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Thue–Morse at multiples of an integer

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ABSTRACT

Let $\mathbf{t} = (t_n)_{n\geqslant 0}$ be the classical Thue–Morse sequence defined by $t_n = s_2(n) \pmod 2$, where s_2 is the sum of the bits in the binary representation of n. It is well known that for any integer $k \geqslant 1$ the frequency of the letter "1" in the subsequence $t_0, t_k, t_{2k}, \ldots$ is asymptotically $1/2$. Here we prove that for any k there is an $n \leqslant k+4$ such that $t_{kn} = 1$. Moreover, we show that n can be chosen to have Hamming weight $\leqslant 3$. This is best in a twofold sense. First, there are infinitely many k such that $t_{kn} = 1$ implies that n has Hamming weight $\geqslant 3$. Second, we characterize all k where the minimal n equals k, $k+1$, $k+2$, $k+3$, or $k+4$. Finally, we present some results and conjectures for the generalized problem, where s_2 is replaced by s_b for an arbitrary base $b \geqslant 2$.

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1. Introduction and preliminaries

Let $s_b(n)$ denote the sum of the digits of n when expressed in base b, and let

$$t_n = s_2(n) \mod 2, \quad n \geqslant 0,$$

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be the Thue–Morse sequence **t**. In an e-mail message dated June 7 2010, Jorge Buescu of the Universidade de Lisboa observed that the Thue–Morse sequence can be regarded as a 2-coloring of the integers, and therefore, by van der Waerden's theorem, must contain arbitrarily long monochromatic arithmetic progressions.[3] (By a *monochromatic arithmetic progression* we mean a series of indices $i, i+j, i+2j, \dots, i+(n-1)j$ such that $t_i = t_{i+j} = \cdots = t_{i+(n-1)j}$.) He then asked, is it true that **t** has no infinite monochromatic arithmetic progressions?

The answer is yes: **t** has no infinite monochromatic arithmetic progressions. This is a consequence of the following result of Gelfond [5], which says that the values of $s_b(n)$ are equally distributed in residue classes, even if the residue class of n is fixed. (A weaker result, applicable in the case of the Thue–Morse sequence, had previously been given by Fine [4].)

Theorem 1. *Let b, r, m be positive integers with $\gcd(b-1, r) = 1$, and let a, c be any integers. Then the number of integers $n \leqslant x$ congruent to $a \bmod k$ such that $s_b(n) \equiv c \pmod r$ is equal to $\frac{x}{kr} + O(x^{\lambda})$ for some $\lambda < 1$ that does not depend on x, k, a, or c.*

Gelfond's theorem, however, concerns the *average* distribution of the values of $s_b(n)$ in residue classes. It suggests the following question: how large can the smallest n be that is congruent to $a \bmod k$ and satisfies $s_b(n) \equiv c \pmod r$?

In this paper we answer the question for the case $a = 0$, k arbitrary, $c = 1$, $b = r = 2$. In other words, we find a bound on the number of terms in a fixed arithmetic progression of the Thue–Morse sequence we have to look at in order to see a "1". We include some weaker results for arbitrary b and give some conjectures.

Remark 1. Jean-Paul Allouche notes that Buescu's original question can also be answered by appealing to a lemma in his paper [1, p. 284]. His lemma states that if a, b, c are integers with $b - c > a$, then $t_{an+b} - t_{an+c}$ cannot be constant for large n. If t_{An+B} were constant for some integers A, B then it would have the same value when replacing n by $n+2$. Thus $t_{An+2A+B} - t_{An+B}$ would be constant and equal to 0, but $2A + B - B = 2A > A$ and we are done.

Remark 2. Dartyge, Luca and Stănică [3] recently investigated another problem on the *pointwise* behavior of s_b on integer multiples, namely, to bound the smallest nontrivial n that is congruent to $0 \bmod k$ and satisfies $s_b(n) = s_b(k)$. For other distributional properties of s_b on integer multiples we refer the interested reader to the bibliographic list in [3].

To begin with, for $k \geqslant 1$ we write

$$\mathcal{N}_k = \{n\colon\ t_{kn} = 1\}, \qquad f(k) = \min\{n\colon\ n \in \mathcal{N}_k\}.$$

The first few values of $(f(k))_{k \geqslant 1}$ are given by

$$1,\ 1,\ 7,\ 1,\ 5,\ 7,\ 1,\ 1,\ 9,\ 5,\ 1,\ 7,\ 1,\ 1,\ 19,\ 1,\ 17,\ 9,\ 1,\ 5 \dots. \tag{1}$$

The function f is of interest because of some old work of Newman [7]. Leo Moser observed that the first 7 multiples of 3 all have an even number of digits in their base-2 expansion. In our notation, this means $f(3) = 7$. Newman showed that among the first multiples of 3, there is always a small preponderance of those with even parity. More precisely, he showed that for all $x \geqslant 2$,

$$\frac{1}{20} \cdot (3x)^{\alpha} < \left|(\mathbb{N}_0 \setminus \mathcal{N}_3) \cap [0, x-1]\right| - \left|\mathcal{N}_3 \cap [0, x-1]\right| < 5 \cdot (3x)^{\alpha}, \tag{2}$$

[3] We do not need the power of van der Waerden's theorem to prove this. For example, as we will see later, if $k = 2^r - 1$ for some $r \geqslant 1$, then $s_2(kn) = r$ for $1 \leqslant n \leqslant k$.

where $\alpha = \log_4 3$. Coquet [2] gave a precise expression for the middle term in (2) that involves a continuous periodic fractal function with a completely explicit Fourier expansion.

From Gelfond's theorem we get that $f(k) < \infty$ for all k. Indeed, a simple observation shows that $f(k) = O(k)$. To see this, we need the following result (see [6] for the base b generalization). For the convenience of the reader we here include a full proof for $b = 2$.

