma} \alpha_*^{\gamma}$ by naturality of the Novikov assembly map when the source group is K-amenable (see [33, Corollary 1.3 in Part 2]). By diagram chasing, one sees that the desired equality $\bar{\beta}_1^{(a)} = v_1^{\Gamma} \bar{\beta}_1^{(t)}$ follows from the analogous result for $\mathbb{Z}$, namely from $\beta_1^{(a)} = v_1^{\mathbb{Z}} \beta_1^{(t)}$, which in turn is a consequence of the well-known fact that the Baum–Connes conjecture holds for the group $\mathbb{Z}$ (see [21, 12.5.9], [33, Section 4 in Part 2] or [42, Example 6.1.6] for a direct proof). $\quad\square$

We have already mentioned in Section 3 that $\beta_1^{(a)} : \Gamma^{ab} \longrightarrow K_1(C_r^* \Gamma)$ is rationally injective (see [8,17]).

10.5. Proof of Theorem 10.1

We treat the topological and the analytical settings together. Consider two elements $\gamma_1, \gamma_2 \in \Gamma$, viewed as (homotopy classes of) pointed continuous maps $S^1 \longrightarrow B\Gamma$. By definition of K-homology with compact supports and of RKK-groups, both $K_1(B\Gamma)$ and $RKK_1(B\Gamma, \mathbb{C})$ are defined as the colimit of $K_1(Y)$ and $KK_1(C(Y), \mathbb{C})$, respectively, with Y running over the compact subspaces of $B\Gamma$. Letting $X := \gamma_1(S^1) \cup \gamma_2(S^1)$, a compact subspace of $B\Gamma$, it is therefore enough to check that the equality

$$(\gamma_1 \gamma_2)_* [S^1]_K = (\gamma_1)_* [S^1]_K + (\gamma_2)_* [S^1]_K$$

holds in $K_1(X)$ and $KK_1(C(X), \mathbb{C})$ respectively, where $\gamma_1 \gamma_2$ stands for the product-loop. Up to homotopy, we may assume that γ_1 and γ_2 are constant on a neighbourhood of the base-point of S^1. The key-point that allows to connect the present situation with what has been done so far, is that the product-loop is nothing but the composition of maps

$$\gamma_1 \gamma_2 = \gamma_1 \# \gamma_2 : \underbrace{S^1 \# S^1}_{=S^1} \xrightarrow{p} S^1 \vee S^1 \xrightarrow{\gamma_1 \vee \gamma_2} X,$$

where we borrow the notation from Proposition 7.7, and where we identify $S^1 \# S^1$ with S^1, as indicated. Bearing in mind the equality $[S^1]_K = -[D]$, what has to be proved is that

$$(\gamma_1 \# \gamma_2)_*[D] = (\gamma_1)_*[D] + (\gamma_2)_*[D],$$

which is precisely the "addition formula" for the Dirac operator of Theorem 2.1. This proves Theorem 10.1 both from the topological and from the analytical viewpoint on $\beta_1^{(t)}$.   □

Remark 10.5. We have spent some time on the analytical proof, because it illustrates a difficulty that, apparently, went unnoticed so far. A detailed and explicit treatment of this difficulty is in fact one of the central themes in these notes. We also point out that the second named author provides in [30] an abstract proof of Theorem 10.1, which is of purely homotopical nature.

11. The second homology of a group and the Baum–Connes map

The present section is subdivided into six subsections and presents the program of Section 3 for the case $j = 2$.

11.1. Notation and Zimmermann's result

Let Σ_g be a closed oriented Riemann surface of genus $g \geqslant 1$, and let $\Gamma_g = \pi_1(\Sigma_g)$ be its fundamental group; Γ_g admits the well-known presentation with $2g$ generators and one relation

$$\Gamma_g = \left\langle a_1, b_1, \dots, a_g, b_g \; \middle| \; \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle.$$

The free group $\mathbb{F}_g$ of rank g is isomorphic to the quotient of Γ_g by the normal subgroup generated by the $a_i b_i^{-1}$'s $(1 \leqslant i \leqslant g)$. It follows that every finitely generated group Γ is a quotient of some Γ_g with g big enough.

This remark was exploited by Zimmermann in [45] to give, for Γ finitely generated, a description of $H_2(\Gamma; \mathbb{Z})$ in terms of pointed continuous maps $\Sigma_g \longrightarrow B\Gamma$ inducing epimorphisms on fundamental groups. We would like to avoid this assumption on Γ. It turns out that all the results and their proofs in Zimmermann's article [45] are valid if one suppresses the surjectivity assumption everywhere. Let us now explain the statements one obtains this way. Denote by $S(\Sigma_g, B\Gamma)$ the set of pointed continuous maps from Σ_g to $B\Gamma$ (*not necessarily* inducing epimorphisms on fundamental groups). Two maps $f_1, f_2 \in S(\Sigma_g, B\Gamma)$ are called *equivalent* if there exists some orientation-preserving pointed homeomorphism h of Σ_g such that f_2 is homotopic to $f_1 \circ h$.

Two maps $f_1 \in S(\Sigma_{g_1}, B\Gamma)$ and $f_2 \in S(\Sigma_{g_2}, B\Gamma)$ are *stably equivalent* if there exists closed oriented Riemann surfaces Σ' and Σ'' such that f_1 and f_2 become equivalent after being extended homotopically trivially to the connected sums $\Sigma_{g_1} \# \Sigma'$ and $\Sigma_{g_2} \# \Sigma''$. More precisely, denoting by y_0 the base-point of $B\Gamma$, we require the applications $f_1 \# y_0$ on $\Sigma_{g_1} \# \Sigma'$ and $f_2 \# y_0$ on $\Sigma_{g_2} \# \Sigma''$ to be equivalent.

Denote by $\Omega(\Gamma)$ the set of stable equivalence classes in $\coprod_{g \geqslant 1} S(\Sigma_g, B\Gamma)$, and by $[f]$ the equivalence class of $f \in S(\Sigma_g, B\Gamma)$. Denote by $[\Sigma_g] \in H_2(\Sigma_g; \mathbb{Z})$ the fundamental class of Σ_g. The following result of Zimmermann [45] will be crucial.

Theorem 11.1 (*Zimmermann*). *For an arbitrary discrete group Γ, the map*

$$Z_\Gamma : \Omega(\Gamma) \longrightarrow H_2(\Gamma; \mathbb{Z}), \quad [f] \longrightarrow f_*[\Sigma_g] \qquad (\textit{for } f \in S(\Gamma_g, B\Gamma))$$

is a well-defined bijection (*here, f_* denotes $H_2(f; \mathbb{Z})$*).

Transferring the group structure of $H_2(\Gamma; \mathbb{Z})$ to $\Omega(\Gamma)$ via this bijection Z_Γ, we get a group structure on $\Omega(\Gamma)$ such that

(1) addition corresponds to connected sum (see Remark 11.2 below);
(2) the zero element is for example given by the class of the constant map in $S(\Sigma_g, B\Gamma)$ (with $g \geqslant 1$ arbitrary);
(3) if $f \in S(\Sigma_g, B\Gamma)$ is such that the homomorphism $\pi_1(f) : \Gamma_g \longrightarrow \Gamma$ factorizes through a free group, then $[f]$ is the zero element;
(4) for $f \in S(\Sigma_g, B\Gamma)$, the opposite of $[f]$ is given by $[f \circ h_-]$, where h_- is an orientation-reversing pointed homeomorphism of Σ_g.

From now on, we shall implicitly identify $H_2(\Gamma; \mathbb{Z})$ with $\Omega(\Gamma)$ by the map Z_Γ, which has become a group isomorphism.

Remark 11.2. Let Γ be a group. Consider $f_1 \in S(\Sigma_{g_1}, B\Gamma)$ and $f_2 \in S(\Sigma_{g_2}, B\Gamma)$, and their classes in $\Omega(\Gamma)$. Up to stable equivalence and up to homotopy, we can suppose that f_1 and f_2 are constant on a handle of Σ_{g_1} and Σ_{g_2} respectively (and therefore also on a small disk). Then, according to Example 7.2 and to Proposition 7.4, the class of $f_1 + f_2$ in $\Omega(B\Gamma)$ is represented by the following two maps:

$$f_1 \# f_2 \in S(\Sigma_{g_1+g_2}, B\Gamma) \qquad \text{and} \qquad f_1 \natural f_2 \in S(\Sigma_{g_1+g_2-1}, B\Gamma),$$

where we identify $\Sigma_{g_1} \# \Sigma_{g_2}$ with $\Sigma_{g_1+g_2}$, and $\Sigma_{g_1} \natural \Sigma_{g_2}$ with $\Sigma_{g_1+g_2-1}$, as usual.

Note that in the whole subsection, we can replace the particular connected CW-complex $B\Gamma$ by an arbitrary connected CW-complex X.

11.2. Topological definition of $\beta_2^{(t)}$

Keeping notation as in Lemma 7.5, we let

$$[\Sigma_g]_K := (\mathrm{ch}_{\mathrm{ev}}^{\mathbb{Z}})^{-1}[\Sigma_g] \in K_0(\Sigma_g)$$

be the (unique) K-homology class with integral Chern character given by the fundamental class (an orientation, and even an auxiliary Kähler structure, is fixed on Σ_g). For $f \in S(\Sigma_g, B\Gamma)$, we denote by $f_*: K_0(\Sigma_g) \longrightarrow K_0(B\Gamma)$ the induced map in K-homology. Now, we set

$$\beta_2^{(t)}: H_2(\Gamma; \mathbb{Z}) \longrightarrow K_0(B\Gamma), \quad [f] \longmapsto f_*[\Sigma_g]_K \qquad (\text{for } f \in S(\Gamma_g, B\Gamma)).$$

It is not at all obvious that $\beta_2^{(t)}$ is well-defined, and that it is a group homomorphism; this will be stated as Theorem 11.4 below.

11.3. Analytical definition of $\beta_2^{(t)}$

Bearing in mind the analytical definition of K-homology, it is interesting to express $[\Sigma_g]_K \in K_0(\Sigma_g)$ in this setting. This is precisely the subject of the next lemma, which follows directly from Lemma 7.9.

Lemma 11.3. *One has* $[\Sigma_g]_K = [\bar{\partial}_g] + (g-1) \cdot \iota_*^{\Sigma_g}[1]$, *where* $\iota^{\Sigma_g}: pt \longrightarrow \Sigma_g$ *is the inclusion of the base-point, and* [1] *is the canonical generator of* $K_0(pt) \cong \mathbb{Z}$.

Let X be a compact subspace of $B\Gamma$ such that $f(\Sigma_g) \subseteq X$, as for example $f(\Sigma_g)$ itself. Now, the K-homology generator $\iota_*^{\Sigma_g}[1]$ is given by the Fredholm module

$$\iota_*^{\Sigma_g}[1] = [\mathbb{C}, \mathrm{ev}_{\Sigma_g}, 0] \in KK_0(C(\Sigma_g), \mathbb{C}),$$

where $\mathrm{ev}_{\Sigma_g}: C(\Sigma_g) \longrightarrow \mathbb{C}$ is evaluation at the base-point of the surface Σ_g. Fix a map $f \in S(\Gamma_g, B\Gamma)$. In the analytic framework, $\beta_2^{(t)}[f]$ is the image of the element

$$f_*\left[L^2(\Lambda^{0,*}T^*\Sigma_g), \mathcal{M}, \bar{\partial}_{\Sigma_g}\right] + (g-1) \cdot [\mathbb{C}, \mathrm{ev}_X, 0]$$