Proposition 1. *Let $t, p \geqslant 1$. For all k with $1 \leqslant k < 2^t$ we have*

$$s_2\bigl(p2^t - k\bigr) = s_2(p-1) + t - s_2(k-1).$$

Proof. Write $k - 1 = \sum_{i=0}^{t-1} \kappa_i 2^i$ with $\kappa_i \in \{0, 1\}$. Then

$$\begin{aligned} s_2\bigl(p2^t - k\bigr) &= s_2\bigl((p-1)2^t + 2^t - k\bigr) = s_2(p-1) + s_2\bigl(\bigl(2^t - 1\bigr) - (k-1)\bigr) \\ &= s_2(p-1) + s_2\left(\sum_{i=0}^{t-1}(1-\kappa_i)2^i\right) = s_2(p-1) + \sum_{i=0}^{t-1}(1-\kappa_i) \\ &= s_2(p-1) + t - \sum_{i=0}^{t-1}\kappa_i = s_2(p-1) + t - s_2(k-1). \qquad \square \end{aligned}$$

Now, consider $n = 2^{2r+1} - 1 \geqslant k$. Then

$$s_2(kn) = s_2\bigl(k2^{2r+1} - k\bigr) = s_2(k-1) + 2r + 1 - s_2(k-1) \equiv 1 \pmod 2,$$

so that $f(k) < 4k$.

Our main result shows that $k = 3$ is the first of an infinite class of integers that maximize $f(k) - k$.

Theorem 2. *For all $k \geqslant 1$ we have*

$$f(k) \leqslant k + 4. \tag{3}$$

Moreover, we have

(i) *$f(k) = k + 4$ if and only if $k = 2^{2r} - 1$ for some $r \geqslant 1$.*
(ii) *There are no k with $f(k) = k + 3$ or $f(k) = k + 2$.*
(iii) *$f(k) = k + 1$ if and only if $k = 6$.*
(iv) *$f(k) = k$ if and only if $k = 1$ or $k = 2^r + 1$ for some $r \geqslant 2$.*

The proof of Theorem 2 is constructive. It allows to show that for all $k \geqslant 1$ we can always find a small n with $t_{kn} = 1$ having Hamming weight at most 3.

Corollary 1. *For all k there is an $n \in \mathcal{N}_k$ with $n \leqslant k + 4$ and $s_2(n) \leqslant 3$.*

This is optimal in the sense that there are infinitely many k such that all $n \in \mathcal{N}_k$ satisfy $s_2(n) \geqslant 3$. In the following proposition we give such an infinite family.

Proposition 2. *Let $r \geqslant 4$ and set $k = 3 \cdot 2^r + 3$. If $s_2(kn) \equiv 1 \pmod 2$ then $s_2(n) \geqslant 3$.*

Proof. Suppose that $n = 2^j$, $j \geqslant 0$. Then $s_2(k \cdot 2^j) = s_2(k) = s_2(2^{r+1} + 2^r + 2 + 1) = 4 \equiv 0 \pmod 2$. Let $n = (2^j + 1) \cdot 2^{j'}$ for $j \geqslant 1$ and $j' \geqslant 0$. Then we have

$$s_2(kn) = s_2\bigl(3 \cdot 2^{r+j} + 3 \cdot 2^r + 3 \cdot 2^j + 3\bigr). \tag{4}$$

If $j = 1$ then $s_2(kn) = s_2(9 \cdot 2^r + 9) = 4$. Suppose $j > 1$ and without loss of generality assume that $r \geqslant j$. If $r > j + 1$ then the summands in (4) are noninterfering. This yields $s_2(kn) = 8$. If $r = j + 1$ then $s_2(kn) = s_2(3 \cdot 2^{2j+1} + 9 \cdot 2^j + 3)$ equals 6 or 4 depending on whether $j > 2$ or $j = 2$. The same is true in the case where $r = j$ and $s_2(kn) = s_2(3 \cdot 2^{2j} + 3 \cdot 2^{j+1} + 3)$. □

The paper is structured as follows. We introduce some useful notation in Section 2 which allows us to perform addition in the binary expansion of integers in a well-arranged way. In Section 3 we shortly outline the idea of the proof of our main result. In Section 4 we state some auxiliary results which are based on a detailed investigation of various cases. Section 5 is devoted to the proof of Theorem 2. We conclude with some results in the general case, where the condition $s_2(kn) \equiv 1 \pmod 2$ is changed to $s_b(kn) \equiv c \pmod r$ (Section 6).

2. Notation

In this section we introduce some notation. If

$$k = \varepsilon_{\ell-1}(k)2^{\ell-1} + \varepsilon_{\ell-2}(k)2^{\ell-2} + \cdots + \varepsilon_0(k)$$

is the canonical base-2 representation of k, satisfying $\varepsilon_j \in \{0, 1\}$ for all $0 \leqslant j < \ell$ and $\varepsilon_{\ell-1}(k) \neq 0$, then we let $(k)_2$ denote the binary word

$$\varepsilon_{\ell-1}(k)\varepsilon_{\ell-2}(k) \cdots \varepsilon_0(k).$$

Additionally, for each $k \in \mathbb{N}$ we let $\ell(k)$ denote the length of $(k)_2$; for $k \geqslant 1$ this is $\ell(k) = \lfloor \log_2 k \rfloor + 1$. If w_1 and w_2 are two binary words, then $w_1 w_2$ denotes the binary word obtained by concatenation. The symbol a^n, $n \geqslant 1$, $a \in \{0, 1\}$ is an abbreviation for the word

$$\overbrace{aa \cdots a}^{n},$$

and a^0 is equal to the empty word. For $a \in \{0, 1\}$, we use the notation $\bar{a} = 1 - a$. We define the function s for all binary words $w = \varepsilon_{j-1} \cdots \varepsilon_0$ by

$$s(w) = \#\{i, 0 \leqslant i < j\colon \varepsilon_i = 1\},$$

and in particular, we have $s((k)_2) = s_2(k) \equiv t_k \pmod 2$. If $1 \leqslant j \leqslant \ell(k)$, we set

$$L_j(k) = \varepsilon_{j-1} \cdots \varepsilon_0,$$

the j least significant bits of k in base 2, and

$$U_j(k) = \varepsilon_{\ell(k)-1} \cdots \varepsilon_{\ell(k)-j},$$

the j most significant bits of k. For example, if $k = 119\,759$, then we have

$$(k)_2 = 1\,1\,1\,0\,1\,0\,0\,1\,1\,1\,1\,0\,0\,1\,1\,1\,1, \quad \ell(k) = 17$$

and

$$\overbrace{1\,1\,1\,0\,1\,0\,0\,1\,1\,1\,1\,0}^{U_{12}(k)}\!\!\!\!\!\!\!\!\!\underbrace{0\,1\,1\,1\,1}_{L_8(k)}.$$

Written in short form, this means that $(k)_2 = 1^3\,0\,1\,0^2\,1^4\,0^2\,1^4$,

$$U_{12}(k) = 1^3\,0\,1\,0^2\,1^4\,0 \quad \text{and} \quad L_8(k) = 1^2\,0^2\,1^4.$$

In what follows, we use the convention that if we are talking about $L_j(k)$ or $U_j(k)$, we assume that $\ell(k) \geqslant j$. Note that for all $k \in \mathbb{N}$ and $j < \ell(k)$ we have

$$s\big(L_{\ell(k)-j}(k)\big) \equiv s\big(U_j(k)\big) + t_k \pmod 2.$$

Furthermore, this function also satisfies

$$s(w_1 w_2) = s(w_1) + s(w_2)$$

for two binary words w_1 and w_2.

3. Idea of proof

It is relatively easy to show that $f(k) = k+4$ if $k = 2^{2r} - 1$ for some $r \geqslant 1$ and $f(k) = k$ if $k = 2^n + 1$ for some $n \geqslant 2$ (see the proofs of Theorem 2 and Lemma 1). Moreover, since $f(k) = f(2k)$ for all $k \geqslant 1$, in order to prove Theorem 2, it suffices to show that $f(k) < k$ for all odd integers k other than those stated above. Thus, we assume in Section 4 that k is an odd integer.

We use two different ideas in order to succeed, depending on the base-2 representation of k. We show for a large set of integers k that there exists an integer $n < k$ with Hamming weight 2 such that $t_{kn} = 1$. To be more precise, we find for such integers k a positive integer $a < \ell(k)$ such that $t_{kn} = 1$ with $n = 2^a + 1 \leqslant 2^{\ell(k)-1} + 1 < k$. For the remaining odd integers k we show that there exist positive odd integers $m < k$ with Hamming weight 2 and $n < k$ with Hamming weight 3 such that

$$t_{kn} \equiv 1 + t_k + t_{km} \pmod 2.$$

This implies that $f(k) < k$ since at least one of the three numbers t_k, t_{km} and t_{kn} has to be equal to 1.

4. Auxiliary results

We have to distinguish several cases according to the beginning and the ending part of the binary expansion of k.

Lemma 1. *Let $k \in \mathbb{N}$ be such that there exists an odd integer $u \geqslant 1$ with $L_{u+1}(k) = 0\,1^u$. Then we have $f(k) \leqslant k$. Furthermore, $f(k) = k$ if and only if $k = 2^r + 1$ for some $r \geqslant 2$.*

Proof. Let $\ell = \ell(k)$ and set $n = 2^{\ell-1} + 1$. In what follows we show that $t_{kn} = 1$. We have

$$(kn)_2 = U_{\ell-(u+1)}(k)\,1\,0^u\,L_{\ell-1}(k).$$

The following figure explains this fact:

$$\begin{array}{r} \cdots 0\,1^{u-1}\,1 \\ 1\cdots \\ \hline \cdots 1\,0^{u-1}\,0\cdots \end{array}.$$

The first line $(\cdots 0\,1^{u-1}\,1)$ corresponds to the expansion of $k2^{\ell-1}$ and the second line $(1\cdots)$ to the expansion of k. By "$\cdots$" we refer to digits that are not important for our argument. Since

$$s\big(U_{\ell-(u+1)}(k)\big) \equiv s\big(L_{u+1}(k)\big) + t_k \equiv u + t_k \pmod 2,$$

and

$$s\big(L_{\ell-1}(k)\big) \equiv s\big(U_1(k)\big) + t_k \equiv 1 + t_k \pmod 2,$$

we obtain

$$t_{kn} \equiv s\big((kn)_2\big) \equiv u + t_k + 1 + 1 + t_k \equiv u \equiv 1 \pmod 2,$$

which shows that $t_{kn} = 1$. The definition of $\ell = \ell(k)$ implies that $2^{\ell-1}+1 \leqslant k$. If $k = 2^{\ell-1}+1$, we have $t_{km} = 0$ for all $1 \leqslant m < k$. Indeed, if $1 \leqslant m < 2^{\ell-1}$, then the 2-additivity of the binary sum-of-digits function s_2 implies

$$s_2(km) = s_2\big(2^{\ell-1}m + m\big) = s_2(m) + s_2(m).$$

Thus we have $t_{km} \equiv s_2(km) \equiv 0 \pmod 2$ for all $1 \leqslant m < 2^{\ell-1}$. If $m = 2^{\ell-1}$, then we clearly have $t_{km} = 0$. This finally proves that $f(k) = k$ if $k = 2^{\ell-1}+1$ and $f(k) < k$ if k satisfies the assumptions of Lemma 1 but $k \neq 2^{\ell-1}+1$. □

Lemma 2. *Let $k \in \mathbb{N}$. If there exists an even integer $u \geqslant 2$ with $L_{u+2}(k) = 1\,0\,1^u$, then we have $f(k) < k$.*

Proof. Set $\ell = \ell(k)$. First, we show that if there exists a positive integer $r \neq u$ such that $U_{r+1}(k) = 1^r\,0$, then $f(k) < k$. If $r < u$, we set $n = 2^{\ell-(r+1)} + 1 < k$. Then we have

$$(kn)_2 = U_{\ell-(u+1)}(k)\,1\,0^{u-(r+1)}\,1^{r-1}\,0\,1\,L_{\ell-r-1}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 0\,1^{u-(r+1)}\,1^{r-1}\,1\,1 \\ 1^{r-1}\,1\,0\cdots \\ \hline \cdots 1\,0^{u-(r+1)}\,1^{r-1}\,0\,1\,\cdots \end{array}.$$