(where f is viewed as a map from Σ_g to X) under the homomorphism

$$KK_0(C(X), \mathbb{C}) \longrightarrow RKK_0(B\Gamma, \mathbb{C})$$

induced by the inclusion of X in $B\Gamma$, where ev_X is evaluation at the base-point of X. Suppose f is Lipschitz; up to homotopy, one can always assume this is the case, with

$f(\Sigma_g)$ as suitable X. Then, letting $f^* : C(X) \longrightarrow C(\Sigma_g)$ take a function ϑ to $\vartheta \circ f$, we see that the $*$-closed subalgebra $\mathrm{Lip}(X)$ of $C(X)$ is dense and that $f^* \mathrm{Lip}(X)$ verifies $f^* \mathrm{Lip}(X) \subseteq \mathrm{Lip}(\Sigma_g)$ and consists therefore of functions that are differentiable almost everywhere by Rademacher's Theorem again (see [43, Theorem 11 A]); as a consequence,

$$f_* \left[L^2(\Lambda^{0,*} T^* \Sigma_g), \mathcal{M}, \bar{\partial}_{\Sigma_g} \right] = \left[L^2(\Lambda^{0,*} T^* \Sigma_g), \mathcal{M} \circ f^*, \bar{\partial}_{\Sigma_g} \right].$$

11.4. Properties of $\beta_2^{(t)}$

Theorem 11.4. *The following map is a well-defined group homomorphism*:

$$\beta_2^{(t)} : H_2(\Gamma; \mathbb{Z}) \longrightarrow K_0(B\Gamma), \quad [f] \longmapsto f_*[\bar{\partial}_g] + (g-1) \cdot \iota_*^{BG}[1],$$

for $f \in S(\Sigma_g, B\Gamma)$, where $\iota^{B\Gamma}$ stands for the inclusion of the base-point of $B\Gamma$.

We postpone the proof to Section 11.5 below, and derive, here, some of its consequences. We also point out that [31] contains a purely homotopical proof of the theorem.

Proposition 11.5. *Let* $\mathrm{ch}_{\mathrm{ev}} : K_0(B\Gamma) \longrightarrow \bigoplus_{n=0}^{\infty} H_{2n}(\Gamma; \mathbb{Q})$ *be the even Chern character. Then, one has* $(\mathrm{ch}_{\mathrm{ev}} \otimes \mathrm{id}_{\mathbb{Q}}) \circ (\beta_2^{(t)} \otimes \mathrm{id}_{\mathbb{Q}}) = \mathrm{id}_{H_2(\Gamma; \mathbb{Q})}.$

Proof. Let $[f] \in H_2(\Gamma; \mathbb{Z})$ be represented by $f \in S(\Sigma_g, B\Gamma)$. By naturality of the Chern character, we have a commutative diagram

$$
\begin{array}{ccc}
K_0(\Sigma_g) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\ f_* \ } & K_0(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \\
{\scriptstyle \mathrm{ch}_{\mathrm{ev}} \otimes \mathrm{id}_{\mathbb{Q}}} \Big\downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \Big\downarrow {\scriptstyle \mathrm{ch}_{\mathrm{ev}} \otimes \mathrm{id}_{\mathbb{Q}}} \\
H_{\mathrm{ev}}(\Sigma_g; \mathbb{Q}) & \xrightarrow{\ f_* \ } & H_{\mathrm{ev}}(B\Gamma; \mathbb{Q})
\end{array}
$$

Then, dropping "$\otimes \mathrm{id}_{\mathbb{Q}}$" and "$\otimes 1$" from the notation, one computes

$$\mathrm{ch}_{\mathrm{ev}} \beta_2^{(t)}[f] = \mathrm{ch}_{\mathrm{ev}} f_*[\Sigma_g]_K = f_* \mathrm{ch}_{\mathrm{ev}}[\Sigma_g]_K = f_*[\Sigma_g] = [f],$$

where the last equality follows from the identification given by Theorem 11.1. $\square$

11.5. Proof of Theorem 11.4

We first show that $\beta_2^{(t)}$ is well-defined. We then prove it is a homomorphism.
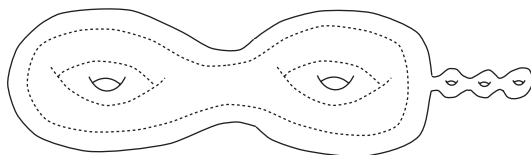
Fig. 3.

We start with the topological setting. Fix $f_1 \in S(\Sigma_{g_1}, B\Gamma)$ and $f_2 \in S(\Sigma_{g_2}, B\Gamma)$. We first show that if $g_1 > g_2$ and if $f_1 = f_2 \# y_0$, then $(f_1)_*[\Sigma_{g_1}] = (f_2)_*[\Sigma_{g_2}]$ in the group $H_2(B\Gamma; \mathbb{Z})$. To do this, we embed Σ_{g_1} and Σ_{g_2} in $\mathbb{R}^3$ in such a way that Σ_{g_1} is contained in a tubular neighbourhood V of Σ_{g_2} (see Fig. 3).

Identifying V with the total space of the normal bundle of Σ_{g_2} yields a projection map $q \colon V \longrightarrow \Sigma_{g_2}$. Clearly, the restriction $q|_{\Sigma_{g_1}} \colon \Sigma_{g_1} \longrightarrow \Sigma_{g_2}$ is a smooth, proper and orientation preserving map; considering the "first handle" (on the left in Fig. 3) of Σ_{g_1} and of Σ_{g_2} (where $q|_{\Sigma_{g_1}}$ is one-to-one and regular), we see that it is of degree one, so that $(q|_{\Sigma_{g_1}})_*[\Sigma_{g_1}] = [\Sigma_{g_2}]$. By naturality and injectivity of the integral Chern character on the K-homology of closed oriented Riemann surfaces (see Lemma 7.5), we deduce that $(q|_{\Sigma_{g_1}})_*[\Sigma_{g_1}]_K = [\Sigma_{g_2}]_K$ in $K_0(\Sigma_{g_2})$. On the other hand, it is clear that the map $f_1 = f_2 \# y_0$ is homotopic to $f_2 \circ q|_{\Sigma_{g_1}}$, hence

$$(f_1)_*[\Sigma_{g_1}]_K = (f_2 \circ q|_{\Sigma_{g_1}})_*[\Sigma_{g_1}]_K = (f_2)_* \circ (q|_{\Sigma_{g_1}})_*[\Sigma_{g_1}]_K = (f_2)_*[\Sigma_{g_2}]_K.$$

It remains to check that, if two maps $f_1, f_2 \in S(\Sigma_g, B\Gamma)$ are equivalent, then $(f_1)_*[\Sigma_g]_K = (f_2)_*[\Sigma_g]_K$ in $K_0(B\Gamma)$. This follows from the fact that orientation-preserving homeomorphisms of Σ_g induce the identity on $K_0(\Sigma_g)$ (again, this can be checked using the integral Chern character and Lemma 7.5). This shows that $\beta_2^{(t)}$ is a well-defined map.

Now, we prove that $\beta_2^{(t)}$ is a group homomorphism still in the topological setting. We fix $f_1 \in S(\Sigma_{g_1}, B\Gamma)$ and $f_2 \in S(\Sigma_{g_2}, B\Gamma)$. Using the first description of the sum in Remark 11.2, we must show that

$$(f_1 \# f_2)_*[\Sigma_{g_1+g_2}]_K = (f_1)_*[\Sigma_{g_1}]_K + (f_2)_*[\Sigma_{g_2}]_K$$

holds in $K_0(B\Gamma)$. We can now exploit Lemma 7.5 to reduce the proof to showing the homological equality

$$(f_1 \# f_2)_*[\Sigma_{g_1+g_2}] = (f_1)_*[\Sigma_{g_1}] + (f_2)_*[\Sigma_{g_2}]$$

in $H_2(B\Gamma; \mathbb{Z})$. This is a special case of Example 7.2 (which was based on the general principle 7.1). This completes the proof in the topological setting. $\quad\square$

We move now to the analytical framework and present the corresponding proof of Theorem 11.4. We first have to show that the map

$$\beta_2^{(t)}: H_2(\Gamma; \mathbb{Z}) \longrightarrow K_0(B\Gamma), \quad [f] \longmapsto f_*[\bar{\partial}_g] + (g-1) \cdot \iota_*^{BG}[1],$$

for $f \in S(\Sigma_g, B\Gamma)$, is well-defined. The proof is subdivided into six steps.

(1) If $q_0: \Sigma_g \longrightarrow pt$ denotes the constant map, then $(q_0)_*[\bar{\partial}_g] = (1-g) \cdot [1]$ holds in $K_0(pt) \cong \mathbb{Z}$. Indeed, the operator $\bar{\partial}_g$ has $1-g$ as index, see [39, p. 27].

(2) The group $K_0(\Sigma_{g_1} \vee \Sigma_{g_2})$ is isomorphic to $\mathbb{Z}^3$ with the elements $\iota_*^{\Sigma_{g_1} \vee \Sigma_{g_2}}[1]$, $[\bar{\partial}_{g_1}]$ and $[\bar{\partial}_{g_2}]$ as generators (using the obvious identifications), where $\iota^{\Sigma_{g_1} \vee \Sigma_{g_2}}$ stands for the inclusion of the base-point, see Lemmas 7.5 and 7.9.

(3) Let x_0 be the base-point of Σ_{g_1} and consider the "crunching" map

$$q := \mathrm{id}_{\Sigma_{g_1}} \vee x_0: \Sigma_{g_1} \vee \Sigma_{g_2} \longrightarrow \Sigma_{g_1}, \quad x \longmapsto \begin{cases} x & \text{if } x \in \Sigma_{g_1}, \\ x_0 & \text{if } x \in \Sigma_{g_2}. \end{cases}$$

Then, under the identifications of (2), $q_*[\bar{\partial}_{g_1}] = [\bar{\partial}_{g_1}]$ and $q_*[\bar{\partial}_{g_2}] = (1-g_2) \cdot \iota_*^{\Sigma_{g_1}}[1]$ hold in the group $K_0(\Sigma_{g_1})$, as follows from (1) for the latter equality.

(4) Let $p: \Sigma_{g_1} \# \Sigma_{g_2} \longrightarrow \Sigma_{g_1} \vee \Sigma_{g_2}$ be the pinching map that "contracts" the identification circle in $\Sigma_{g_1} \# \Sigma_{g_2}$ to the base-point of $\Sigma_{g_1} \vee \Sigma_{g_2}$. Then, under the identifications of (2), the following equality holds:

$$p_*[\bar{\partial}_{g_1+g_2}] = [\bar{\partial}_{g_1}] + [\bar{\partial}_{g_2}] - \iota_*^{\Sigma_{g_1} \vee \Sigma_{g_2}}[1] \in K_0(\Sigma_{g_1} \vee \Sigma_{g_2}).$$

This equality is the "tricky" part of the present proof (and it is precisely here that the proof becomes of analytical nature properly speaking—of course, this can also be directly established in the topological framework, using the integral Chern character of Lemmas 7.5 and 7.9, thus yielding a second topological proof of the well-definiteness). Let K be a small closed neighbourhood of the base-point x_0 in $\Sigma_{g_1} \vee \Sigma_{g_2}$ (K is contractible), and let $K' := p^{-1}(K)$ be the corresponding closed tubular neighbourhood of the identification circle $p^{-1}(x_0)$ in $\Sigma_{g_1} \# \Sigma_{g_2}$ (K is homotopy equivalent to S^1). Let U and U' be the complements of K and K' in $\Sigma_{g_1} \vee \Sigma_{g_2}$ and $\Sigma_{g_1} \# \Sigma_{g_2}$, respectively. We can assume that the map $p|_{U'}: U' \longrightarrow U$ is an isometry. The short exact sequences of C^*-algebras