Since $s(U_{\ell-(u+1)}(k)) \equiv s(L_{u+1}(k)) + t_k \equiv u + t_k \pmod 2$ and $s(L_{\ell-r-1}(k)) \equiv s(U_{r+1}) + t_k \equiv r + t_k$, we get

$$t_{kn} \equiv u + t_k + 1 + (r-1) + 1 + r + t_k \equiv u + 1 \pmod 2. \tag{5}$$

This shows that $f(k) < k$ if $r < u$ since u is even. If $r > u$, we set $n = 2^{\ell-u} + 2^{\ell-u-1} + 1$. Since $\ell - u < \ell - 1$ we have $n < k$. We get

$$(kn)_2 = U_{\ell+2-(u+2)}(3k)\,1\,0\,1^{u-2}\,0^2 L_{\ell-(u+1)}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 1\,0\,1\,1^{u-2}\,1 \\ \cdots 1\,0\,1^{u-2}\,1\,1 \\ 1\,1^{u-2}\,1\,1\,\cdots \\ \hline \cdots 1\,0\,1^{u-2}\,0\,0\,\cdots \end{array}.$$

Noting that $3k$ has $\ell+2$ digits, i.e., $\ell(3k)=\ell+2$, we obtain

$$\begin{aligned} t_{kn} &\equiv s\big(L_{u+2}(3k)\big)+t_{3k}+1+(u-2)+s\big(U_{u+1}(k)\big)+t_k \\ &\equiv s\big(L_{u+2}(3k)\big)+t_{3k}+u-1+u+1+t_k \\ &\equiv s\big(L_{u+2}(3k)\big)+t_{3k}+t_k \pmod 2. \end{aligned}$$

Since

$$L_{u+2}(3k)=0^2\,1^{u-2}\,0\,1,$$

we have $t_{kn}\equiv 1+t_k+t_{3k} \pmod 2$. As we have seen in Section 3, this implies $f(k)<k$.

For the rest of the proof we assume that $U_{u+1}(k)=1^u\,0$ for some integer $u\geqslant 2$. If $(k)_2=1^u\,0\,1^u$, then it is easy to see that $f(k)=3$. Thus we can assume that there exists a positive integer v such that $L_{v+u+2}(k)=0\,1^v\,0\,1^u$. If v is odd, then we set $n=2^{\ell-(u+1)}+1<k$. We get

$$(kn)_2=U_{\ell-(v+u+2)}(k)\,1\,0^{v+1}\,1^{u-2}\,0\,1\,L_{\ell-(u+1)}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 0\,1^v\,0\,1^{u-2}\,1\,1 \\ 1\,1^{u-2}\,1\,0\,\cdots \\ \hline \cdots 1\,0^v\,0\,1^{u-2}\,0\,1\,\cdots \end{array}.$$

This implies

$$\begin{aligned} t_{kn} &\equiv s\big(L_{v+u+2}(k)\big)+t_k+1+(u-2)+1+s\big(U_{u+1}(k)\big)+t_k \\ &\equiv u+v+u\equiv 1 \pmod 2. \end{aligned}$$

If v is even, we have two cases to consider: $u\geqslant 4$ and $u=2$.

If $u\geqslant 4$, we set $n=2^{\ell-u}+2^{\ell-u-1}+1<k$. Then we have

$$(kn)_2=U_{\ell+2-(v+u+2)}(3k)\,0\,1^v\,0\,1^{u-3}\,0\,1^2 L_{\ell-(u+1)}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 0\,1\,1^{v-1}\,0\,1\,1^{u-3}\,1\,1 \\ \cdots 0\,1^{v-1}\,1\,0\,1^{u-3}\,1\,1\,1 \\ 1\,1^{u-3}\,1\,1\,0\,\cdots \\ \hline \cdots 0\,1^{v-1}\,1\,0\,1^{u-3}\,0\,1\,1\,\cdots \end{array}.$$

Note that

$$L_{v+u+2}(3k)=0\,1^{v-1}\,0^2\,1^{u-2}\,0\,1. \tag{6}$$

Thus we get

$$\begin{aligned} t_{kn} &\equiv s\big(L_{v+u+2}(3k)\big) + t_{3k} + v + (u-3) + 2 + s\big(U_{u+1}(k)\big) + t_k \\ &\equiv (v-1) + (u-2) + 1 + t_{3k} + v + (u-3) + 2 + u + t_k \\ &\equiv 1 + t_{3k} + t_k \pmod 2, \end{aligned}$$

and we obtain $f(k) < k$.

Now we consider the case $u = 2$. In order to complete the proof of the lemma, it remains to show that $f(k) < k$ for integers k with $U_3(k) = 1^2\,0$ and $L_{v+4}(k) = 0\,1^v\,0\,1^2$ for an even positive integer v. If $U_4(k) = 1^2\,0\,1$, then we set $n = 2^{\ell-4} + 1$. Here we get

$$(kn)_2 = U_{\ell-(v+4)}(k)\,1\,0^{v-1}\,1\,0^3\,L_{\ell-4}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 0\,1^{v-1}\,1\,0\,1\,1 \\ 1\,1\,0\,1\,\cdots \\ \hline \cdots 1\,0^{v-1}\,1\,0\,0\,0\,\cdots \end{array},$$

and we obtain

$$\begin{aligned} t_{kn} &\equiv s\big(L_{v+4}(k)\big) + t_k + 1 + 1 + s\big(U_4(k)\big) + t_k \\ &\equiv (v+2) + 3 \equiv 1 \pmod 2. \end{aligned}$$