$$0 \longrightarrow C_0(U') \xrightarrow{i} C(\Sigma_{g_1} \# \Sigma_{g_2}) \xrightarrow{r} C(K') \longrightarrow 0$$

and

$$0 \longrightarrow C_0(U) \xrightarrow{j} C(\Sigma_{g_1} \vee \Sigma_{g_2}) \xrightarrow{s} C(K) \longrightarrow 0$$

give rise to the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(K') & \xrightarrow{\ i_*\ } & K_0(\Sigma_{g_1}\#\Sigma_{g_2}) & \xrightarrow{\ r_*\ } & K_0(U') & & \\
& & \cong \Big\downarrow (p|_{K'})_* & & \Big\downarrow p_* & & \cong \Big\downarrow (p|_{U'})_* & & \\
0 & \longrightarrow & K_0(K) & \xrightarrow{\ j_*\ } & K_0(\Sigma_{g_1}\vee\Sigma_{g_2}) & \xrightarrow{\ s_*\ } & K_0(U) & \longrightarrow & 0
\end{array}
$$

Note that $K_0(K)$ and $K_0(K')$ are both isomorphic to $K_0(pt)\cong\mathbb{Z}$, and it is for this reason that $(p|_{K'})_*$ is an isomorphism and that both i_* and j_* are injective. Now, since, for each g, $\bar\partial_g$ is a symmetric elliptic operator on a Riemannian manifold, Proposition [21, Proposition 10.8.8] (which is of purely analytical nature) can be applied, and we have

$$
\begin{aligned}
s_* \circ p_*[\bar\partial_{g_1+g_2}] &= p_* \circ i_*[\bar\partial_{g_1+g_2}] && \text{(by commutativity of the diagram)}\\
&= p_*[\bar\partial_{g_1+g_2}|_{U'}] && \text{(by [21, Proposition 10.8.8])}\\
&= p_*[\bar\partial_{g_1}|_{U'}] + p_*[\bar\partial_{g_2}|_{U'}] && \text{(by the local description of } \bar\partial_g)\\
&= [\bar\partial_{g_1}|_U] + [\bar\partial_{g_2}|_U] && \text{(since } p|_{U'} \text{ is an isometry)}\\
&= s_*[\bar\partial_{g_1}] + s_*[\bar\partial_{g_2}] && \text{(by [21, Proposition 10.8.8]).}
\end{aligned}
$$

Therefore, it follows that

$$
p_*[\bar\partial_{g_1+g_2}] - [\bar\partial_{g_1}] - [\bar\partial_{g_2}] \in \mathrm{Ker}(s_*) = \mathrm{Im}(j_*) = \mathbb{Z}\cdot\iota^{\Sigma_{g_1}\vee\Sigma_{g_2}}[1].
$$

The determination of the corresponding integer λ (which we have to show is -1) amounts to the determination of the indices, namely

$$
\begin{aligned}
\lambda &= \mathrm{Index}(\bar\partial_{g_1+g_2}) - \mathrm{Index}(\bar\partial_{g_1}) - \mathrm{Index}(\bar\partial_{g_2}) = (1 - g_1 - g_2) - (1 - g_1) - (1 - g_2)\\
&= -1,
\end{aligned}
$$

by [39, p. 27], and we are done.

(5) By (3) and (4), using the same notation, one has the following equality:

$$
(q \circ p)_*[\bar\partial_{g_1+g_2}] = [\bar\partial_{g_1}] - g_2\cdot\iota_*^{\Sigma_{g_1}}[1] \ \in \ K_0(\Sigma_{g_1}).
$$

(6) We now really establish the well-definiteness of $\beta_2^{(t)}$. To verify it, we must show that if two maps

$$
f_1\colon \Sigma_{g_1} \longrightarrow B\Gamma \qquad \text{and} \qquad f_2\colon \Sigma_{g_1}\#\Sigma_{g_2} \longrightarrow B\Gamma
$$

are related by the equality $f_2 = f_1 \# y_0$, with y_0 standing for the base-point of $B\Gamma$, then

$$(f_2)_*[\bar{\partial}_{g_1+g_2}] + (g_1 + g_2 - 1) \cdot \iota_*^{B\Gamma}[1] = (f_1)_*[\bar{\partial}_{g_1}] + (g_1 - 1) \cdot \iota_*^{B\Gamma}[1]$$

holds in $K_0(B\Gamma)$. The key observation is that $f_2 = f_1 \circ q \circ p$, so, by virtue of (5),

$$(f_2)_*[\bar{\partial}_{g_1+g_2}] = (f_1)_*[\bar{\partial}_{g_1}] - g_2 \cdot \iota_*^{B\Gamma}[1]$$

and we can conclude.

Finally, we show that $\beta_2^{(t)}$ is a group homomorphism in the analytical setting. Again, we fix $f_1 \in S(\Sigma_{g_1}, B\Gamma)$ and $f_2 \in S(\Sigma_{g_2}, B\Gamma)$. Consider the compact subspace $X := f_1(\Sigma_{g_1}) \cup f_2(\Sigma_{g_2})$ of $B\Gamma$. Using the second description of the sum in Remark 11.2, according to Lemma 11.3, we must show that

$$(f_1 \natural f_2)_* \left((1 - (g_1 + g_2 - 1)) \cdot \iota_*^{\Sigma_{g_1} \natural \Sigma_{g_2}}[1] \right) = (f_1)_* \left((1 - g_1) \cdot \iota^{\Sigma_{g_1}}[1] \right)$$
$$+ (f_2)_* \left((1 - g_2) \cdot \iota_*^{\Sigma_{g_2}}[1] \right)$$

and that

$$(f_1 \natural f_2)_*[\bar{\partial}_{g_1+g_2-1}] = (f_1)_*[\bar{\partial}_{g_2}] + (f_2)_*[\bar{\partial}_{g_2}]$$

in $KK_0(C(X), \mathbb{C})$. For the first equality, it suffices to note that $f_1 \natural f_2$, f_1 and f_2 are pointed maps, so that this reduces to an equality of integers. The second is the content of Theorem 2.2, that we have proved both in the topological and in the analytical settings.

11.6. Definition of the map $\beta_2^{(a)}$ and connection with $\beta_2^{(t)}$

We now construct the map $\beta_2^{(a)} : H_2(\Gamma; \mathbb{Z}) \longrightarrow K_0(C_r^*\Gamma)$. Denote by $C^*\Gamma$ the full C^*-algebra of the group Γ, and by $\lambda_\Gamma : C^*\Gamma \twoheadrightarrow C_r^*\Gamma$ the canonical epimorphism. It is well-known that the Novikov assembly map factors through the K-theory of the full C^*-algebra (see [22] or [33, Section 2.3 in Part 2]), i.e. for $i = 0, 1$, there is a homomorphism

$$\tilde{v}_i^\Gamma : K_i(B\Gamma) \longrightarrow K_i(C^*\Gamma) \quad \text{such that} \quad v_i^\Gamma = (\lambda_\Gamma)_* \circ \tilde{v}_i^\Gamma.$$

For a map $f \in S(\Sigma_g, B\Gamma)$, we denote by the same symbol the associated group homomorphism $\pi_1(f) : \Gamma_g \longrightarrow \Gamma$, and also the corresponding $*$-homomorphism $C^*(\pi_1(f)) :$

$C^*\Gamma_g \longrightarrow C^*\Gamma$ (the latter being well-defined thanks to the universal property of the full C^*-algebra). We define

$$\beta_2^{(a)} \colon H_2(\Gamma; \mathbb{Z}) \longrightarrow K_0(C_r^*\Gamma), \quad [f] \longmapsto (\lambda_\Gamma \circ f)_* \, \tilde{v}_0^{\Gamma_g}[\Sigma_g]_K \quad (\text{for } f \in S(\Sigma_g, B\Gamma)).$$

Theorem 11.6. *The map* $\beta_2^{(a)}$ *is a well-defined group homomorphism satisfying the equality* $\beta_2^{(a)} = v_0^\Gamma \circ \beta_2^{(t)}$.

Proof. For $f \in S(\Sigma_g, B\Gamma)$, we have to show that $v_0^\Gamma \beta_2^{(t)}[f] = (\lambda_\Gamma \circ f)_* \, \tilde{v}_0^{\Gamma_g}[\Sigma_g]_K$ in $K_0(C_r^*\Gamma)$. The result will follow since, by Theorem 11.4, the left-hand side only depends on the class $[f]$ of f in $H_2(\Gamma; \mathbb{Z})$, and moreover $\beta_2^{(t)}$ is a group homomorphism. Now, the map $\tilde{v}_i^\Gamma$ is natural with respect to *arbitrary* group homomorphisms (and not just injective ones, see [33, Theorem 1.1 in Part 2]), so that

$$(\lambda_\Gamma \circ f)_* \, \tilde{v}_0^{\Gamma_g}[\Sigma_g]_K = (\lambda_\Gamma)_* \, f_* \, \tilde{v}_0^{\Gamma_g}[\Sigma_g]_K = (\lambda_\Gamma)_* \, \tilde{v}_0^\Gamma f_*[\Sigma_g]_K$$

$$= v_0^\Gamma f_*[\Sigma_g]_K = v_0^\Gamma \beta_2^{(t)}[f].$$

This completes the proof. $\quad\square$

In the unbounded analytical description of KK-theory in the sense of [2], the 'universal' class $\tilde{v}_0^{\Gamma_g}[\Sigma_g]_K \in K_0(C^*\Gamma_g) = KK_0(\mathbb{C}, C^*\Gamma_g)$ is given by the unbounded Kasparov triple

$$\tilde{v}_0^{\Gamma_g}[\Sigma_g]_K = [\mathcal{E}_g, \bar{\partial}_g] = [\mathcal{E}_g, \pi_\circ, \bar{\partial}_g] \in KK_0(\mathbb{C}, C^*\Gamma_g),$$

where $\mathcal{E}_g$ is defined as we next explain and $\pi_\circ \colon \mathbb{C} \longrightarrow \mathcal{L}_{C^*\Gamma_g}(\mathcal{E}_g)$ is the unit. Letting $\widetilde{\Sigma}_g$ be the universal cover of Σ_g, $\mathcal{E}_g$ is the separation-completion of the algebra $\Gamma_c(\Lambda^{0,*}T^*\widetilde{\Sigma}_g)$ of compactly supported smooth sections of the vector bundle $\Lambda^{0,*}T^*\widetilde{\Sigma}_g$ over $\widetilde{\Sigma}_g$ with respect to the $C^*\Gamma_g$-valued scalar product determined by

$$\langle \xi_1 | \xi_2 \rangle (\sigma) := \langle \xi_1 | \sigma \cdot \xi_2 \rangle_{L^2(\widetilde{\Sigma}_g, \Lambda^{0,*}T^*\widetilde{\Sigma}_g)},$$

for $\xi_1, \xi_2 \in \Gamma_c(\Lambda^{0,*}T^*\widetilde{\Sigma}_g)$ and $\sigma \in \Gamma_g$ (acting on $\Gamma_c(\Lambda^{0,*}T^*\widetilde{\Sigma}_g)$ in the usual way, via deck transformations), compare with D. Kucerovsky's Appendix to [33]. It follows that for $f \in S(\Sigma_g, B\Gamma)$, we have

$$\beta_2^{(a)}[f] = [\mathcal{E}_g', \bar{\partial}_g] = [\mathcal{E}_g', \pi_\circ', \bar{\partial}_g] \in KK_0(\mathbb{C}, C_r^*\Gamma),$$