If $U_5(k) = 1^2\,0^3$, we set $n = 2^{\ell-4} + 2^{\ell-5} + 1$. It follows that

$$(kn)_2 = U_{\ell+2-(v+4)}(3k)\,1\,0^{v-1}\,1\,0^2\,1\,L_{\ell-5}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 0\,1\,1^{v-2}\,1\,0\,1\,1 \\ \cdots 0\,1^{v-2}\,1\,1\,0\,1\,1 \\ 1\,1\,0\,0\,0\,\cdots \\ \hline \cdots 1\,0^{v-2}\,0\,1\,0\,0\,1\,\cdots \end{array},$$

and we obtain

$$\begin{aligned} t_{kn} &\equiv s\big(L_{v+4}(3k)\big) + t_{3k} + 1 + 1 + 1 + s\big(U_5(k)\big) + t_k \\ &\equiv v + t_{3k} + 1 + 2 + t_k \equiv 1 + t_{3k} + t_k \pmod 2. \end{aligned}$$

Here we used Eq. (6) and we get $f(k) < k$.

If $U_5(k) = 1^2\,0^2\,1$, we set $n = 2^{\ell-3} + 2^{\ell-1} + 1$. It is easy to see that $\ell(5k) = \ell + 2$ or $\ell(5k) = \ell + 3$. We have

$$(kn)_2 = U_{\ell(5k)-(v+4)}(5k)\,1\,0^{v+3}\,L_{\ell-5}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 0\,1\,1\,1^{v-2}\,0\,1\,1 \\ \cdots 0\,1^{v-2}\,1\,1\,0\,1\,1 \\ 1\,1\,0\,0\,1\cdots \\ \hline \cdots 1\,0^{v-2}\,0\,0\,0\,0\,0\cdots \end{array}.$$

Since

$$L_{v+4}(5k) = 0\,1^{v-2}\,0^2\,1^3,$$

we obtain

$$\begin{aligned} t_{kn} &\equiv s\big(L_{v+4}(5k)\big) + t_{5k} + 1 + s\big(U_5(k)\big) + t_k \\ &\equiv (v+1) + t_{5k} + 1 + 3 + t_k \equiv 1 + t_{5k} + t_k \pmod 2. \end{aligned}$$

Again, the considerations of Section 3 show that $f(k) < k$. □

Lemma 3. *Let $k \in \mathbb{N}$. If there exist an even integer $u \geqslant 2$ and a positive integer $r \neq u$ such that $L_{u+2}(k) = 0^2\,1^u$ and $U_{r+1}(k) = 1^r\,0$, then we have $f(k) < k$.*

Proof. Let $\ell = \ell(k)$. If $r < u$, we set $n = 2^{\ell-r-1} + 1 < k$. In exactly the same manner as at the beginning of the proof of Lemma 2 (see Eq. (5)), we see that $t_{kn} = 1$ and thus, $f(k) < k$.

If $r > u$, we set $n = 2^{\ell-u-1} + 1 < k$. Then we get

$$(kn)_2 = U_{\ell-(u+2)}(k)\,1\,0\,1^{u-1}\,0\,L_{\ell-(u+1)}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 0\,0\,1^{u-1}\,1 \\ 1\,1^{u-1}\,1\,\cdots \\ \hline \cdots 1\,0\,1^{u-1}\,0\,\cdots \end{array}.$$

Similarly as before, we have $s(U_{\ell-(u+2)}(k)) \equiv s(L_{u+2}(k)) + t_k \equiv u + t_k \pmod 2$ and $s(L_{\ell-(u+1)}(k)) \equiv s(U_{u+1}) + t_k \equiv u + 1 + t_k$ (note that $u + 1 \leqslant r$). We obtain

$$t_{kn} \equiv u + t_k + 1 + (u-1) + (u+1) + t_k \equiv u + 1 \equiv 1 \pmod 2.$$

This shows the desired result. □

Lemma 4. *Let $k \in \mathbb{N}$. If there exist an even integer $u \geqslant 2$ and a positive integer $s < u - 1$ such that $L_{u+2}(k) = 0^2\,1^u$ and $U_{u+s+1}(k) = 1^u\,0^s\,1$, then we have $f(k) < k$.*

Proof. Let $\ell = \ell(k)$ and set $n = 2^{\ell-1} + 2^{u-1} + 1$. Since k is odd and starts with at least two 1's, we see that $n < k$. We have

$$(nk)_2 = U_{\ell-(u+2)}(k)\,1\,0^{u+s+1}\,L_{\ell(km)-(s+u+1)}(mk),$$

where $m = 2^{u-1} + 1 < k$ and $\ell(km) = \ell + u$, as illustrated below:

$$\begin{array}{r} \cdots 0\,0\,1^{u-1}\,1 \\ 1^{u-1}\,1\,0^s\,1\,\cdots \\ 1\,1^s\,1\,\cdots \\ \hline \cdots 1\,0\,0^{u-1}\,0\,0^s\,\cdots \end{array}.$$

We have

$$U_{s+u+1}(km) = 1\,0^{u-1}\,1\,0^s.$$

In particular, we obtain

$$s\big(L_{\ell(km)-(s+u+1)}(mk)\big) \equiv s\big(U_{s+u+1}(km)\big) + t_{km} \equiv t_{km} \pmod 2,$$

and we get

$$t_{nk} \equiv s\big(L_{u+2}(k)\big) + t_k + 1 + t_{km} \equiv u + 1 + t_k + t_{km} \equiv 1 + t_k + t_{km} \pmod 2.$$

As before, we get $f(k) < k$, which proves Lemma 4. □

Lemma 5. *Let $k \in \mathbb{N}$. If there exist an even integer $u \geqslant 2$ and a positive integer $t \geqslant 2$ such that $L_{u+t+2}(k) = 0\,1\,0^t\,1^u$ and $U_{2u-1}(k) = 1^u\,0^{u-1}$, then we have $f(k) < k$.*

Proof. Let $\ell = \ell(k)$ and let us first assume that $2 \leqslant t \leqslant u-1$. We set $n = 2^{\ell-(u+t)} + 1 < k$. Then we get

$$(kn)_2 = U_{\ell-(u+t+2)}(k)\,1\,0^{t+1}\,1^{u-(t+1)}\,0\,1^t\,L_{\ell-(u+t)}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 0\,1\,0^t\,1^{u-(t+1)}\,1\,1^t \\ 1^t\,1^{u-(t+1)}\,1\,0^t\,\cdots \\ \hline \cdots 1\,0\,0^t\,1^{u-(t+1)}\,0\,1^t\,\cdots \end{array}.$$

Since $u < u + t \leqslant 2u - 1$, we have

$$s\big(L_{\ell-(u+t)}(k)\big) \equiv s\big(U_{u+t}(k)\big) + t_k \equiv u + t_k \pmod 2.$$

This implies

$$t_{kn} \equiv s\big(L_{u+t+2}(k)\big) + t_k + 1 + \big(u - (t+1)\big) + t + u + t_k \equiv u + 1 \equiv 1 \pmod 2.$$

If $t = u$ and $U_{2u}(k) = 1^u\,0^{u-1}\,1$, we again set $n = 2^{\ell-(u+t)} + 1 < k$. This time we can write

$$(kn)_2 = U_{\ell-(u+t+2)}(k)\,1\,0^{t+u+1}\,L_{\ell-(u+t)}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 0\,1\,0^t\,1^{u-1}\,1 \\ 1^t\,0^{u-1}\,1\,\cdots \\ \hline \cdots 1\,0\,0^t\,0^{u-1}\,0\,\cdots \end{array},$$

and we get

$$t_{kn} \equiv s\big(L_{u+t+2}(k)\big) + t_k + 1 + s\big(U_{u+t}(k)\big) + t_k \equiv (u+1) + 1 + (u+1) \equiv 1 \pmod 2.$$

Alternatively, if $t = u$ and $U_{2u+1}(k) = 1^u\,0^u\,a$ for some $a \in \{0,1\}$, then we set $n = 2^{\ell-(t+u+1)} + 1 < k$. Since we have (recall that $\bar{a} = 1 - a$)

$$(kn)_2 = U_{\ell-(u+t+2)}(k)\,1\,0\,1^{t-1}\,a\,\bar{a}^u\,L_{\ell-(u+t+1)}(k),$$

as illustrated below,

$$\begin{array}{r} \cdots 0\,1\,0^{t-1}\,0\,1^{u-1}\,1 \\ 1\,1^{t-1}\,0\,0^{u-1}\,a\,\cdots \\ \hline \cdots 1\,0\,1^{t-1}\,a\,\bar{a}^{u-1}\,\bar{a}\,\cdots \end{array},$$

and we finally obtain

$$\begin{aligned} t_{kn} &\equiv s\big(L_{u+t+2}(k)\big) + t_k + 1 + (t-1) + a + au + s\big(U_{u+t+1}(k)\big) + t_k \\ &\equiv (u+1) + 1 + (t-1) + a(u+1) + (u+a) \\ &\equiv 1 \pmod 2. \end{aligned}$$

This shows the desired result. □

Lemma 6. *Let $k \in \mathbb{N}$. If there exist an even integer $u \geqslant 2$ and positive integers $t \geqslant 2$ such that $L_{u+t+2}(k) = 1\,1\,0^t\,1^u$ and $U_{2u-1}(k) = 1^u\,0^{u-1}$, then we have $f(k) < k$.*

Proof. Let $\ell = \ell(k)$. First, we consider the case $2 \leqslant t \leqslant u-1$. Set $n = 2^{\ell-(t+u-1)} + 2^{\ell-(t+u)} + 1 < k$. Then we have

$$(kn)_2 = U_{\ell+2-(t+u+2)}(3k)\,1\,0^{t-1}\,1\,0\,1^{u-(t+1)}\,0\,1^{t-2}\,0\,1\,L_{\ell-(u+t)}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 1\,1\,0\,0^{t-2}\,0\,1\,1^{u-(t+1)}\,1\,1^{t-2}\,1 \\ \cdots 1\,1\,0^{t-2}\,0\,0\,1^{u-(t+1)}\,1\,1^{t-2}\,1\,1 \\ 1^{t-2}\,1\,1\,1^{u-(t+1)}\,1\,0^{t-2}\,0\,0\,\cdots \\ \hline \cdots 1\,0\,0^{t-2}\,1\,0\,1^{u-(t+1)}\,0\,1^{t-2}\,0\,1\,\cdots \end{array}.$$

Noting that $\ell(3k) = \ell + 2$, we obtain

$$\begin{aligned} t_{kn} &\equiv s\big(L_{u+t+2}(3k)\big) + t_{3k} + \big(u - (t+1)\big) + (t-2) + s\big(U_{u+t}(k)\big) + t_k + 3 \\ &\equiv s\big(L_{u+t+2}(3k)\big) + t_{3k} + u + u + t_k \\ &\equiv s\big(L_{u+t+2}(3k)\big) + t_{3k} + t_k \pmod 2. \end{aligned}$$

Since

$$L_{u+t+2}(3k) = 0\,1\,0^{t-2}\,1\,0\,1^{u-2}\,0\,1, \tag{7}$$

we have $t_{kn} \equiv 1 + t_{3k} + t_k \pmod 2$ and consequently $f(k) < k$.