where $\pi_\circ' : \mathbb{C} \longrightarrow \mathcal{L}_{C_r^*\Gamma}(\mathcal{E}_g')$ is the unit, and $\mathcal{E}_g'$ is the separation-completion of $\Gamma_c(\Lambda^{0,*} T^* \widetilde{\Sigma}_g)$ with respect to the $C_r^*\Gamma$-valued scalar product determined by

$$\langle \xi_1 | \xi_2 \rangle (\gamma) := \sum_{\sigma \in \pi_1(f)^{-1}(\gamma)} \langle \xi_1 | \sigma \cdot \xi_2 \rangle_{L^2(\widetilde{\Sigma}_g, \Lambda^{0,*} T^* \widetilde{\Sigma}_g)},$$

for $\xi_1, \xi_2 \in \Gamma_c(\Lambda^{0,*} T^* \widetilde{\Sigma}_g)$ and $\gamma \in \Gamma$, see [33, Section 3 in Part 2]. This provides a purely analytical description of $\beta_2^{(a)}$. See also [32, Section 3] for information on $\beta_2^{(a)}[f]$ in connection with group homology and algebraic K-theory, described therein via an element $v_2[\Sigma_g, f]$ lying in a suitable quotient of $K_2^{\mathrm{alg}}(\mathbb{Z}\Gamma)$.

12. The case of two-dimensional groups

Recall that we call a group Γ *two-dimensional* if its classifying space has the homotopy type of a CW-complex (not necessarily finite) of dimension $\leqslant 2$.

Examples of two-dimensional groups abound:

(1) *Surface groups*: The Baum–Connes conjecture was proved for those groups by Kasparov [25].
(2) *Torsion-free one-relator groups*: For this class, the Baum–Connes conjecture was established in [6].
(3) *Knot groups*: By [6], they also satisfy the Baum–Connes conjecture.
(4) Groups acting freely co-compactly on a two-dimensional Euclidean building: These groups have Kazhdan's property (T) (see [46] for an elegant proof of this fact). For groups acting on $\tilde{A}_2$-buildings (in particular co-compact torsion-free lattices in $\mathrm{PGL}_3(F)$, with F a local field), the Baum–Connes conjecture is an outstanding result of Lafforgue [27]. For other cases (e.g. co-compact torsion-free lattices in the symplectic group $\mathrm{Sp}_4(F)$, F a local field), the Baum–Connes conjecture is still open. Let us mention however that, in these cases, it is known by work of Kasparov and Skandalis [26] that the Novikov assembly map v_*^Γ is injective.
(5) It was shown by Champetier [11] that there is a certain genericity of two-dimensional groups among finitely presentable groups. Indeed, fix the finite generating set X and the number k of relations. Among groups $\Gamma = \langle X \mid r_1, \ldots, r_k \rangle$ generated by X and on k relations, the proportion of two-dimensional groups goes to 1 as $\max \{|r_1|, \ldots, |r_k|\} \to +\infty$ (see [11, pp. 199–200]); moreover, for $k = 2$, there is genericity in the stronger sense of Gromov, namely, the proportion of two-dimensional groups goes to 1 even as $\min \{|r_1|, |r_2|\} \to +\infty$ (see [11, Theorem 4.13]).
(6) The following result is proved by Wise in [44]. Suppose given an arbitrary finitely presentable group Γ. Then, there exists a compact negatively curved two-dimensional simplicial complex X and a finitely generated normal subgroup N of $\pi_1(X)$ such that $\pi_1(X)/N \cong \Gamma$. Negative curvature implies that X is acyclic and therefore a model for $B\pi_1(X)$; as a consequence, $\pi_1(X)$ is a 2-dimensional group. In particular,

any finitely presentable group is a quotient of some (finitely presentable) two-dimensional group.

What is special about two-dimensional groups in our context comes from the canonical "identification" between the integral homology of the group and the K-homology of its classifying space, see Lemma 7.5.

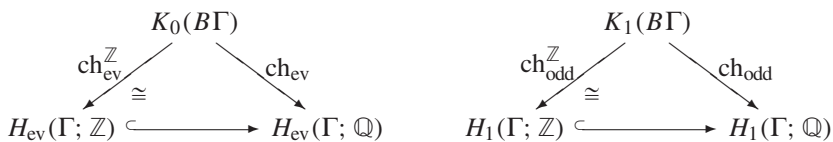**Lemma 12.1.** *Let Γ be a two-dimensional group. Then the maps*

$$\beta_{\mathrm{ev}}^{(\mathrm{t})} \colon \mathbb{Z} \oplus H_2(\Gamma; \mathbb{Z}) \xrightarrow{\cong} K_0(B\Gamma), \quad (m, [f]) \longmapsto m \cdot \iota_*^{B\Gamma}[1] + \beta_2^{(\mathrm{t})}[f]$$

and

$$\beta_1^{(\mathrm{t})} \colon H_1(\Gamma; \mathbb{Z}) \xrightarrow{\cong} K_1(B\Gamma), \quad \gamma^{\mathrm{ab}} \longmapsto \gamma_*[S^1]_K = -\gamma_*[D],$$

are isomorphisms, as indicated.

Proof. Since $B\Gamma$ is at most two-dimensional, we have first that its integral homology is torsion-free (so that it injects into its rational homology), and, second, by Lemma 7.5, we have commutative diagrams

$$
\begin{array}{ccc}
& K_0(B\Gamma) & \\
{}^{\mathrm{ch}_{\mathrm{ev}}^{\mathbb{Z}}}\swarrow & & \searrow^{\mathrm{ch}_{\mathrm{ev}}} \\
H_{\mathrm{ev}}(\Gamma; \mathbb{Z}) \overset{\cong}{\hookrightarrow} & & H_{\mathrm{ev}}(\Gamma; \mathbb{Q})
\end{array}
\qquad
\begin{array}{ccc}
& K_1(B\Gamma) & \\
{}^{\mathrm{ch}_{\mathrm{odd}}^{\mathbb{Z}}}\swarrow & & \searrow^{\mathrm{ch}_{\mathrm{odd}}} \\
H_1(\Gamma; \mathbb{Z}) \overset{\cong}{\hookrightarrow} & & H_1(\Gamma; \mathbb{Q})
\end{array}
$$

By Propositions 10.3 and 11.5, the maps $\beta_1^{(\mathrm{t})}$ and $\beta_2^{(\mathrm{t})}$ are, rationally, right-inverses of the Chern character ch_* in the corresponding degrees. A corresponding result holds for $\beta_0^{(\mathrm{t})}$, see Section 3. By diagram chase, it follows that $\beta_{\mathrm{ev}}^{(\mathrm{t})}$ and $\beta_1^{(\mathrm{t})}$ are isomorphisms, as was to be shown. $\quad\square$

From this lemma and Theorems 10.4 and 11.6, we immediately get the following reformulation of the Baum–Connes conjecture for two-dimensional groups.