If $t = u$, then we have to consider three different cases. If $U_{2u}(k) = 1^u\,0^{u-1}\,1$ we again set $n = 2^{\ell-(t+u-1)} + 2^{\ell-(t+u)} + 1 < k$. This time we get

$$(kn)_2 = U_{\ell+2-(t+u+2)}(3k)\,1\,0^t\,1^u\,0\,L_{\ell-(2u)}(k),$$

as illustrated below,

$$\begin{array}{r} \cdots 1\,1\,0\,0^{t-1}\,1\,1^{u-1} \\ \cdots 1\,1\,0^{t-1}\,0\,1^{u-1}\,1 \\ 1^{t-1}\,1\,0^{u-1}\,1\,\cdots \\ \hline \cdots 1\,0\,0^{t-1}\,1\,1^{u-1}\,0\,\cdots \end{array},$$

which yields

$$\begin{aligned} t_{kn} &\equiv s\big(L_{u+t+2}(3k)\big) + t_{3k} + 1 + u + s\big(U_{2u}(k)\big) + t_k \\ &\equiv s\big(L_{u+t+2}(3k)\big) + t_{3k} + 1 + u + (u+1) + t_k \\ &\equiv s\big(L_{u+t+2}(3k)\big) + t_{3k} + t_k \pmod 2. \end{aligned}$$

As above (see Eq. (7)), we get that $f(k) < k$. If $t = u$ and $U_{2u+1}(k) = 1^u\,0^u\,1$, then we set $n = 2^{\ell-(t+u)} + 2^{\ell-(t+u)-1} + 1 < k$. We have

$$(kn)_2 = U_{\ell+2-(t+u+2)}(3k)\,1^2\,0^t\,1^{u-1}\,0\,L_{\ell-(2u+1)}(k),$$

as illustrated below:

$$\begin{array}{r} \cdots 1\,1\,0\,0^{t-1}\,1\,1^{u-1} \\ \cdots 1\,1\,0^{t-1}\,0\,1^{u-1}\,1 \\ 1\,1^{t-1}\,0\,0^{u-1}\,1\,\cdots \\ \hline \cdots 1\,1\,0^{t-1}\,0\,1^{u-1}\,0\,\cdots \end{array}.$$

Note that the dots in the first and second line of the figure have to be erased if $k = 51$. (The binary representation of 51 is given by $(51)_2 = 1\,1\,0\,0\,1\,1$.) We get

$$\begin{aligned} t_{kn} &\equiv s\big(L_{u+t+2}(3k)\big) + t_{3k} + 2 + (u-1) + s\big(U_{2u+1}(k)\big) + t_k \\ &\equiv s\big(L_{u+t+2}(3k)\big) + t_{3k} + 2 + (u-1) + (u+1) + t_k \\ &\equiv s\big(L_{u+t+2}(3k)\big) + t_{3k} + t_k \pmod 2. \end{aligned}$$

Using Eq. (7), we obtain $f(k) < k$. If $t = u$ but $U_{2u+1}(k) = 1^u\,0^{u+1}$, then we choose $n = 2^{\ell-1} + 2^u + 1$. Since k is odd and starts with at least two 1's, we again obtain that $n < k$. This leads us to

$$(kn)_2 = U_{\ell-(u+2)}(k)\,1\,0\,1^{u-1}\,0\,1^{u-1}\,L_{\ell+u-(2u)}(km),$$

where $m = 2^u + 1 < k$, as illustrated below:

$$\begin{array}{l} \cdots 0\,0\,1^{u-1}\,1 \\ 1\,1^{u-1}\,0\,0^{u-1}\,0\,\cdots \\ \phantom{\cdots 0\,0\,1\,1^{u-1}\,0\,}1\,1^{u-1}\,0\,\cdots \\ \hline \cdots 1\,0\,1^{u-1}\,0\,1^{u-1}\,\cdots \end{array}.$$

Note that $\ell(km) = \ell + u$ and

$$U_{2u}(km) = 1^{2u}.$$

We obtain

$$\begin{aligned} t_{kn} &\equiv s\big(L_{u+2}(k)\big) + t_k + 1 + (u-1) + (u-1) + s\big(U_{2u}(km)\big) + t_{km} \\ &\equiv u + t_k + 1 + 2u + t_{km} \\ &\equiv 1 + t_k + t_{km} \pmod 2, \end{aligned}$$

which shows $f(k) < k$ in this case, too.

In order to prove the lemma, it remains to consider the case $t > u$. We set $n = 2^{\ell-1} + 2^{\ell-u-1} + 1 < k$ and we get

$$(kn)_2 = U_{\ell+u-(2u+1)}(km)\,1\,0^{u-1}\,1^{u-1}\,0\,1\,L_{\ell-(u+1)}(k),$$

as illustrated below,

$$\begin{array}{l} \cdots 0\,1^{u-1}\,1 \\ \cdots 0\,0^{u-1}\,0\,1^{u-2}\,1\,1 \\ \qquad\qquad 1\,1^{u-2}\,1\,0\,\cdots \\ \hline \cdots 1\,0^{u-1}\,1\,1^{u-2}\,0\,1\,\cdots \end{array},$$

where $m = 2^u + 1 < k$. Since

$$L_{2u+1}(km) = 0\,1^{2u},$$

we obtain

$$\begin{aligned} t_{kn} &\equiv s\big(L_{2u+1}(km)\big) + t_{km} + 1 + (u-1) + 1 + s\big(U_{u+1}(k)\big) + t_k \\ &\equiv 2u + t_{km} + u - 1 + u + t_k \\ &\equiv 1 + t_k + t_{km} \pmod 2. \end{aligned}$$

The same argument as before finally shows the desired result. □

5. Proof of Theorem 2

Proof of Theorem 2. As already noted in Section 3, we have $f(k) = f(2k)$ for all $k \geqslant 1$. Consequently, it suffices to show that $f(k) \leqslant k + 4$ for odd integers k.

If $k = 2^{2r+1} - 1$, $r \geqslant 0$, then $(k)_2 = 1^{2r+1}$ and we trivially have $f(k) = 1$. If $k = 2^{2r} - 1$, $r \geqslant 1$, then we will show that $f(k) = k + 4$. In order to do this, we note that the binary sum-of-digits function s_2 satisfies the relation

$$s_2\big(a2^k - b\big) = s_2(a-1) + k - s_2(b-1)$$

for all positive integers a, b, k with $1 \leqslant b < 2^k$. Thus we have for all $1 \leqslant m \leqslant k$,

$$t_{km} \equiv s_2\big(2^{2^r} m - m\big) = s_2(m-1) + 2r - s_2(m-1) \equiv 0 \pmod 2.$$

If $m = 2^{2r}$ or $m = 2^{2r} + 2$, then we clearly have $t_{km} = 0$ since $t_{km/2} = 0$. If $m = 2^{2m} + 1$, then $km = 2^{4r} - 1$ and consequently $t_{km} = 0$. If $m = 2^{2r} + 3$, then

$$t_{km} \equiv s_2\left(2^{4r} + 2^{2r+1} - 3\right) \equiv 1 + (2r+1) - s_2(2) \equiv 1 \pmod 2,$$

which finally proves $f(k) = k + 4$ for $k = 2^{2r} - 1$, $r \geqslant 1$. If k is a positive integer different from ones already considered, then there exist positive integers r, s, t and u such that

$$U_{r+s+1}(k) = 1^r\,0^s\,1 \quad \text{and} \quad L_{t+u+1} = 1\,0^t\,1^u.$$

If u is odd, then Lemma 1 implies that $f(k) \leqslant k$ where equality occurs if and only if $k = 2^r + 1$ for some $r \geqslant 2$. If u is even but $t = 1$ or $r \neq u$, then Lemma 2 and Lemma 3 imply $f(k) < k$. Let us assume that u is even, $t \geqslant 2$ and $r = u$. Then there exists $a \in \{0, 1\}$, such that

$$L_{t+u+2} = a\,1\,0^t\,1^u.$$

If $s < u - 1$, then Lemma 4 implies $f(k) < k$. Contrarily, if $s \geqslant u - 1$, then Lemma 5 (if $a = 0$) or Lemma 6 (if $a = 1$) yields $f(k) < k$. Hence we have for all positive integers k,

$$f(k) \leqslant k + 4,$$

where equality occurs if and only if $k = 2^{2r} - 1$, $r \geqslant 1$. Note that an even positive integer $2m$ cannot satisfy $f(2m) = 2m + 4$, since we then would get $f(m) = f(2m) = 2m + 4 \leqslant m + 4$ and consequently $m \leqslant 0$. Moreover, we see that for odd integers k there exist no solutions to the equation

$$f(k) = k + \alpha$$

for $\alpha = 0, 1, 2, 3$, except in the case $\alpha = 0$ where we have $f(k) = k$ if and only if $k = 2^r + 1$ for some $r \geqslant 2$ or $k = 1$. If $k = 2m$ is even, then $f(2m) = 2m + \alpha$ implies $f(m) = 2m + \alpha \leqslant m + 4$. Hence this can only happen if $m \leqslant 4 - \alpha$. We see that there exist no solutions to $f(k) = k + \alpha$ for $\alpha = 2$ and $\alpha = 3$, there are no even solutions for $\alpha = 0$ and the only solution to $f(k) = k + 1$ is $k = 6$ (compare with Eq. (1)). This finally proves Theorem 2. □

Remark 3. By a similar case analysis it might be possible to prove that

$$\min\{n\colon\ t_{kn} = 0\} \leqslant k + 2.$$

However, it does not seem possible to obtain this bound in a direct way from the bound (2).

6. Some weak general results

Given the generality of Gelfond's theorem, it is natural to try to bound the minimal n such that $n \equiv a \pmod k$ and $s_b(n) \equiv c \pmod r$. Here we only get a weaker upper bound.

Proposition 3. *Let b, r, k be positive integers with $\gcd(b-1, r) = 1$, and let c be any integer. Then there exists a non-negative integer $n < b^r k$ such that $s_b(kn) \equiv c \pmod r$.*

Proof. We claim that if $1 \leqslant k \leqslant b^t$, then $s_b(k(b^t - 1)) = (b-1)t$. To see this, note that for $p, t \geqslant 1$ and all k with $1 \leqslant k < b^t$ we have

$$s_b\left(pb^t - k\right) = s_b(p-1) + (b-1)t - s_b(k-1).$$

This is a direct generalization of Proposition 1, and a complete proof can be found in [6]. Let s be the smallest integer such that $k \leqslant b^s$. Then $b^{s-1} < k$. Choose $t \in \{s, s+1, \ldots, s+r-1\}$ such that $(b-1)t \equiv c \pmod{r}$. This is possible since $\gcd(b-1, r) = 1$. Then $s_b(k(b^t - 1)) = (b-1)t \equiv c \pmod{r}$, as desired. Furthermore, $b^t \leqslant b^{s+r-1} \leqslant b^r b^{s-1} < b^r k$. Thus we can take $n = b^t - 1$. □

Corollary 2. *Let b, r, k be positive integers with $\gcd(b-1, r) = 1$, and let a, c be any integers. Then there exists an integer $n < b^{r+1}k^3$ such that $n \equiv a \pmod{k}$ and $s_b(n) \equiv c \pmod{r}$.*

Proof. Without loss of generality we can assume $0 \leqslant a < k$. As in the proof of Proposition 3 let s be the smallest integer such that $b^s \geqslant k$, so $b^{s-1} < k$. From Proposition 3 we know that there exists an integer t such that $s_b(k(b^t - 1)) \equiv (c-a) \pmod{r}$, and $b^t < b^r k$. Then clearly $s_b(kb^s(b^t - 1) + a) \equiv c \pmod{r}$, so we can take $n = kb^s(b^t - 1) + a$. Then $n < b^{r+1}k^3$. □

In the setting of Proposition 3 we conjecture that a similar phenomenon takes place as we have seen in the case of the classical Thue–Morse sequence.

Conjecture 3. *Let b, r be positive integers with $\gcd(b-1, r) = 1$, and let c be any integer. There exists a constant C, depending only on b and r such that for all $k \geqslant 1$ there exists $n \leqslant k + C$ with $s_b(kn) \equiv c \pmod{r}$. Furthermore, we can take $C \leqslant b^{r+c}$.*

We guess that this conjecture is hard to prove. In the case of the Thue–Morse sequence we used a detailed case study to succeed. In principle, in each of our lemmas we make use of a new idea to get a parity change. We do not see how this extends to the general setting where b, r and c all vary freely.

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