Proposition 12.2. *For a two-dimensional group Γ, the Baum–Connes conjecture is equivalent to the following statement: the maps*

$$\beta_{\mathrm{ev}}^{(\mathrm{a})} \colon \mathbb{Z} \oplus H_2(\Gamma; \mathbb{Z}) \longrightarrow K_0(C_r^*\Gamma), \ (m, [f]) \longmapsto m \cdot [1] + \beta_2^{(\mathrm{a})}[f]$$

and

$$\beta_1^{(\mathrm{a})} \colon H_1(\Gamma; \mathbb{Z}) = \Gamma^{\mathrm{ab}} \longrightarrow K_1(C_r^*\Gamma), \quad \gamma^{\mathrm{ab}} \longmapsto [\gamma] = \big[\mathrm{Diag}(\gamma, 1, 1, \ldots)\big],$$

are isomorphisms.

We single out one consequence of *surjectivity* of the Baum–Connes assembly map, consistent with the philosophy that surjectivity implies analytical results.

Corollary 12.3. *Let* Γ *be a two-dimensional group. Suppose that the assembly map* $v_1^{\Gamma} \colon K_1(B\Gamma) \longrightarrow K_1(C_r^*\Gamma)$ *is onto. Then every element of* $\mathrm{GL}_{\infty}(C_r^*\Gamma)$ *lies in the same path-component as a diagonal matrix* $\mathrm{Diag}(\gamma, 1, 1, \ldots)$, *for some* $\gamma \in \Gamma$.

Proof. Since $K_1(C_r^*\Gamma)$ is by definition the group of path-components of $\mathrm{GL}_{\infty}(C_r^*\Gamma)$, the result follows from the previous one together with the very definition of $\beta_1^{(a)}$. $\quad\square$

Remarks 12.4.

(1) Suppose that Γ is a two-dimensional group. One may rephrase the previous corollary by saying that the quotient group $K_1(C_r^*\Gamma) \big/ \langle [\gamma] \mid \gamma \in \Gamma \rangle$ is zero if and only if v_1^{Γ} is surjective. Now, observe that for an arbitrary discrete group G, the class $[-1] \in K_1(C_r^*G)$ of the diagonal matrix $\mathrm{Diag}(-1, 1, 1, \ldots)$ is zero; indeed, this class lies in the image of the canonical homomorphism $K_1(\mathbb{C}) \longrightarrow K_1(C_r^*G)$ and $K_1(\mathbb{C}) = 0$. In particular, v_1^{Γ} is surjective if and only if the group

$$\mathrm{Wh}^{\mathrm{top}}(\Gamma) := K_1(C_r^*\Gamma) \, \langle [\pm\gamma] \mid \gamma \in \Gamma \rangle = K_1(C_r^*\Gamma) \Big/ \langle [\gamma] \mid \gamma \in \Gamma \rangle$$

vanishes. The definition of this quotient is somewhat reminiscent of the definition of the *Whitehead group* in algebraic K-theory (hence our notation):

$$\mathrm{Wh}(\Gamma) := K_1^{\mathrm{alg}}(\mathbb{Z}\Gamma) \, \langle [\pm\gamma] \mid \gamma \in \Gamma \rangle \,,$$

see e.g. [37]. It follows from [32, Theorem 1.1] that the map $\beta_1^{(a)}$ factorizes through the algebraic K-group $K_1^{\mathrm{alg}}(\mathbb{Z}\Gamma)$ (for an arbitrary group Γ). Therefore, we can also deduce from this all that for our two-dimensional group Γ, the following three statements are implied by the surjectivity of v_1^{Γ}:
   (a) the canonical map $K_1^{\mathrm{alg}}(\mathbb{Z}\Gamma) \longrightarrow K_1(C_r^*\Gamma)$ is surjective;
   (b) the canonical map $\mathrm{Wh}(\Gamma) \longrightarrow \mathrm{Wh}^{\mathrm{top}}(\Gamma)$ is surjective;
   (c) $\mathrm{Wh}^{\mathrm{top}}(\Gamma) = 0$.
   It would be of great interest to study these three properties independently of the Baum–Connes conjecture, and for a larger class of groups.
(2) Let Γ be a discrete group. If M is a closed oriented manifold equipped with a continuous map $M \longrightarrow B\Gamma$, then all higher signatures of M coming via f from classes lying in the subring of $H^*(\Gamma; \mathbb{Q})$ generated by $H^j(\Gamma; \mathbb{Q})$ with $j \leqslant 2$ are oriented homotopy invariants of M: this is an unpublished result of Connes, Gromov and Moscovici (see however [20]); a complete proof is now available, see [29, Corollary 0.3]. As a corollary, the usual Novikov Conjecture in topology holds for a two-dimensional group. It is not clear to us whether v_0^{Γ} is rationally injective for Γ a two-dimensional group. With no doubt, this would constitute a useful result.

Acknowledgments

Thanks are due to P. Baum, N. Higson, P. Julg and W. Lück for helpful conversations. We feel considerably indebted to G. Skandalis for his precious and generous help.

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